Goldman flows on the moduli space of flat $SU(2)$-connections over a nonorientable surface.

David Klein

July 30, 2007

Basics

Surface Σ, trivial G-bundle $\boxed{\begin{array}{rl} G \to & G \times \Sigma \\ & \quad\downarrow \\ & \quad\Sigma \end{array}}$

Flat connections $\mathcal{A}_{\text{flat}}(\Sigma) \subset \Omega^1(\Sigma) \otimes \mathfrak{g}$

Gauge transformations $\mathcal{G}(\Sigma) = C^\infty(\Sigma, G)$

Based gauge transformations $\mathcal{G}(\Sigma, p) \subset \mathcal{G}(\Sigma)$

Moduli space $\mathcal{M}(\Sigma) = \mathcal{A}_{\text{flat}}(\Sigma)/\mathcal{G}(\Sigma)$

$$= (\mathcal{A}_{\text{flat}}(\Sigma)/\mathcal{G}(\Sigma, p))/G$$
$$\cong \text{Hom}(\pi_1(\Sigma, p), G)/G$$

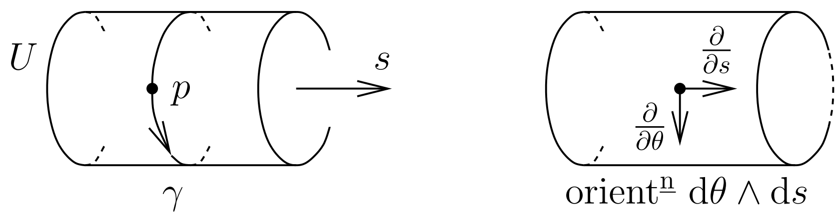
Fact: $\mathcal{A}_{\text{flat}}(\Sigma)/\mathcal{G}(\Sigma, p) \cong \text{Hom}(\pi_1(\Sigma, p), G)$.

For this talk, we work exclusively with $G = SU(2)$, with inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g} = \mathfrak{su}(2)$:

$$\begin{aligned} \langle \zeta, \eta \rangle &= -\tfrac{1}{2}\text{tr}(\zeta\eta) \\ &= -\tfrac{1}{8}\text{tr}(\text{ad}_\zeta \circ \text{ad}_\eta). \end{aligned}$$

Goldman Flows

Let $U \cong S^1 \times (-1, 1)$ be an embedded oriented cylinder with coordinates (θ, s), and consider the simple closed oriented curve γ with base point p, shown here:



If Σ is oriented then assume that the embedding is orientation preserving.

If f is *any* smooth conjugation $\mathbb{R}$-valued invariant function defined on a subset $G \setminus X$ of G, then the associated Goldman flow $\{\Xi_t\}_{t\in\mathbb{R}}$ is an $\mathbb{R}$-action on

$$\mathcal{M}_\gamma = \{A \in \mathcal{A}_{\text{flat}}(\Sigma) \ : \ \text{Hol}_\gamma A \notin X\}/\mathcal{G}(\Sigma).$$

Although any one orbit is periodic, the periods usually differ from orbit to orbit.

In this talk, we only considered the function

$$f(g) = \text{Arccos}(\tfrac{1}{2}\text{tr}(g)),$$

which produces an S^1-action. The function

$$f_0(g) = \text{tr}(g),$$

for instance, produces an $\mathbb{R}$-action that is defined on the entire moduli space $\mathcal{M}(\Sigma)$. The fixed point set of this $\mathbb{R}$-action is exactly where the S^1-action is not defined.

Let $\mathcal{S}_\gamma$ be the set of flat connections whose holonomy along γ is not central:

$$\begin{aligned}\mathcal{S}_\gamma &= \{A \in \mathcal{A}_{\text{flat}}(\Sigma) \ : \ \text{Hol}_\gamma A \neq \pm 1\} \\ &= \{A \in \mathcal{A}_{\text{flat}}(\Sigma) \ : \ \tfrac{1}{2}\text{tr}(\text{Hol}_\gamma A) \neq \pm 1\};\end{aligned}$$

$\mathcal{S}_\gamma$ is an open dense subset of $\mathcal{A}_{\text{flat}}(\Sigma)$.

