Representing abstract semigroups as semigroups of projections of C^*-algebras

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Some motivation

Part of the invariant used in classification is K_0, constructed out of the monoid (i.e. semigroup with 0) $V(\cdot)$.

For A,

$$V(A) = \{[p] \mid p \text{ a projn in } M_\infty(A)\},$$

where [ ] is usual MvN equivalence.

In general, $V(A)$ is a <u>conical</u> monoid:

$$x + y = 0 \text{ if and only if } x = y = 0.$$

Restricting to the real rank zero class, it further enjoys the following property:

Definition. A monoid M is termed a <u>refinement</u> monoid if

$$x_1 + x_2 = y_1 + y_2 \text{ in } M \Rightarrow \exists z_{ij} \text{ with}$$

	x_1	x_2
y_1	z_{11}	z_{12}
y_2	z_{21}	z_{22}

So we can ask:

Question 1: Which conical, refinement monoids M appear as $V(A)$ for A with real rank zero?

Digression. Machinery developed by G. Bergman in the mid 70's allows to find, given any conical monoid with order unit, a (generally non-commutative) ring R such that $V(R) \cong M$.

(Here, $V(R)$ is the monoid of iso classes of fg projective, say, right modules.)

Even in the purely algebraic case, it is of interest to determine which conical refinement monoids are representable as $V(R)$ where R is a von Neumann regular ring.

[Wehrung, 1998] **Size matters!** $\exists$ a conical refinement monoid of size $\aleph_2$ not representable as $V(R)$ for any regular ring R.

So can ask instead:

Question 2: Which conical, countable refinement monoids are representable as $V(A)$ for a real rank zero A?

Graph algebras

Setup. Take $E = (E^0, E^1, r, s)$ a graph (or quiver), with source and range maps

$$s\colon E^1 \to E^0\,,\ e \mapsto s(e)$$

$$r\colon E^1 \to E^0\,,\ e \mapsto r(e)\,.$$

Assume E is countable and row-finite (i.e. every vertex emits finitely many edges, i.e. $|s^{-1}(v)| < \infty$).

The graph C^*-algebra $C^*(E)$ is the C^*-algebra generated by a universal Cuntz-Krieger family: a set of p/w orthogonal projns

$$\{p_v \mid v \in E^0\}$$

and partial isometries

$$\{s_e \mid e \in E^1\}$$

with

1) $s_e^* s_e = p_{r(e)}$

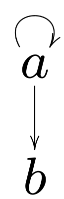
2) $p_v = \sum_{e \in s^{-1}(v)} s_e s_e^*$.

Graph monoids

Definition. Given a row-finite graph E, define the <u>graph monoid</u>

$$M(E) = \langle a_v, v \in E^0 \mid a_v = \sum_{e \in s^{-1}(v)} a_{r(e)} \rangle$$

For example, if E



then

$$M(E) = \langle a, b \mid a = a + b \rangle \, .$$

Write $\mathbb{N}_0 = \{0, 1, 2, ...\}$ and $\mathbb{N} = \{1', 2', 3', ...\}$. Then we can identify

$$M(E) \to \mathbb{N}_0 \sqcup \mathbb{N} \, , \; a \mapsto 1', \; b \mapsto 1 \, .$$

In fact, $M(E)$ is the projn monoid of the Toeplitz algebra.

Basic structural results

Theorem. [Ara-Moreno-Pardo, 07] For a row-finite graph E, the natural monoid homomorphism

$$\gamma_E \colon M(E) \to V(C^*(E)), a_v \mapsto [p_v]$$

is an isomorphism.

Moreover:

Theorem. [Ara-Moreno-Pardo, 07] For a row-finite graph E, the monoid $M(E)$ is always a refinement monoid, and is also <u>separative</u>.

[$a + c = b + c$ with $c \leq na$ and $c \leq mb$ implies $a = b$.]

So graph algebras provide us with refinement monoids, although not all $C^*(E)$ have real rank zero:

Theorem. [Jeong-Park, 02] $C^*(E)$ has real rank zero iff E satisfies condition (K): every vertex on a cycle has at least two cycles based on it.

