Strong "UCT"-classes of non-simple C*-algebras

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Some related papers

1. M.Rørdam, EK: *Purely infinite C*-algebras: ideal-preserving zero homotopies*, GAFA 15 (2005), 377-415. (concerning: Regular Abelian subalgebras of s.p.i. separable nuclear C*-algebras).

2. H.Harnisch, EK: *The inverse problem for primitive ideal spaces*, 2005, SFB478-preprint 399, Uni.Münster. (concerning: Topological characterization of primitive ideal spaces of separable nuclear C*-algebras and reconstruction from data (A, X).) available on the web: www.math.uni-muenster.de SFB478-Preprint server

3. EK: '*The non-commutative Michael selection principle and the classification of non-simple C*-algebras* (german; pp. 92-141 in "C*-algebras" Springer, 2000) (concerning: KK-theory applications

to reconstruction and applications, and passage from the nuclear to the exact case.)

Matricially o-convex cones and actions

Definition 1. *A* point-norm closed *cone* $\mathcal{C}$ *of completely positive maps from* A *into* B *is called a* matricially operator-convex cone *("m.o.c.c") if* $\mathcal{C}$ *is invariant under the operations*
(OC1): $b_1^* V_1(\,\cdot\,)b_1 + b_2 * V_2(\,\cdot\,)b_2 \in \mathcal{C}$ *for* $V_1, V_2 \in \mathcal{C}$, $b_1, b_2 \in B$, *(i.e.,* $\mathcal{C}$ *is operator-convex) and*
(OC2): $c^* V \otimes \mathrm{id}_n(r^*(\,\cdot\,)r)c \in \mathcal{C}$ *for all* $V \in \mathcal{C}$, $n = 1, 2, \ldots$, *columns* $c \in M_{n,1}(B)$ *and rows* $r \in M_{1,n}(A)$*. (i.e.,* $\mathcal{C}$ *is matricial).*

If $S \subset \mathrm{CP}(A, B)$*, then* S generates *the m.o.c.c.* $\mathcal{C} := \mathcal{C}(S)$*, that is, the point-norm closure of the smallest convex subset* $M \subset \mathrm{CP}(A, B)$ *invariant under the operations (OC2).*

Denote by $\mathcal{C}_2 \circ \mathcal{C}_1$ *(resp. by* $\mathcal{C}_1 \otimes \mathcal{C}_3$ *) the m.o.c.c. that is generated by the* set $S := \{V_2 \circ V_1\,;\ V_j \in \mathcal{C}_j\}$

(resp. by $\mathcal{S} := \{V_1 \otimes V_3\,;\ V_j \in \mathcal{C}_j\}$) for m.o.c.c.'s $\mathcal{C}_1 \subset \mathrm{CP}(A, B)$ and $\mathcal{C}_2 \subset \mathrm{CP}(B, C)$. and $\mathcal{C}_3 \subset \mathcal{CP}(C, D)$.

Examples of m.o.c.cones are $\mathrm{CP}(\Omega; A, B)$ (the Ω-equivariant c.p. maps) and $\mathrm{CP}_{\mathrm{rn}}(\Omega, A, B)$ (Ω-residually nuclear maps) for actions $\Psi_A\colon \Omega \to \mathcal{I}(A)$ of lattices Ω on A and Ψ_B on B. The maps Ψ_A are general monotone increasing maps from the lattice Ω into the lattice of ideals $\mathcal{I}(A) \cong \mathbb{O}(\mathrm{Prim}(A))$. (There are m.o.c.c. that do not come from such a construction.) But, if A is separable and exact, B is separable and $\mathcal{C} \subset \mathrm{CP}_{\mathrm{nuc}}(A, B)$, then $\mathcal{C} = \mathrm{CP}(\Omega; A, B)$ for $\Omega := \mathbb{O}(X)$ and a suitable l.s.c. action $\Psi\colon \Omega \to \mathcal{I}(A)$ of X on A. Write: $\mathcal{C} = \mathrm{CP}_{\mathrm{nuc}}(X; A, B)$.