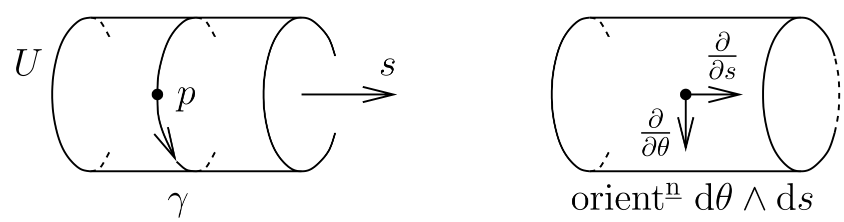
Let $\mathcal{M}_\gamma = \mathcal{S}_\gamma / \mathcal{G}(\Sigma) \subset \mathcal{M}(\Sigma)$.

Define an $\mathbb{R}$-valued function f_γ on $\mathcal{S}_\gamma$,

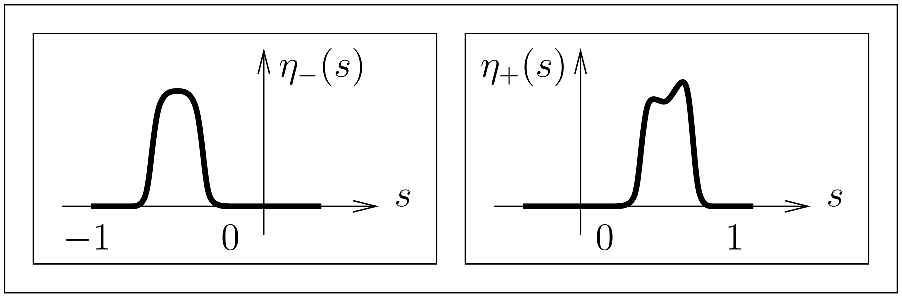
$$f_\gamma(A) = \text{Arccos}\left(\tfrac{1}{2}\text{tr}(\text{Hol}_\gamma A)\right);$$

f_γ is $\mathcal{G}(\Sigma)$-invariant, and may thus be viewed as a function on $\mathcal{M}_\gamma$.

Our goal is to define the Goldman flow $\{\Xi_t\}_{t\in\mathbb{R}}$ associated to the curve $\gamma \subset U$; it is a periodic $\mathbb{R}$-action on $\mathcal{M}_\gamma$, i.e. an S^1-action.



Let $\eta_-(s)$, (resp. $\eta_+(s)$), be a smooth bump function that integrates to 1 and is compactly supported on $(-1,0)$, (resp. $(0,1)$).



Define a logarithm ℓ on $G \smallsetminus \{\pm 1\}$ by requiring that $\exp(\ell(g)) = g$ and $\det(\ell(g)) < \pi$. Define the *normalized* logarithm F on $G \smallsetminus \{\pm 1\}$:

$$F(g) = \frac{\ell(g)}{\sqrt{\langle \ell(g), \ell(g) \rangle}}.$$

Lemma. *Suppose* $A \in \mathcal{S}_\gamma \subset \mathcal{A}_{\mathrm{flat}}(\Sigma)$*. There is a unique based gauge transformation* $u \in \mathcal{G}(U,p)$ *on the cylinder* U *such that* $u.(A|_U) = \frac{\mathrm{d}\theta}{-2\pi} \otimes \ell(\mathrm{Hol}_\gamma A)$.

For $t \in \mathbb{R}$, define

$$\Xi_t^\pm(A) = A + \eta_\pm(s)\mathrm{d}s \otimes \mathrm{Ad}_{u^{-1}}(tF(\mathrm{Hol}_\gamma A)),$$

where the second term, a $\mathfrak{g}$-valued 1-form on U, extends by zero to 1-form on Σ.