Changing the question

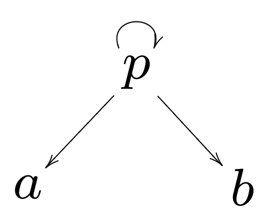
We can reformulate the question

Question 3: Which countable, conical, refinement monoids are isomorphic to $M(E)$ for a row-finite graph E?

We shall also restrict attention to fg monoids.

Some negative situations:

1) Consider the graph E



so $M(E) = \langle p, a, b \mid p = p + a + b \rangle$.

This is not partially ordered $\because$ $p \le p + a \le p + a + b = p$, yet $p \ne p + a$.

Its antysimmetrization is

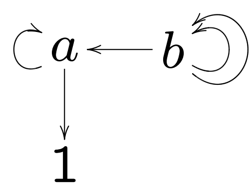
$$M_0 = \langle p, a, b \mid p = p + a = p + b \rangle \,,$$

still fg, refinement, conical. But

Theorem 1. (Ara-P-Wehrung, 07) M_0 is <u>not</u> a graph monoid.

2) Let $\mathbb{Z}^\infty = \mathbb{N}_0 \cup \{\infty\}$ with addition $x + \infty = \infty$ for all x.

Consider the graph E:



We see that $M(E) = \langle a, b, 1 \mid a = a + 1 \,, b = 2b + a \rangle$

Note $2(a + b) = a + a + 2b = a + b$ and

$a + b + 1 = (a + 1) + b = a + b$.

So can define

$$\epsilon \colon \mathbb{Z}^\infty \to M(E) \,,\ 1 \mapsto 1 \,,\ \infty \mapsto a + b \,,$$

and

$$\rho\colon M(E) \to \mathbb{Z}^{\infty}\,,\ 1 \to 1\,,\ a, b \mapsto \infty\,.$$

So $\rho\epsilon = \text{id}$ but $\epsilon\rho \neq \text{id}$.

Hence $\mathbb{Z}^{\infty}$ is a retract of $M(E)$ with E finite.

Let us see the following:

Theorem 2. (Ara-P-Wehrung, 07). $\mathbb{Z}^\infty$ is <u>not</u> a graph monoid of any finite graph.

Proof. Spse otherwise, hence $\exists$ a finite graph E and

$$\varphi\colon M(E) \to \mathbb{Z}^\infty \text{ an isomorphism.}$$

Put

$$U = \{v \in E^0 \mid \varphi(v) < \infty\} = \{x_1, \ldots, x_m\}$$

$$V = \{v \in E^0 \mid \varphi(v) = \infty\} = \{y_1, \ldots, y_n\}.$$

Put

$$\alpha_{i,i'} = |\{e \in E^1 \mid e\colon x_i \to x_{i'}\}|$$

$$\beta_{j,j'} = |\{e \in E^1 \mid e\colon y_j \to y_{j'}\}|$$

$$\gamma_{i,j} = |\{e \in E^1 \mid e\colon y_j \to x_i\}|$$

<u>Note:</u> No need to consider $x_i \to y_j$ as would get $x_i = y_j + a$ so $\infty > \varphi(x_i) = \varphi(y_j) + \varphi(a) = \infty$.

Hence, get a presentation of $M(E)$:

$$x_i = \sum_{i'=1}^{m} \alpha_{i,i'} x_{i'}$$

$$y_j = \sum_{i=1}^{m} \gamma_{i,j} x_i + \sum_{j'=1}^{n} \beta_{j,j'} y_{j'}$$

i.e. in obvious matrix notation:

$$(\dagger)\ \ X_0 = AX_0\,,\ Y_0 = BY_0 + CX_0\,.$$

Need a

Lemma. For any abelian group G, the only column vectors X, Y in G that satisfy (†) are $X = Y = 0$.

