Another construction of type II_1 factors with prescribed countable fundamental group

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The fundamental group $\mathcal{F}(M)$

Let M be a type II_1 factor, and $t > 0$. Let $n \geq t$, and choose in $M_n(\mathbf{C}) \otimes M$ a projection p of normalized trace t/n. Define

$$M^t = p(M_n(\mathbf{C}) \otimes M)p.$$

Note that M^t is a type II_1 factor and it does not depend on the choice of n and p. Moreover, $(M^s)^t \simeq M^{st}$, $\forall s, t > 0$.

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Equivalent definition. On $M \otimes B(\ell^2)$, fix a f.n.s. trace Tr. For $\theta \in \mathrm{Aut}(M \otimes B(\ell^2))$, denote by $\mathrm{mod}(\theta)$ the unique $\lambda > 0$ such that $\mathrm{Tr}(\theta) = \lambda\,\mathrm{Tr}$. Then $\mathcal{F}(M) = \{\mathrm{mod}(\theta) : \theta \in \mathrm{Aut}(M \otimes B(\ell^2))\}$.

Review on some computations of $\mathcal{F}(M)$

- $\mathcal{F}(\mathcal{R}) = \mathbf{R}^*_+$ (Murray & von Neumann, 1943). It is a consequence of the uniqueness of the AFD type II_1 factor. Moreover, it can be realized through a one-parameter trace-scaling action (θ_s) on $\mathcal{R} \otimes B(\ell^2)$.

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- $\mathcal{F}(L(\Gamma))$ is countable for Γ an ICC c.d. Kazhdan group (Connes, 1979). This is the first occurrence of a rigidity phenomenon in vN algebras.
- $\mathcal{F}(L(\mathbf{F}_\infty)) = \mathbf{R}^*_+$ (Rădulescu, 1991). This is proved using free probability. Once again, it can be realized through a one-parameter trace-scaling action on $L(\mathbf{F}_\infty) \otimes B(\ell^2)$.

Review on some computations of $\mathcal{F}(M)$

- $\mathcal{F}(L(G)) = \{1\}$, for $G = \mathbf{Z}^2 \rtimes \mathrm{SL}(2, \mathbf{Z})$ (Popa, 2001). This is proved using the tension between the relative property (T) of the pair $(\mathbf{Z}^2 \rtimes \mathrm{SL}(2, \mathbf{Z}), \mathbf{Z}^2)$ and the Haagerup property of $\mathrm{SL}(2, \mathbf{Z})$ and Gaboriau's results.

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- $\forall S \subset \mathbf{R}^*_+$ countable subgroup, $\exists G \curvearrowright \mathcal{R}$, such that $\mathcal{F}(\mathcal{R} \rtimes G) = S$ (Popa, 2003). The Connes-Størmer Bernoulli shift $G \curvearrowright \bigotimes_G (B(\ell^2), \psi_S)$ is mixing and malleable. Realize $\mathcal{R}$ as the centralizer of $\bigotimes_G (B(\ell^2), \psi_S)$ and restrict the action.

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- $\mathcal{F}(*_{s \in S} L(G)^s) = S$ (Ioana, Peterson & Popa, 2005).

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- $\mathcal{F}(*_{s \in S} L(G)^s) = S$ (Ioana, Peterson & Popa, 2005).
- $\forall S \subset \mathbf{R}^*_+$ countable subgroup, $\exists G \curvearrowright L(\mathbf{F}_\infty)$, such that $\mathcal{F}(L(\mathbf{F}_\infty) \rtimes G) = S$ (H, 2007). This time the free Bogoliubov shift $G \curvearrowright (T_S, \varphi_S)$ on the free Araki-Woods factor is mixing and malleable (in a free sense). Realize $L(\mathbf{F}_\infty)$ as the centralizer of (T_S, φ_S) and restrict the action.