If A and B are $\mathrm{C}_0(Y)$ algebras, then the natural action of $\Omega := \mathbb{O}(Y)$ on A an B are given by $\Psi_A\colon U \mapsto \mathrm{C}_0(U)A \in \mathcal{I}(A)$ and similar Ψ_B. A map $T \in \mathrm{CP}(A, B)$ is in $\mathrm{CP}(\Omega; A, B)$ iff T is $\mathrm{C}_0(Y)$-

modular.

It can happen that $\mathrm{CP}(Y; A, B) = \{0\}$: Consider *e.g.*, the action of $Y := [0, 1]$ on $B := \mathrm{C}(Y)$ and on $A := \mathrm{C}(\{0, 1\}^{\mathbb{N}})$ given by the natural $C(Y)$-algebra structures $B = \mathrm{C}[0, 1]$ and $A \supset \mathrm{C}[0, 1] \cong C^*(1, f)$-algebra where f is the continuous map $f(\alpha_1, \alpha_2, \ldots) := \sum_n \alpha_n 2^{-n}$. (The action of $\Omega = \mathbb{O}(Y)$ on A is given by the inverse $\Psi_B(U) := f^{-1}(U)$ of f, and $\Psi_B = \mathrm{id}$.)

For a continuous map λ from $[0, 1]$ into a finite T_0 space Z one always has that $\mathrm{CP}(Z; A, B)$ is infinite-dimensional, where the action $\Phi_A \colon \mathbb{O}(Z) \to \mathcal{I}(A)$ is given by $\Phi_A(V) := \Psi_B(\lambda^{-1}(V))$ and similarly Φ_B. (Here, λ^{-1} could be replaced by any monotone increasing map from $\mathbb{O}(Z)$ to $\mathbb{O}[0, 1]$.)

In some special cases (but with arbitrary topological spaces X, Z), one has that $\mathrm{CP}(X; A_1, A_2) = \mathrm{CP}(Z; A_1, A_2)$, provided that

- there is a continuous map $\lambda\colon X \to Z$ such that for the corresponding action $\Psi_{A_j}(\lambda^{-1}(V))) = \Phi_{A_j}(V)$ holds,

- $\lambda^{-1}(\mathbb{O}(Z))$ contains a basis of the topology of X , and

- the actions of X are *upper semi-continuous* (see below).

If Ω is a *complete* lattice (i.e., $\bigvee$=l.u.b. and $\bigwedge$=g.l.b. exist inside Ω itself, so as $e.g.,$ for $\Omega = \mathbb{O}(X)$) $\Psi_A\colon \Omega \to \mathcal{I}(A)$ will be called *lower semi-continuous* if $\Psi_A(\bigwedge U_n) = \bigcap \Psi(U_n)$ — in particular $A(U \wedge V) = A(U) \cap A(V)$ in relaxed notation — , and *upper semi-continuous* (respectively *monotone upper semi-continuous*) if $\Psi_A(\bigvee U_n)$ = closure of $\sum \Psi_A(U_n)$ — in particular $A(U \vee V) = A(U) + A(V)$ —, (respectively if $\Psi_A(\bigvee U_n)$ = closure of $\bigcup \Psi_A(U_n)$) for $U_1 \leq U_2 \leq \cdots$ in Ω.

We have seen: Upper semi-continuous actions of X usually have small $\mathrm{CP}(X; A, B)$ that do not allow one to rediscover the action itself.

It is not difficult to see:
If X is a T_0 space and X contains an open quasi-compact subset, then X can not act lower s.c. and monotone upper s.c. at the same time on a separable purely infinite C^*-algebra that does not contain a projection.

The action of $\mathbb{O}(Y)$ defined for a $\mathrm{C}_0(Y)$-algebra A is always upper semi-continuous.

The prototype of a *lower* semi-continuous action of $\mathrm{Prim}(B)$ on $\mathrm{Prim}(A)$ should be given by a an action $\Psi_A(J) := h^{-1}(h(A) \cap \mathcal{M}(B, J))$ for some *-morphism $h\colon A \to \mathcal{M}(B)$. Unfortunately such h does not exist in general, i.e., in general also the lower s.c. actions can not produce sufficiently many A–B-bi-modules that allow one to rediscover the action.

But one has at least the following useful result (in the opposite direction), where F denotes the free group on countably many generators.