Proof of Lemma. (†) gives a presentation of $M(E)$, hence of $\mathbb{Z}^\infty$ inside G, where the x's are integers and the y's are infinite. Thus:

$$y_j + x_i = y_j = 2y_j \Rightarrow x_i = y_j = 0 \forall i, j\,.$$

□

Now put $G = \mathbb{Q}$ and $X = 0$.

Lemma $\Rightarrow$ the only $Y \in M_{n,1}(\mathbb{Q})$ that satisfies $Y = BY$ is $Y = 0$.

$\therefore\ I - B$ is invertible.

But now, in (†) we have $X_0 = AX_0$ with $X_0 \neq 0$ and if we put $Y := (I - B)^{-1} C X_0$ we verify that

$B(I - B)^{-1} = (I - B)^{-1} - I$ and $Y = BY + CX_0$

Hence (X_0, Y) satisfies (†), a contradiction as $X_0 \neq 0$. □

Main result (I)– turning the negative into positive

Definition. An element $p \in M$ is <u>prime</u> if

$$0 \neq p \text{ and } p \leq a_1 + a_1 \Rightarrow p \leq a_1 \text{ or } p \leq a_2.$$

M is <u>primely generated</u> if every element is a sum of primes.

E.g. $\mathbb{Z}^\infty$. The prime elements here are 1, ∞ and is clearly primely generated.

Theorem. [Brookfield 01] Every fg refinement monoid M is primely generated. And every element in M is either *regular* ($2x \leq x$) or *free* ($nx \leq mx \Rightarrow n \leq m$).

For any prime p, denote

$$L_{\text{free}}(M,p) =$$

$$\{q \text{ free prime} \mid q < p, \text{ and } \nexists r \text{ prime } q < r < p\}$$

Main result (II)

Theorem 2. [Ara-P-Wehrung] Let M be a fg refinement monoid which is partially ordered. Then the following are equivalent:

(i) M is a graph monoid.

(ii) For every free prime, $|L_{\text{free}}(M,p)| \leq 1$.

Corollary. $\mathbb{Z}^{\infty}$ is a graph monoid !

Some ingredients in the proof.

Definition. An <u>ideal</u> of M is an subset I such that $x+y \in I$ iff $x, y \in I$.

An ideal I defines a congruence:

$$x \sim y \iff x+z = y+t \text{ for } z, t \in I$$

Denote $M/I = M/\sim$.

Lemma. [Ara-Moreno-Pardo] The class of graph monoids is closed under ideals and quotients.

(i) $\Rightarrow$ (ii)

Spse $\exists$ p free such that $|L_{\text{free}}(M, p)| \geq 2$.

$\therefore$ $\exists a, b$ free primes such that $a < p$, $b < p$ and nothing lies in between.

Write $a + a' = p$. Note p prime and $a < p$ imply $p \leq a'$. Thus

$$p \leq a + p \leq a + a' + p = p \Rightarrow a + p = p$$

Likewise, $p = b + p$, $\therefore M$ contains a copy of

$M_0 = \langle a, b, p \mid p = p{+}a = p{+}b \rangle$ (not a graph monoid).

Put $N = \{x \in M \mid x \leq np$ for some $n\}$, an ideal of M, and

$$I = \{x \in N \mid a \nleq x, b \nleq x\},$$

an ideal of N.

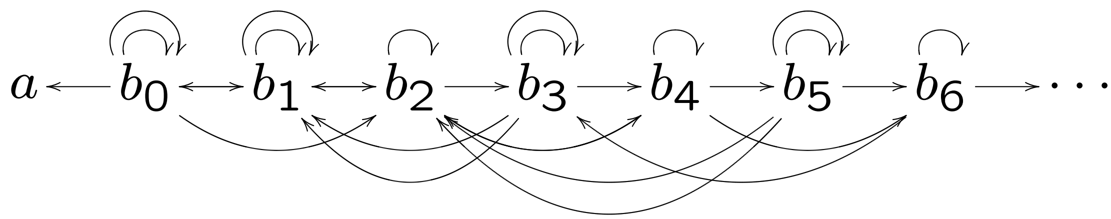
Note $M_0 \subseteq N$, hence $\exists$ ϵ: $M_0 \to N/I$.