Almost periodic vN algebra $(\mathcal{M}, \varphi)$

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Set $\mathrm{Sp}(\mathcal{M}, \varphi) = \{\gamma > 0; \mathcal{M}_\gamma \neq 0\}$, necessarily countable. $(\mathcal{M}, \varphi)$ is said to be almost periodic if

$$L^2(\mathcal{M}, \varphi) = \bigoplus_{\gamma \in \mathrm{Sp}(\mathcal{M}, \varphi)} L^2(\mathcal{M}_\gamma).$$

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Examples

Any finite vN algebra (N, τ). Any type I factor $(B(H), \psi)$ endowed with a f.n. state. Any factor of type III_λ, $0 < \lambda < 1$, endowed with a $\frac{2\pi}{\log \lambda}$-periodic f.n. state. Any tensor/free product of a.p. vN algebras.

Haagerup property for finite vN algebras

Let (P, τ) be a finite vN algebra. Denote $\|x\|_2 = \tau(x^*x)^{1/2}$, $\forall x \in P$. A sequence of n.c.p. maps $\phi_n : P \to P$ is said to be a τ-deformation if $\phi_n(1) \leq 1$, $\tau \circ \phi_n \leq \tau$, $\forall n \in \mathbf{N}$ and $\lim_n \|\phi_n(x) - x\|_2 = 0, \forall x \in P$.

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Definition (Choda, 1983)

A finite vN algebra (P, τ) is said to have the Haagerup property if there exists a τ-deformation $\phi_n : P \to P$ such that the corresponding bounded operator T_{ϕ_n} on $L^2(P, \tau)$ is compact, $\forall n \in \mathbf{N}$.

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Examples

Any amenable finite vN algebra. Any interpolated free group factor $L(\mathbf{F}_t)$, $\forall 1 < t \leq \infty$. And of course, for any G c.d. group, we have

$$L(G) \text{ has the H. property} \iff G \text{ has the H. property.}$$

Relative property (T) for finite vN algebras

Definition (Popa, 2001)

An inclusion $B \subset N$ of finite vN algebras is said to be rigid or to have the relative property (T) if $\exists \tau$ (or equivalently $\forall \tau$) a f.n. trace on N such that for any τ-deformation $\phi_n : N \to N$,

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(G, H) has the relative property (T) $\Longleftrightarrow L(H) \subset L(G)$ is rigid. Examples are given by $(\mathbf{Z}^2 \rtimes \Gamma, \mathbf{Z}^2)$ where $\Gamma \subset \mathrm{SL}(2, \mathbf{Z})$ is a non-amenable subgroup; $(G \times \Gamma, G)$, with G a property (T) group, and Γ any c.d. group.

Popa's intertwining-by-bimodules device

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- $\exists n \geq 1$, $\exists \gamma > 0$, $\exists p$ *a projection in* $M_n(\mathbf{C}) \otimes B$, $\exists v \in M_{1,n}(\mathbf{C}) \otimes \mathcal{M}$ *a non-zero partial isometry which is a* γ*-eigenvector for* φ, *and* $\exists \theta : A \to p(M_n(\mathbf{C}) \otimes B)p$ *a* $*$*-homomorphism such that* $v^* v \leq p$ *and* $xv = v\theta(x), \forall x \in A$.

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- $\exists \gamma > 0$*, and* $\exists \mathcal{H}$ *a non-zero* A*-*B*-subbimodule of* $L^2(\mathcal{M}_\gamma)$ *which is finitely generated as a right* B*-module.*

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- *$\exists \gamma > 0$, and $\exists \mathcal{H}$ a non-zero A-B-subbimodule of $L^2(\mathcal{M}_\gamma)$ which is finitely generated as a right B-module.*
- *There is no sequence of unitaries (u_n) in A, such that $\|E_B(x^* u_n y)\|_2 \to 0$, $\forall x, y \in \mathcal{M}$.*

We shall denote $A \prec_{\mathcal{M}} B$, and say that A embeds into B inside $\mathcal{M}$.