Theorem 2. [Separation for m.o.c.cones] *For every m.o.c.c. $\mathcal{C} \subset \mathrm{CP}(A, B)$ there exists a lower s.c. action of $Z := \mathrm{Prim}(B \otimes^{max} C^*(F))$ on $A \otimes^{max} C^*(F)$ such that $T \in \mathrm{CP}(A, B)$ is in $\mathcal{C}$ if and only if*

$$T \otimes \mathrm{id} \in \mathrm{CP}(Z; A \otimes^{max} C^*(F), B \otimes^{max} C^*(F)).$$

Corollary 3. *If B is nuclear, or if A is exact* and $\mathcal{C} \subset \mathrm{CP}_{\mathrm{nuc}}(A, B)$ *then*

$$\mathcal{C} = \mathrm{CP}_{\mathrm{rn}}(X; A, B)$$

for the lower s.c. action of $X := \mathrm{Prim}(B)$ on A given by

$$\Psi^{\mathcal{C}}(J) := \{a \in A\,;\ V(a) \in J \ \ \forall V \in \mathcal{C}\,\}.$$

The opposite direction is crucial: Given a lower s.c. action Ψ of $\mathrm{Prim}(B)$ on A for separable exact A. Show the existence of $\mathcal{C}$ such that $\Psi = \Psi^{\mathcal{C}}$ (*after that* it follows from the above corollary that $\mathcal{C} = \mathrm{CP}_{\mathrm{rn}}(X; A, B)$). This can be done; the proof needs some m.o.c.c.-related KK-theory.

A Hilbert A-B-module (E, ϕ) is *$\mathcal{C}$-compatible* if every map $a \in A \mapsto \langle d(a)x, x\rangle \in B$ is in $\mathcal{C}$. Each Hilbert A-B-module $(\mathcal{H}, \phi\colon A \to \mathcal{L}(\mathcal{H}))$ defines a m.o.c.c. $\mathcal{C}(H, d) :=$ the smallest m.o.c.c. containing all c.p. maps $V\colon a \in A \mapsto \langle \phi(a)x, x\rangle \in B$ for $x \in \mathcal{H}$ ("generalized" vector states). It induces:

Proposition 4. *There is a* natural bijection *between m.o.c.c.'s $\mathcal{C} \subset \mathrm{CP}(A, B)$ and classes of of Hilbert A-B-modules that are closed under (infinite) Hilbert module sums and isometric module morphisms.*

It leads to a natural definition of cone-depending KK-theory (or "$\mathcal{C}$–equivariant" KK-theory).

$\mathcal{C}$–depending KK-, Ext- and R(ørdam)–groups.

Define KK-groups depending on m.o.c. cones $\mathcal{C} \subset \mathrm{CP}(A, B)$:
If A and B are separable algebras, equipped with gradings β_A and β_B and $\mathcal{C} = \beta_A \circ \mathcal{C} = \mathcal{C} \circ \beta_B$, then consider the Abelian semi-group $\mathbb{E}(\mathcal{C}; A, B)$ of unitary equivalence classes of graded Kasparov modules (E, ϕ, F) with countably generated $\mathcal{C}$-compatible Hilbert A-B-module (E, ϕ). The ϕ-compact perturbations of the derivatives F define an equivalence relation $\sim_{sp}$ on $\mathbb{E}(\mathcal{C}; A, B)$ that is compatible with addition. They define a semigroup $SKK(\mathcal{C}; A, B)$.

If A and B are stable and trivially graded, then we can define the semigroups $\mathrm{SExt}(\mathcal{C}; A, B)$ and $\mathrm{SR}(\mathcal{C}; A, B)$ of unitary equivalence classes (by unitaries in $\mathcal{M}(B)$ respectively in $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B)))$ of Busby invariants of extensions $h\colon A \to \mathrm{Q}(B) := \mathcal{M}(B)/B$ and $h\colon A \to \mathrm{Q}(\mathbb{R}_+, B) := \mathrm{C_b}(\mathbb{R}_+, B)/\,\mathrm{C_0}(\mathbb{R}_+, B)$ that have completely positive lifts $V\colon A \to \mathcal{M}(B)$ respectively $V\colon A \to \mathrm{C_b}(\mathbb{R}_+, B)$ that are "locally" in $\mathcal{C}$, i.e., $b^*V(\cdot)b \in \mathcal{C}$ for all $b \in B$ respectively $V(\cdot)(t) \in \mathcal{C}$ for all $t \in \mathbb{R}_+$.