Aside:
If $A|_U = \frac{\mathrm{d}\theta}{-2\pi} \otimes \begin{pmatrix} i\alpha & \\ & -i\alpha \end{pmatrix}$, where $\alpha \in (0,\pi)$, then

$$\Xi_t^\pm(A)|_U = \frac{\mathrm{d}\theta}{-2\pi} \otimes \begin{pmatrix} i\alpha & \\ & -i\alpha \end{pmatrix} + \eta_\pm(s)\mathrm{d}s \otimes \begin{pmatrix} it & \\ & -it \end{pmatrix}.$$

$\{\Xi_t^+\}_{t\in\mathbb{R}}$ and $\{\Xi_t^-\}_{t\in\mathbb{R}}$ define $\mathbb{R}$-actions on $\mathcal{S}_\gamma$ satisfying the following conditions:

(i) The $\mathbb{R}$-actions $\Xi_t^\pm$ have "support" in U in the following sense: $\Xi_t^-(A) = A$ outside of some compact subset of $S^1 \times (-1, 0) \subset U$, and $\Xi_t^+(A) = A$ outside of some compact subset $S^1 \times (0, 1) \subset U$.

(ii) Ξ_t^- and Ξ_t^+ are $\mathcal{G}(\Sigma)$-equivariant: if $A \in \mathcal{S}_\gamma$ and $\psi \in \mathcal{G}(\Sigma)$, then $\Xi_t^\pm(\psi.A) = \psi.(\Xi_t^\pm(A))$.

(iii) If $A \in \mathcal{S}_\gamma$ and $t \in \mathbb{R}$, then there exists $\psi \in \mathcal{G}(\Sigma)$ such that $\psi.\Xi_t^-(A) = \Xi_t^+(A)$.

(iv) If $\mathrm{d}(f_\gamma)_A$ is the tangent map of f_γ at A, then

$$\mathrm{d}(f_\gamma)_A(B) = \int_U \left\langle \left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Xi_t^\pm A\right) \wedge B \right\rangle ,$$

for $B \in T_A\mathcal{S}_\gamma = T_A\mathcal{A}_{\text{flat}}(\Sigma)$.

Theorem (Goldman, Jeffrey and Weitsman)**.** *The $\mathbb{R}$-actions Ξ_t^+ and Ξ_t^- on $\mathcal{S}_\gamma$ define a (common) $\mathbb{R}$-action $\{\Xi_t\}_{t\in\mathbb{R}}$ on $\mathcal{M}_\gamma$; the action Ξ_t is periodic, with period π if $\Sigma \smallsetminus \gamma$ is disconnected and period 2π if $\Sigma \smallsetminus \gamma$ is connected. Thus Ξ_t defines an S^1-action on $\mathcal{M}_\gamma$, called a Goldman flow.*

Remark:

If Σ is compact and oriented then there is a symplectic form on $\mathcal{M}(\Sigma)$ given by

$$\omega_{[A]}([B],[C]) = \int_M \langle B \wedge C \rangle,$$

(see, for instance, Atiyah and Bott's paper *The Yang-Mills equation over Riemann surfaces*, or McDuff and Salamon's text *Introduction to symplectic topology*); conditions (i) and (iv) imply that

$$\begin{aligned} \mathrm{d}(f_\gamma)_{[A]}([B]) &= \int_M \left\langle \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Xi_t^{\pm} A\right) \wedge B \right\rangle \\ &= \omega_{[A]}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Xi_t[A], [B]\right), \end{aligned}$$

and so $\{\Xi_t\}_{t\in\mathbb{R}}$ is the (periodic) flow of the Hamiltonian vector field on $\mathcal{M}_\gamma$ with Hamiltonian f_γ.

Aside:

$$\mathrm{d}_A = \mathrm{d} - [A \wedge \cdot] : \Omega^k(\Sigma) \otimes \mathfrak{g} \to \Omega^{k+1}(\Sigma) \otimes \mathfrak{g}$$

$$\begin{aligned} T_A \mathcal{A}_{\text{flat}}(\Sigma) &= \{B \in \Omega^1(\Sigma) \otimes \mathfrak{g} \mid \mathrm{d}_A B = 0\} \\ &= \{\mathrm{d}_A \text{ 1-cocycles}\} \end{aligned}$$

$$\begin{aligned} T_A(\mathcal{G}(\Sigma).A) &= \{\mathrm{d}_A f \mid f \in C^\infty(\Sigma) \otimes \mathfrak{g}\} \\ &= \{\mathrm{d}_A \text{ 1-coboundaries}\} \end{aligned}$$