Popa's intertwining-by-bimodules device

Note that if $B = \mathbf{C}$, we have $E_{\mathbf{C}} = \varphi 1$ and

$$\begin{aligned} A \nprec_{\mathcal{M}} \mathbf{C} &\iff \exists (u_n) \text{ in } A, \varphi(x^* u_n y) \to 0, \forall x, y \in \mathcal{M} \\ &\iff A \text{ is diffuse} \end{aligned}$$

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Theorem (H, 2007)

Let $(\mathcal{M}, \varphi)$ be an a.p. vN algebra, let $A \subset \mathcal{M}^\varphi$ and $\mathcal{B}$ globally invariant under (σ_t^φ). TFAE:

- $\exists \gamma > 0$, *and* $\exists \mathcal{H}$ *a non-zero A-$\mathcal{B}^\varphi$-subbimodule of $L^2(\mathcal{M}_\gamma)$ which is finitely generated as a right $\mathcal{B}^\varphi$-module.*
- *There is no sequence of unitaries (u_n) in A such that $\|E_{\mathcal{B}}(x^* u_n y)\|_2 \to 0$, $\forall x, y \in \mathcal{M}$.*

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Using this device, we generalized two results of Ioana, Peterson & Popa.

Controlling relative commutants in a free product

Let $\mathcal{A} \subset \mathcal{M}$ be an inclusion of vN algebras. A unitary $u \in \mathcal{U}(\mathcal{M})$ is said to normalize $\mathcal{A}$ if $u\mathcal{A}u^* = \mathcal{A}$. The normalizer of $\mathcal{A}$ inside $\mathcal{M}$ is the vN subalgebra of $\mathcal{M}$ generated by such unitaries.

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Theorem (H, 2007)

For $i = 1, 2$, let $(\mathcal{M}_i, \varphi_i)$ be a.p. vN algebras, and let $N \subset \mathcal{M}_i^{\varphi_i}$, $\varphi_{1|N} = \varphi_{2|N}$. Set $\mathcal{M} = \mathcal{M}_1 *_N \mathcal{M}_2$ and φ the free product state. Let $Q \subset \mathcal{M}_1^{\varphi_1}$ be such that $Q \nprec_{\mathcal{M}_1} N$. Then, any Q-$\mathcal{M}_1^{\varphi_1}$-subbimodule of $L^2(\mathcal{M}, \varphi)$ finitely generated as a right $\mathcal{M}_1^{\varphi_1}$-module is contained in $L^2(\mathcal{M}_1)$.

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This theorem implies in particular that $Q' \cap \mathcal{M} \subset \mathcal{M}_1$, and more generally that the (quasi-)normalizer of Q inside $\mathcal{M}$ is contained in $\mathcal{M}_1$!

Intertwining rigid subalgebras in a free product

Theorem (H, 2007)

For $i = 1, 2$, let $(\mathcal{M}_i, \varphi_i)$ be a.p. vN algebras, and let $N \subset \mathcal{M}_i^{\varphi_i}$, $\varphi_{1|N} = \varphi_{2|N}$. Set $\mathcal{M} = \mathcal{M}_1 *_N \mathcal{M}_2$ and φ the free product state. Let $Q \subset \mathcal{M}^\varphi$ be a vN subalgebra such that the inclusion is rigid. Then, there exists $i \in \{1, 2\}$ such that $Q \prec_{\mathcal{M}} \mathcal{M}_i$.

Using this result and the previous one, we can prove the main result of this talk.

Main theorem

A type II_1 factor N is said to be w-rigid if there exists a vN subalgebra $B \subset N$ such that the inclusion is rigid, B is diffuse, and $B \subset N$ is regular, i.e., the normalizer of B inside N is precisely N.

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Theorem (H, 2007)

Let N be a w-rigid factor such that $\mathcal{F}(N) = \{1\}$. Let $(\mathcal{A}, \psi)$ be an a.p. vN algebra such that $\mathcal{A}^\psi$ has the Haagerup property. Write $(\mathcal{M}, \varphi) = (N, \tau) * (\mathcal{A}, \psi)$, and $M = \mathcal{M}^\varphi$. Then, M is a type II_1 factor and $\mathcal{F}(M)$ is the subgroup of $\mathbf{R}^*_+$ generated by $\mathrm{Sp}(\mathcal{A}, \psi)$.

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Examples

- We can take $N = L^\infty(\mathbf{T}^2) \rtimes \Gamma$, with $\Gamma \subset \mathrm{SL}(2, \mathbf{Z})$ non-amenable.