Definition 5. *Let* $\mathrm{KK}(\mathcal{C}; A, B)$ *denote the Grothendieck group of* $\mathbb{E}(\mathcal{C}; A, B)/\sim_{sp}$.

Define Rørdam groups $\mathrm{R}(\mathcal{C}; A, B)$*, and Extension groups* $\mathrm{Ext}(\mathcal{C}; A, B)$ *similar (for trivially graded* A *and* B*),*

Suppose that A is separable, B is σ-unital Then it follows (almost) straight from the definitions and Kasparov's original approach, and from the fact that $\mathrm{CP_{in}}(B) \circ \mathcal{C} \circ \mathrm{CP_{in}}(A) = \mathcal{C}$ for all m.o.c.c.s $\mathcal{C} \subset \mathrm{CP}(A, B)$:

- There are natural semigroup morphisms

$$\mathrm{Hom}(A,B)\cap\mathcal{C}\to\mathrm{SR}(\mathcal{C};A,B)\to\mathrm{SExt}(\mathcal{C};A,B)$$

- With $\mathcal{C}' := \mathcal{C}\otimes\mathrm{CP}(\mathbb{C},\mathbb{C}_{(1)})$, there is a natural isomorphism

$$\mathrm{Ext}(\mathcal{C};A,B)\cong\mathrm{KK}(\mathcal{C}';A,B_{(1)})\,.$$

- One can tensor elements of $\mathrm{KK}(\mathcal{C};A,B)$ with elements of $\mathrm{KK}(C,D)$ for nuclear separable C and D, i.e., there is a natural morphism

$$\mathrm{KK}(\mathcal{C};A,B)\otimes_{\mathbb{Z}}\mathrm{KK}(C,D)\to\mathrm{KK}(\mathcal{C}_{C,D};A\otimes C,B\otimes D)\,,$$

where $\mathcal{C}_{C,D}$ denotes the cone of $T\in\mathrm{CP}(A\otimes C;B\otimes D)$ with $\mathrm{id}\otimes f(T(\cdot\otimes c))\in\mathcal{C}$ for all $c\in C_+$ and $f\in D^*_+$.

- $\mathrm{KK}(\mathcal{C};A,B)$ is *homotopy-invariant*,

- the usual Kasparov product defines a morphism

$$\mathrm{KK}(\mathcal{C}_1; A, B) \times \mathrm{KK}(\mathcal{C}_2; B, C) \to \mathrm{KK}(\mathcal{C}_2 \circ \mathcal{C}_1; A, C)\,,$$

  and satisfies Bott periodicity, i.e.

$$\mathrm{KK}(\mathcal{C}; A, B) \cong \mathrm{KK}(\mathcal{C}(\mathbb{R}^2); A, S^2 B)\,.$$

- in particular: If a locally quasi-compact T_0 space X acts on A, B and C then for $\mathcal{C}_1 := \mathrm{CP}_{rn}(X; A, B)$ and $\mathcal{C}_2 := \mathrm{CP}_{rn}(X; B, C)$ the above formulas lead to a bi-additive map

$$\mathrm{KK}(X; A, B) \times \mathrm{KK}(X; B, C) \to \mathrm{KK}(X; A, C)\,.$$

- Additivity:

$$\mathrm{KK}(\mathcal{C}_1 + \mathcal{C}_2; A_1 \oplus A_2, B) \cong \mathrm{KK}(\mathcal{C}_1; A_1, B) \oplus \mathrm{KK}(\mathcal{C}_2; A_2, B)\,,$$