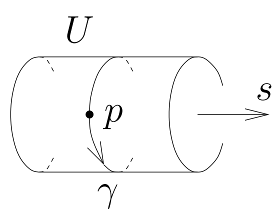
$$T_{[A]}\mathcal{M}(\Sigma) = H^1(\Omega^\bullet(\Sigma) \otimes \mathfrak{g}, \mathrm{d}_A)$$

$\Xi_t^{\pm}$ and holonomy

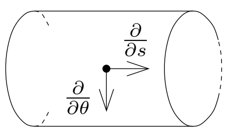
Suppose $\sigma : [0,1] \to \Sigma$ is a curve that either has both endpoints at p, or has one endpoint at p and one endpoint in $\Sigma \setminus U$; suppose further that σ does not otherwise intersect γ. Given $A \in \mathcal{S}_\gamma$, let

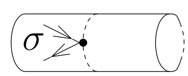
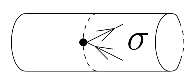
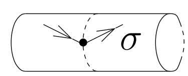
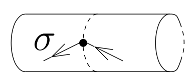
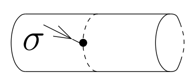
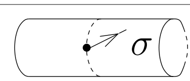
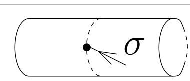
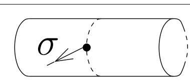
$$\zeta_t = \exp(tF(\mathrm{Hol}_\gamma A)),$$

where $F(g)$ is the normalized logarithm; then the holonomy of $\Xi_t^{\pm}(A)$ along σ is given in the following table.



orient$^{\underline{\mathrm{n}}}$ $\mathrm{d}\theta \wedge \mathrm{d}s$



	$\mathrm{Hol}_\sigma(\Xi_t^-(A))$	$\mathrm{Hol}_\sigma(\Xi_t^+(A))$
	$\mathrm{Ad}_{\zeta_t}(\mathrm{Hol}_\sigma A)$	$\mathrm{Hol}_\sigma A$
	$\mathrm{Hol}_\sigma A$	$\mathrm{Ad}_{\zeta_t^{-1}}(\mathrm{Hol}_\sigma A)$
	$(\mathrm{Hol}_\sigma A)(\zeta_t^{-1})$	$(\zeta_t^{-1})(\mathrm{Hol}_\sigma A)$
	$(\zeta_t)(\mathrm{Hol}_\sigma A)$	$(\mathrm{Hol}_\sigma A)(\zeta_t)$
	$(\mathrm{Hol}_\sigma A)(\zeta_t^{-1})$	$\mathrm{Hol}_\sigma A$
	$\mathrm{Hol}_\sigma A$	$(\zeta_t^{-1})(\mathrm{Hol}_\sigma A)$
	$\mathrm{Hol}_\sigma A$	$(\mathrm{Hol}_\sigma A)(\zeta_t)$
	$(\zeta_t)(\mathrm{Hol}_\sigma A)$	$\mathrm{Hol}_\sigma A$

Σ compact and nonorientable...