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Examples

- We can take $N = L^\infty(\mathbf{T}^2) \rtimes \Gamma$, with $\Gamma \subset \mathrm{SL}(2, \mathbf{Z})$ non-amenable.
- Many examples of such $\mathcal{A}$'s do exist: all amenable vN algebras endowed with an a.p.f.n. state; a.p. free Araki-Woods factors (in the sense of Shlyakhtenko) endowed with their free quasi-free state...

Sketch of the proof: part I

M is a type II_1 factor. Since N is diffuse, $N \not\prec_{\mathcal{M}} \mathbf{C}$, thus (Th.Cont.) $\Longrightarrow N' \cap \mathcal{M} \subset N$. So, $\mathcal{Z}(M) \subset \mathcal{Z}(N) = \mathbf{C}$ and M is a type II_1 factor.

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$\mathrm{Sp}(\mathcal{A}, \psi) \subset \mathcal{F}(M)$. Denote by $\Gamma \subset \mathbf{R}^*_+$ the subgroup generated by $\mathrm{Sp}(\mathcal{A}, \psi)$, and note that $\Gamma = \mathrm{Sp}(\mathcal{M}, \varphi)$. The inclusion $\Gamma \subset \mathcal{F}(M)$ is quite simple. Indeed, take $\gamma \in \mathrm{Sp}(\mathcal{A}, \psi)$ and v a non-zero partial isometry in $\mathcal{A}_\gamma$. Write $p = v^* v$, $q = vv^*$. Note that $p, q \in \mathcal{A}^\psi$ and $\psi(q) = \gamma\psi(p)$. Consequently, $\mathrm{Ad}(v)$ yields a $*$-isomorphism between pMp and qMq, and then $\gamma \in \mathcal{F}(M)$.

Sketch of the proof: part I

M is a type II_1 factor. Since N is diffuse, $N \not\prec_{\mathcal{M}} \mathbf{C}$, thus (Th.Cont.) $\Longrightarrow N' \cap \mathcal{M} \subset N$. So, $\mathcal{Z}(M) \subset \mathcal{Z}(N) = \mathbf{C}$ and M is a type II_1 factor.

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$\mathcal{F}(M) \subset \Gamma$: the hard part. Let $t \in \mathcal{F}(M)$ such that $t \geq 1$, and let $\theta : M \to M^t$ be a $*$-isomorphism. Since N is w-rigid, let $B \subset N$ be such that the inclusion is rigid, B is diffuse and regular in N. Now, we are moving $B \subset N \subset M$ through the isomorphism θ...

Sketch of the proof: part II

Denote $Q = \theta(B)$, $P = \theta(N)$, $Q \subset P \subset M^t$. Then, we are going back to M. Since $s = 1/t \leq 1$, choose a projection $q \in Q$, such that $Q^s := qQq$ and $P^s := qPq$. We regard $Q^s \subset P^s \subset M$. Note that the inclusion $Q^s \subset P^s$ is rigid, Q^s is diffuse and P^s is the (quasi-)normalizer of Q^s inside M.

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Thus, (Th.Intertw.) $\Longrightarrow Q^s \prec_{\mathcal{M}} N$ or $Q^s \prec_{\mathcal{M}} \mathcal{A}$. But actually $Q^s \not\prec_{\mathcal{M}} \mathcal{A}$. Indeed otherwise, once again thanks to (Th.Cont.), we would have $Q^s \subset P^s \subset \mathcal{A}^\psi$ (modulo amplification!!!). This situation is impossible because $\mathcal{A}^\psi$ has the Haagerup property.

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Avoiding some technical details, we can then show that $\exists n \geq 1$, $\exists \gamma \in \mathrm{Sp}(\mathcal{M}, \varphi)$ and $\exists v \in M_n(\mathbf{C}) \otimes \mathcal{M}$ a non-zero partial isometry which is a γ-eigenvector and such that $vv^* = 1_P = 1_{M^t}$ and $v^* P v \subset N^{t/\gamma}$. The same proof as before with θ^{-1} instead of θ can show that actually $v^* P v = N^{t/\gamma}$. Thus, $N \simeq N^{t/\gamma}$, $t/\gamma \in \mathcal{F}(N)$ and so $t = \gamma \in \mathrm{Sp}(\mathcal{M}, \varphi) = \Gamma$.