- half-exactness:
  If $J \lhd A$ are σ-unital, $\pi\colon A \to B := A/J$ and $\mathcal{C}_1 \subset \mathrm{CP}(D,A)$, $\mathcal{C}_0 := \mathrm{CP}_{\mathrm{in}}(A,J)\circ\mathcal{C}_1$, $\mathcal{C}_2 := \pi\circ\mathcal{C}_1$, then

$$\mathrm{KK}(\mathcal{C}_0; D, J) \to \mathrm{KK}(\mathcal{C}_1; D, A) \to \mathrm{KK}(\mathcal{C}_2; D, B)\,,$$

  is exact. On the other side,

$$\mathrm{KK}(\mathcal{C}_0; J, D) \to \mathrm{KK}(\mathcal{C}_1; A, D) \to \mathrm{KK}(\mathcal{C}_2\circ\mathcal{C}_1; B, D)$$

  is exact if the cones $\mathcal{C}_j$ satisfy $\mathcal{C}_0 = \mathcal{C}_1|J$ and

$$\mathcal{C}_2 \circ \pi = \{V \in \mathcal{C}_1\,;\; V|J = 0\}.$$

The notion of $\mathrm{KK}(\mathcal{C};\cdot,\cdot)$-equivalence.

One has $\mathcal{C}$-dependent "split-additivity": Suppose that $h\colon B \to A$ is a (grading-preserving) split morphism for $\pi := \pi_J$. Then the m.o.c. cone $\mathcal{C}_1 := \mathcal{C}(g) \subset \mathrm{CP}(J \oplus B, A)$ generated by

$$g : (j,b) \in J \oplus B \to \mathrm{diag}(j, h(b)) \in M_2(A)$$

is the same as the sum of $\mathrm{CP}_{inn}(J,A)$ and $\mathcal{C}(h)$, and the cone $\mathcal{C}_2 \subset \mathrm{CP}(A, J \oplus B)$ generated by by the Kasparov $(A, J \oplus B)$–module $z := ((J \oplus B) \oplus (J \oplus B)^{op}, (k \oplus h) \oplus (k \circ h \circ \pi \oplus 0), F)$, where F is the flip $((j_1, b_1), (j_2, b_2)) \mapsto ((j_2, b_2), (j_1, b_1))$, has the property that $\mathrm{CP}_{inn}(J \oplus B, J \oplus B) = \mathcal{C}_2 \circ \mathcal{C}_1$, $\mathrm{CP}_{inn}(A, A) \subset \mathcal{C}_2 \circ \mathcal{C}_1$ and $[g \otimes_A z] = [\mathrm{id}] \in \mathrm{KK}(\mathrm{CP}_{inn}; J \oplus B, J \oplus B)$ $[z \otimes_{J \oplus B} g] = [\mathrm{id}] \in \mathrm{KK}(\mathcal{C}_2 \circ \mathcal{C}_1; A, A)$.

Definition 6. *Given* $\mathcal{C}_1 \subset \mathrm{CP}(A, B)$, $\mathcal{C}_2 \subset \mathrm{CP}(B, A)$ *with* $\mathrm{CP}_{\mathrm{in}}(A) \subset \mathcal{C}_2 \circ \mathcal{C}_1$ *and* $\mathrm{CP}_{\mathrm{in}}(B) \subset \mathcal{C}_2 \circ \mathcal{C}_1$ *if there are* $z \in \mathrm{KK}(\mathcal{C}_1; A, B)$ *and* $v \in \mathrm{KK}(\mathcal{C}_2; B, A)$ *such that* $z \otimes_B v = [\mathrm{id}_A]$ *and* $v \otimes_A z = [\mathrm{id}_B]$ *in* $\mathrm{KK}(\mathcal{C}_2 \circ \mathcal{C}_1; A, A)$ *and* $\mathrm{KK}(\mathcal{C}_1 \circ \mathcal{C}_2; B, B)$ *respectively, then we call* z *a* $\mathrm{KK}(\mathcal{C}; \cdot, \cdot)$*-equivalence.* A *and* B *will be called* $\mathrm{KK}(\mathcal{C}; \cdot, \cdot)$*-equivalent.*

Theorem 7. *Suppose that* A *and* B *are* stable *and separable, and that* $\mathcal{C}_1 \subset \mathrm{CP}(A, B)$ *is an m.o.c.c., and that there exists a non-degenerate *-monomorphism* $h_1 \colon A \to B$ *such that* $h_1 \oplus h_1$ *is unitarily equivalent to* h_1 *and generates* $\mathcal{C}_2$*,*

(i) then the natural semi-group morphism from the semi-group of unitary equivalence classes $\mathrm{Hom}(A, B) \cap \mathcal{C}_1$ *into* $\mathrm{KK}(\mathcal{C}_1; A, B)$ *(induced by* $\varphi \mapsto [\varphi]$*) is* surjective*, and*