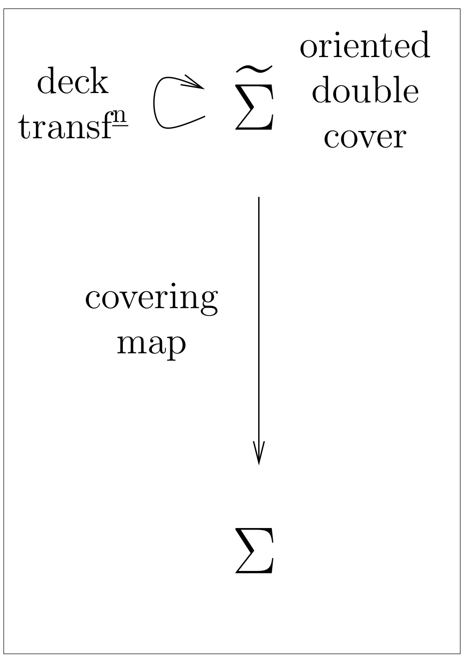
Pullback by the covering map induces

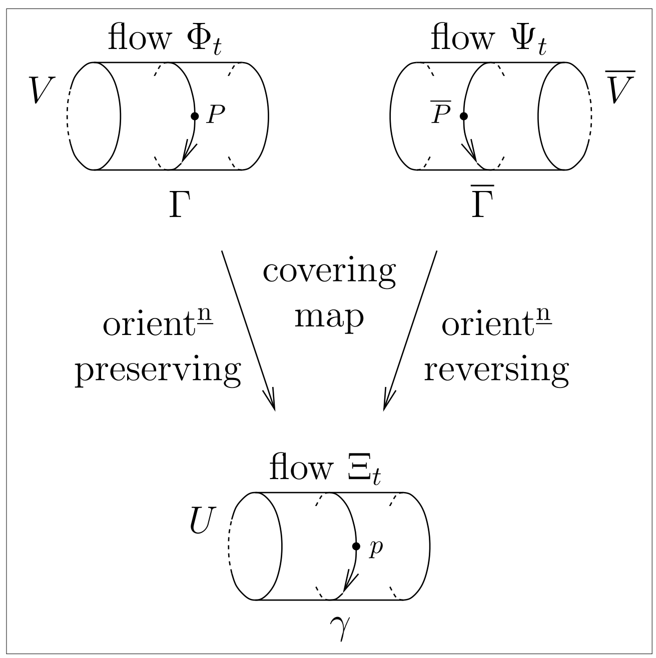
$$\iota : \mathcal{M}(\Sigma) \to \mathcal{M}(\widetilde{\Sigma}),$$

and pullback by the deck transformation induces

$$\tau : \mathcal{M}(\widetilde{\Sigma}) \to \mathcal{M}(\widetilde{\Sigma}).$$

In Nan-Kuo Ho's paper *The real locus of an involution map on the moduli space of flat connections on a Riemann surface*, the fixed point set of τ, $\mathcal{M}(\widetilde{\Sigma})^\tau$, is shown to be a Lagrangian submanifold of $\mathcal{M}(\widetilde{\Sigma})$.





Theorem (K, 2007)**.**

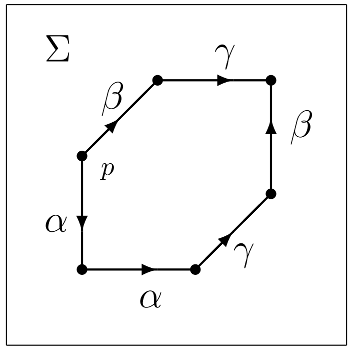
(i) $\Phi_t \circ \Psi_{-t}$ *preserves* $\mathcal{M}(\widetilde{\Sigma})^\tau$,

(ii) $\Phi_t \circ \Psi_{-t}$ *preserves* $\iota(\mathcal{M}(\Sigma))$,

(iii) $(\Phi_t \circ \Psi_{-t}) \circ \iota = \iota \circ \Xi_t$.

Example

Suppose the compact and nonorientable surface Σ has Euler characteristic $\chi(\Sigma) = -1$.



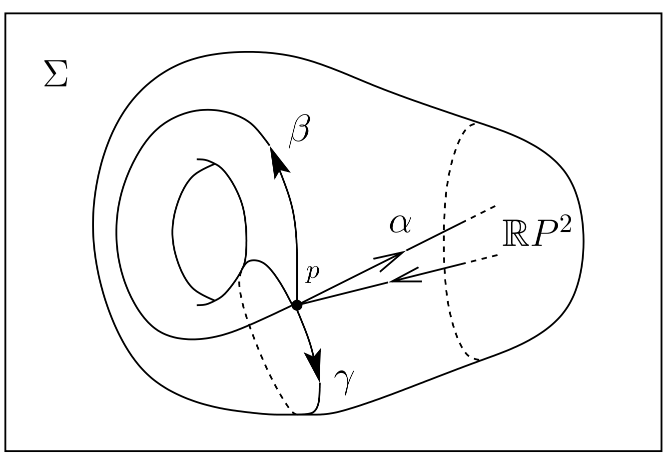
$$\pi_1(\Sigma, p) = \langle \gamma, \beta, \alpha \mid \gamma\beta\gamma^{-1}\beta^{-1}\alpha^2 = 1 \rangle$$