Lemma. ϵ is an isomorphism.

$\therefore N/I$ is not a graph monoid, hence neither is M.

(ii) ⇒ (i)

Start with $\mathbb{Z}^\infty$. What is its graph presentation?



The presentation (‡) for $M(E)$ is:

$$b_0 = 2b_0 + b_1 + b_2 + a$$

$$b_1 = b_0 + 2b_1 + b_2$$

$$b_2 = b_2 + b_1 + b_3 + b_4$$

$$b_3 = 2b_3 + 2b_1 + b_4$$

$$b_4 = b_4 + b_2 + b_5 + b_6$$

$$b_5 = 2b_5 + 2b_2 + b_6$$

$$b_6 = b_6 + b_3 + b_7 + b_8$$

Define $\varphi\colon M(E) \to \mathbb{Z}^\infty$ by $a \mapsto 1$ and $b_i \mapsto \infty$ for all i, clearly surjective.

To prove 1-1, enough to show $a + b_0 = b_0 = b_n = 2b_0$ for all n.

Known. Such an $M(E) \hookrightarrow \prod G \cup \{\infty\}$ where the G's are abelian groups with usual addition.

Thus enough to prove:

Claim. For any abelian group G, if a, b_n in $G \cup \{\infty\}$ satisfy (‡), then $a + b_0 = b_0 = b_n = 2b_0$.

To ease notation write $x \sim y$ if $x \leq ny$ and $y \leq mx$. From (‡) get:

$$b_1 \leq b_0 \text{ \& } b_0 \leq b_1 \Rightarrow b_0 \sim b_1$$

$$b_2 \leq b_1 \text{ \& } b_1 \leq b_2 \Rightarrow b_1 \sim b_2$$

...so $b_i \sim b_j$ for all i, j.

$\therefore$ one $b_i = \infty$ iff all are and the rels hold trivially.

May thus assume all b_i's are in G.

1st equation:

$b_0 = 2b_0+b_1+b_2+a \Rightarrow a \in G$ and $b_0+b_1+b_2+a = 0$

2nd equation:

$b_1 = b_0 + 2b_1 + b_2 \Rightarrow b_0 + b_1 + b_2 = 0$ hence $a = 0$

3rd and 4th eqns imply:

$$0 = b_1 + b_3 + b_4$$

$$0 = b_3 + 2b_1 + b_4\,, \text{ so also } b_1 = 0$$

5th and 6th eqns imply similarly that $b_2 = 0$, hence $b_0 + b_1 + b_2 = 0$ ensures $b_0 = 0$.

Continuing in this way, find that $a = b_n = 0$ for all n, hence rels also hold in a trivial way. $\Box$

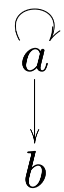
Two more questions

The construction for $\mathbb{Z}^\infty$ gives a graph with condition (K), hence is representable with a graph C^*-algebra with real rank zero. The result however motivates.

Question 3. If M satisfies the conditions of the thm, when can we find a graph E with condition (K) such that $M(E) \cong M$?

Question 4. Even if E does not satisfy (K), is it possible to represent $M(E)$ as $V(A)$ for A with real rank zero?

Take E as:



so $M(E) = \langle a, b \mid a = a + b \rangle$. Spse A has real rank zero and $V(A) = M(E)$.

Let $I_0 = \{x \in M(E) \mid x \leq nb \text{ some } n\}$, an ideal, and $I_0 = V(I) \cong \mathbb{N}_0$ for an ideal of A; so I is simple and so elementary.

This implies $K_1(I) = 0$.

Also $V(A/I) = V(A)/V(I) \cong \mathbb{N}_0$, so that similarly as before $K_1(A/I) = 0$. That implies:

$$0 \to K_0(I) \to K_0(A) \to K_0(A/I) \to 0$$

is exact, which is impossible as all three groups are $\mathbb{Z}$.