(ii) $[\psi] = [\varphi]$ *holds in* $\mathrm{KK}(\mathcal{C}_1; A, B)$ *if and only if*

$\psi \oplus h_1$ and $\varphi \oplus h_1$ are unitarily homotopic (i.e. there is a norm-continuous map $t \in [0, \infty) \mapsto u(t)\mathcal{U}(\mathcal{M}(B))$ with $u(0) = 1$ and $\lim u(t)^*(\varphi(a) \oplus h_1(a))u(t) = \psi(a) \oplus h_1(a)$ for all $a \in A$).

Corollary 8. *If, in addition to the assumptions of the last theorem, $\mathcal{C}_2 \subset \mathrm{CP}(B, A)$ is an m.o.c.c. such that there is non-degenerate *-morphism $h_2 \colon B \to A$ which generates $\mathcal{C}_2$ and is unitarily equivalent to $h_2 \oplus h_2$, then:*
There is an isomorphism φ from A onto B with $\varphi \in \mathcal{C}_1$ and $\varphi^{-1} \in \mathcal{C}_2$ if and only if $\mathrm{id}_A \in \mathcal{C}_2 \circ \mathcal{C}_1$ and $\mathrm{id}_B \in \mathcal{C}_1 \circ \mathcal{C}_2$ and there are $z_1 \in \mathrm{KK}(\mathcal{C}_1; A, B)$ and $z_2 \in \mathrm{KK}(\mathcal{C}_2; B, A)$ with $z_1 \otimes_A z_2 = [\mathrm{id}_B]$ in $\mathrm{KK}(\mathcal{C}_1 \circ \mathcal{C}_2; B, B)$ and $z_2 \otimes_B z_1 = [\mathrm{id}_A]$ in $\mathrm{KK}(\mathcal{C}_2 \circ \mathcal{C}_1; A, A)$.

Examples:
If A and B are $\mathrm{C}(Y)$-algebras then $\mathrm{KK}(\mathcal{C}; A, B)$ is the same as $\mathcal{R}\,\mathrm{KK}^G(Y; A, B)$ in the sense of Kasparov (for the trivial group G or for trivial G-actions), if

$\mathcal{C} := \mathrm{CP}(Y; A, B)$, the m.o.c. cone of c.p. $\mathrm{C}_0(Y)$-module maps from A into B.
If A is exact, then $\mathrm{KK}(\mathrm{CP}_{\mathrm{nuc}}(A, B); A, B)$ is the same as $\mathrm{KK}_{nuc}(A, B)$ in the sense of *G. Skandalis*.

Applications of Thm.7 and Cor.8 to classification problems:

Suppose that A and B are separable and stable. To apply the above Theorem one needs to know when there is a "universal" Hilbert (A, B)-module that rediscovers a given map Ψ from $\mathbb{O}(\mathrm{Prim}(B))$ into $\mathbb{O}(\mathrm{Prim}(A)$, *e.g.*, coming from a homeomorphism from $\mathrm{Prim}(A)$ onto $\mathrm{Prim}(B)$.

Thus, a basic problem is the question of how well the cone $\mathcal{C} := \mathrm{CP}_{\mathrm{rn}}(X; A, B)$ rediscovers an given action Ψ of $X := \mathrm{Prim}(B)$ on A, i.e., if, for each $J \in \mathcal{I}(B)$, $b \in J$ and $\varepsilon > 0$, there is a Ψ-residually nuclear map $V \colon A \to B$ and $a \in \Psi(J)$ such that $\|V(a) - b\| < \varepsilon$. A necessary condition is that Ψ is lower semi-continuous (i.e., $J \to \|\Psi(J) + a\|$ defines a lower semi-continuous function on X). This is equivalent to $\Psi = \Psi^{\mathcal{C}}$ for a suitable residually nuclear m.o.c.c. $\mathcal{C} \subset \mathrm{CP}_{\mathrm{nuc}}(A, B)$.