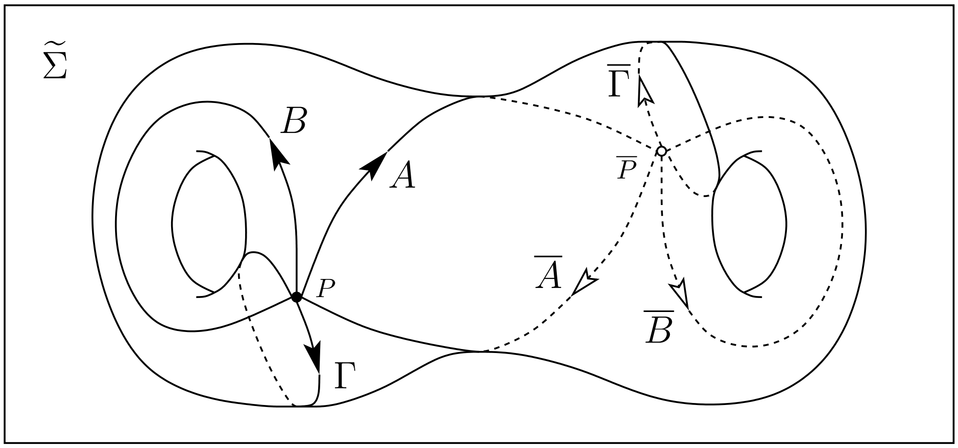
$$\begin{aligned} \text{Let } \mathcal{R} &= \{(c, b, a) \in G^3 \ : \ cbc^{-1}b^{-1}a^2 = 1\} \\ &\cong \mathrm{Hom}(\pi_1(\Sigma, p), G) \\ &\cong \mathcal{A}_{\mathrm{flat}}(\Sigma)/\mathcal{G}(\Sigma, p), \end{aligned}$$

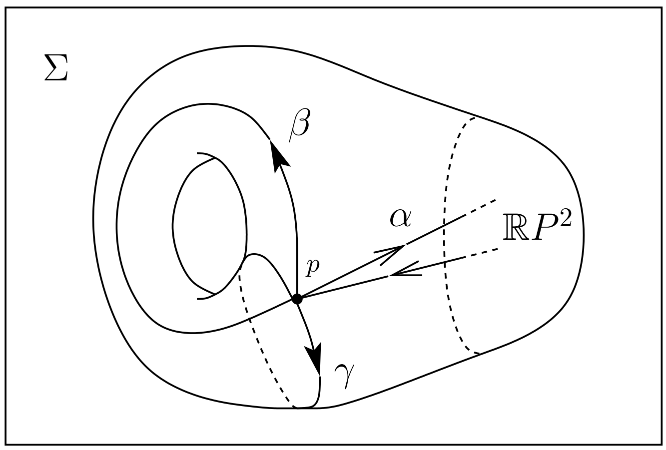
and let G act on $\mathcal{R}$:

$$g.(c, b, a) = (gcg^{-1}, gbg^{-1}, gag^{-1}).$$

We identify $\mathcal{M} := \mathcal{R}/G \cong \mathcal{M}(\Sigma)$.







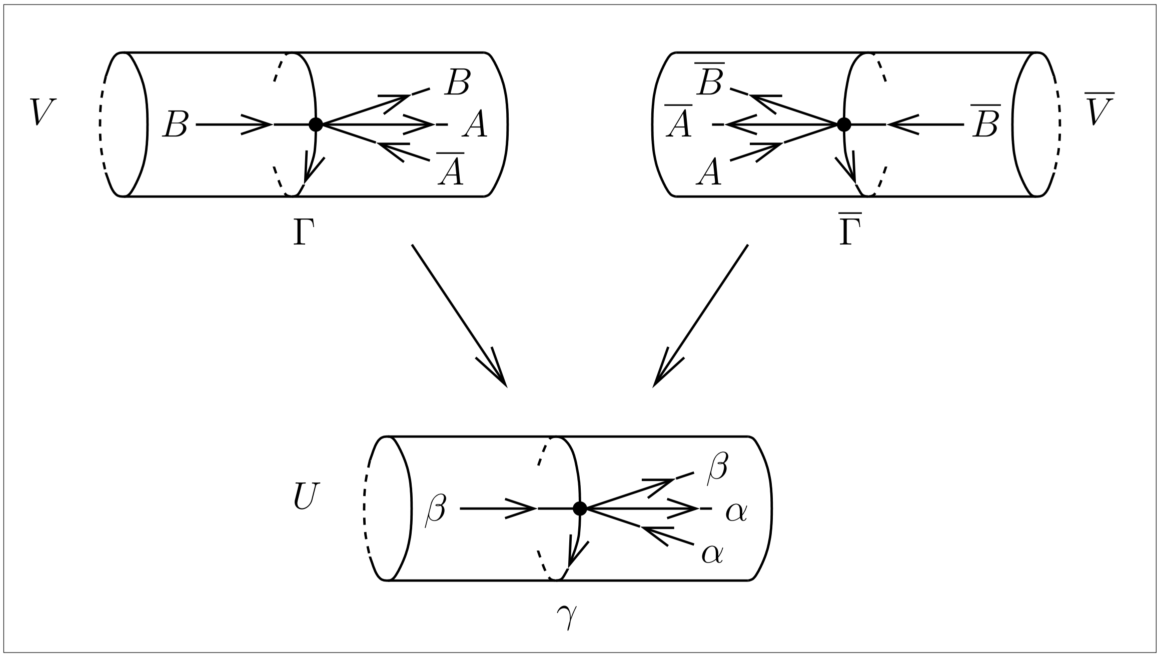
$$\text{Let } \widetilde{\mathcal{R}} = \left\{ \begin{array}{l} (c, b, a, \overline{c}, \overline{b}, \overline{a}) \in G^6 \ : \\ \quad cbc^{-1}b^{-1}a\overline{a} = 1, \overline{c}\overline{b}\overline{c}^{-1}\overline{b}^{-1}\overline{a}a = 1 \end{array} \right\}$$
$$\cong \mathcal{A}_{\mathrm{flat}}(\widetilde{\Sigma})/\mathcal{G}(\widetilde{\Sigma}, P, \overline{P}),$$

and let $G \times G$ act on $\widetilde{\mathcal{R}}$:

$$(g, h).(c, b, a, \overline{c}, \overline{b}, \overline{a}) = (gcg^{-1}, gbg^{-1}, gah^{-1}, h\overline{c}h^{-1}, h\overline{b}h^{-1}, h\overline{a}g^{-1}).$$

We identify $\widetilde{\mathcal{M}} := \widetilde{\mathcal{R}}/(G \times G) \cong \mathcal{M}(\widetilde{\Sigma})$.

For $t \in \mathbb{R}$ and $g \in G \smallsetminus \{\pm 1\}$, let $\zeta(g) = \exp(tF(g))$.



The $\mathbb{R}$-action Ξ_t^- on $\mathcal{R}_\gamma \subset \mathcal{R}$,

$$\mathcal{R}_\gamma := \{(c, b, a) \in \mathcal{R} \mid c \neq \pm 1\},$$

$$\Xi_t^-(c, b, a) = (c, b(\zeta_t^{-1}(c)), a),$$

covers the Goldman flow Ξ_t on $\mathcal{M}_\gamma = \mathcal{R}_\gamma / G$.

The $\mathbb{R}$-action Φ_t^- on $\widetilde{\mathcal{R}}_\Gamma \subset \widetilde{\mathcal{R}}$,

$$\widetilde{\mathcal{R}}_\Gamma := \{(c, b, a, \overline{c}, \overline{b}, \overline{a}) \in \widetilde{\mathcal{R}} \mid c \neq \pm 1\},$$

$$\Phi_t^-(c, b, a, \overline{c}, \overline{b}, \overline{a}) = (c, b(\zeta_t^{-1}(c)), a, \overline{c}, \overline{b}, \overline{a}),$$

covers the Goldman flow Φ_t on $\widetilde{\mathcal{M}}_\Gamma = \widetilde{\mathcal{R}}_\Gamma / (G \times G)$.

The $\mathbb{R}$-action Ψ_t^+ on $\widetilde{\mathcal{R}}_{\overline{\Gamma}} \subset \widetilde{\mathcal{R}}$,

$$\widetilde{\mathcal{R}}_{\overline{\Gamma}} := \{(c, b, a, \overline{c}, \overline{b}, \overline{a}) \in \widetilde{\mathcal{R}} \mid \overline{c} \neq \pm 1\} \subset \widetilde{\mathcal{R}},$$

$$\Psi_t^+(c, b, a, \overline{c}, \overline{b}, \overline{a}) = (c, b, a, \overline{c}, \overline{b}(\zeta_t(\overline{c})), \overline{a}),$$

covers the Goldman flow Ψ_t on $\widetilde{\mathcal{M}}_{\overline{\Gamma}} = \widetilde{\mathcal{R}}_{\overline{\Gamma}} / (G \times G)$.

Since the cylinders V and $\overline{V}$ in $\widetilde{\Sigma}$ are disjoint, the flows Φ_t^+ and Ψ_t^- commute. The $\mathbb{R}$-action $\Phi_t^- \circ \Psi_{-t}^+$ on $\widetilde{\mathcal{R}}_\Gamma \cap \widetilde{\mathcal{R}}_{\overline{\Gamma}}$ covers a periodic $\mathbb{R}$-action $\Phi_t \circ \Psi_{-t}$ on $\widetilde{\mathcal{M}}_\Gamma \cap \widetilde{\mathcal{M}}_{\overline{\Gamma}}$:

$$\begin{aligned}&\Phi_t^- \circ \Psi_{-t}^+(c, b, a, \overline{c}, \overline{b}, \overline{a}) \\ &\quad = (c, b(\zeta_t^{-1}(c)), a, \overline{c}, \overline{b}(\zeta_{-t}(\overline{c})), \overline{a})\end{aligned}$$

Lemma (Nan-Kuo Ho)**.** *For $x \in G$, let*

$$\mathcal{N}_x = \left\{ \begin{array}{l} (c, b, a, c, b, ax) \in \widetilde{\mathcal{R}} \mid \\ \quad xcx^{-1} = c,\ xbx^{-1} = b,\ xax^{-1} = a \end{array} \right\};$$

the fixed point set of τ is

$$\mathcal{M}(\widetilde{\Sigma})^\tau = \bigcup_{x \in G} (\mathcal{N}_x / (G \times G)),$$

and the image of ι is

$$\iota(\mathcal{M}(\Sigma)) = \mathcal{N}_1 / (G \times G).$$

Theorem (K)**.**

(i) $\Phi_t \circ \Psi_{-t}$ *preserves* $\mathcal{M}(\widetilde{\Sigma})^\tau$,

(ii) $\Phi_t \circ \Psi_{-t}$ *preserves* $\iota(\mathcal{M}(\Sigma))$,

(iii) $(\Phi_t \circ \Psi_{-t}) \circ \iota = \iota \circ \Xi_t$.

To prove the first two statements, it suffices to show that the flow $\Phi_t^- \circ \Psi_{-t}^+$ on $\widetilde{\mathcal{R}}$ preserves $\mathcal{N}_x$, for each $x \in G$.

To prove the last statement, work with the maps $\Phi_t^- \circ \Psi_{-t}^+$ on $\widetilde{\mathcal{R}}$ and Ξ_t^- on $\mathcal{R}$; alternatively, work with the gauge theoretic description: use a bump function $\eta_-(s)$ with support in $(-1, 0)$ to define Ξ_t and Φ_t, and use $\eta_+(s) := \eta_-(-s)$ to define Ψ_t.