One has only the following partial results (Harnisch-K, Rørdam-K, K):
The answer is positive if $B \otimes \mathcal{O}_\infty$ contains a regular Abelian C^*–subalgebra C, A is arbitrary, and Ψ is lower s.c. A C^*–subalgebra C of $B \otimes \mathcal{O}_\infty$ is "regular" if the map $\Psi_C \colon \mathcal{I}(B) \ni J \to C \cap J \in \mathcal{I}(C)$ is injective and continuous. The latter happens here if and only if $C \cap J_1 + C \cap J_2 = C \cap (J_1 + J_2)$. Thus this action satisfies the stronger assumptions of Ralf Meyer.

The above described results together show then that B satisfies this condition if B is nuclear, and – finally – even if B is exact. (The question is open for $A = \mathrm{C}[0, 1]$, B arbitrary.)

Theorem 9. *Suppose $H \colon A \to \mathcal{M}(B)$ is a non-degenerate nuclear monomorphism, A and B are stable and separable, B strongly purely infinite.*

If the action of $\mathrm{Prim}(B)$ *on* A *is monotone upper semi-continuous, then there exists a non-degenerate nuclear embedding* $h_0\colon A \to B$ *such that* h_0 *and* $h_0 \oplus h_0$ *are unitarily homotopic, and that* $\delta_\infty \circ h_0$ *and* $\delta_\infty \circ H$ *are unitarily homotopic in* $\mathcal{M}(B)$*.*

The action of $\mathrm{Prim}(B)$ is given here by $J \to H^{-1}(H(A) \cap \mathcal{M}(B, J))$.

With $h_0\colon A \to B$ we can apply the Theorem to the realization of elements of $\mathrm{KK}(\mathrm{Prim}(B), A, B)$ by monomorphisms $h\colon A \to B$.

Definition 10. *A separable* B *is in the "strong UCT class" if* $B \otimes \mathcal{O}_\infty$ *contains a "regular" Abelian C*-subalgebra* A *such that* $A \hookrightarrow B$ *defines in* $\mathrm{KK}(X; A, B)$ *a* $\mathrm{KK}(X; \cdot, \cdot)$*–equivalence of* A *and* B *(where* $X := \mathrm{Prim}(B)$*). (The "weak" UCT class should allow in addition extensions, inductive limits, and should start with regular type I subalgebras.)*

If such A exists, it is *not* uniquely determined, but it has the property that A and the action of X on A determine $B \otimes \mathcal{O}_\infty \otimes \mathbb{K}$ up to isomorphisms *if B is nuclear*, i.e. there is a canonical reconstruction of B from (A, X) if B is strongly purely infinite, separable, stable and nuclear. (Note that the action of $\mathrm{Prim}(B)$ on A now satisfies the additional requirements of Ralf Meyer.) Explicitly:

Theorem 11. [HH-EK, Reconstruction] *Suppose that A is separable, nuclear and stable, that Ω is a sub-lattice of $\mathcal{I}(A) \cong \mathbb{O}(\mathrm{Prim}(A))$ such that $\mathrm{Prim}(A), \emptyset \in \Omega$, $\bigcup U_n, (\bigcap U_n)^\circ \in \Omega$ for every sequence $U_1, U_2, \ldots$ in Ω. Then there is a* non-degenerate *-mono*morphism $H_0 \colon A \to \mathcal{M}(A)$ with following properties:*

(i) *The infinite repeat $\delta_\infty \circ H_0$ is unitarily equivalent to H_0.*

(ii) For every $U \in \mathbb{O}(\mathrm{Prim}(A))$ *holds* $H_0(J(V)) = H_0(A) \cap \mathcal{M}(A, J(U))$ *where* $V \in \Omega$ *is given by* $V = \bigcup\{W \in \Omega\,;\ W \subset U\}$.

The H_0 *is uniquely determined up to unitary homotopy, i.e., if* $H_1 \colon A \to \mathcal{M}(A)$ *also satisfies the requirements (i) and (ii) then there is a continuous path* $t \in \mathbb{R}_+ \to U(t) \in \mathcal{U}(\mathcal{M}(A))$ *such that* $U(t)^* H_2(a) U(t) - H_0(a) \in A$ *for all* $a \in A$ *and* $t \in \mathbb{R}_+$ *and* $\lim_{t\to\infty} U(t)^* H_2(a) U(t) = H_0(a)$.

The Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{H}}$ *of the Hilbert* A*-*A*-module* $\mathcal{H} := (A, H_0)$ *is stable and strongly purely infinite; and it is the same as the* C**-Fock algebra of* $\mathcal{H}$.

The natural embedding of A *into* $\mathcal{O}_{\mathcal{H}}$ *defines a lattice isomorphism from* Ω *onto* $\mathbb{O}(\mathrm{Prim}(\mathcal{O}))$ *and a* $\mathrm{KK}(\mathcal{C}; \cdot, \cdot)$*-equivalence.*

If a locally compact group G *acts on* A *by* $\alpha \colon G \to$

$\mathrm{Aut}(A)$ *and* $\alpha(g)(J) \in \Omega$ *for all* $J \in \Omega$*, then* H_0 *can be found such that in addition,* H_0 *is* G*-equivariant (i.e.,* $\gamma(g)\left(H_0(a)b\right) = H_0\left(\gamma(g)(a)\right)\gamma(g)(b)$*) with respect to an action* $\gamma\colon G \to \mathrm{Aut}(A)$ *of* G *on* A *that is outer conjugate to* α*. In particular,* G *acts on* $\mathcal{O}_{\mathcal{H}}$ *and in a way that is compatible with the* $\mathrm{KK}(\Omega; \cdot, \cdot)$*-equivalence from* A *into* $\mathcal{O}_{\mathcal{H}}$*.*

If A is of type I, then $\mathcal{O}_{\mathcal{H}}$ is a $\mathbb{Z}$-crossed product of an inductive limit of type I C^*-algebras by an automorphism.

The generalization of the proofs for simple classification to the non-simple case is related to the (non-trivial) fact that *nuclear* (or exact) B with $B \otimes \mathcal{O}_2 \cong B$ have the strong UCT property: It says that a T_0 space X is the primitive ideal space $\mathrm{Prim}(B)$ of a separable *nuclear* C*-algebra B if and only if

(PN1) the topology of X is second countable, (PN2) every prime closed subset of X is the closure of a point,

(PN3) there exists a locally compact space Y and a continuous map $\varphi\colon Y \to X$ that is *pseudo-open* (:= for every decreasing sequence $U_1 \supset U_2 \supset \cdots$ of open subsets of X, the inverse image $\varphi^{-1}(V)$ of the interior V of $\bigcap_n U_n$ is the interior of $\bigcap_n \varphi^{-1}(U_n)$) and *pseudo-epimorphic* (:= the intersection of $\varphi(Y)$ with different open subsets of X is different).

(Note that φ with (PN3) is an open epimorphism if X is a T_1-space.)

One takes $\Omega := \varphi^{-1}(\mathbb{O}(X))$. The existence of the corresponding universal module $H_0\colon \mathrm{C}_0(Y) \otimes \mathbb{K} \to \mathcal{M}(\mathrm{C}_0(Y) \otimes \mathbb{K})$ can be deduced directly from the Bartle-Graves-Michael selection theorem.

The (re-)construction of $B \otimes \mathbb{K}$ with $\mathrm{Prim}(B) \cong X$

from $\pi : Y \to X$ shows that *for every second countable locally compact group G and every continuous action α of G on X there is a continuous action of G on $B \otimes \mathcal{O}_2 \otimes \mathbb{K}$ that induces α.* In particular, $\mathrm{Aut}(B \otimes \mathcal{O}_2 \otimes \mathbb{K}) \to \mathrm{Homeo}(X)$ is a topological group *epimorphism* that has a sort of local splitting property.

Application to examples suggested by *Chris Phillips*: Let $G = \mathbb{Z}^m \times \mathbb{R}^n$ a locally compact non-compact second countable Abelian group. Then there is an action α of the dual group $\Gamma = \mathbb{T}^m \times \mathbb{R}^n$ (of G) on $\mathcal{O}_2 \otimes \mathbb{K}$ such that $A := (\mathcal{O}_2 \otimes \mathbb{K}) \rtimes \Gamma$

- is a *prime* strongly purely infinite C^*-algebra,
- A has quasi-compact primitive ideal space $\mathrm{Prim}(A) \cong G \cup \{\infty\}$ given by the (nontrivial) closed subsets consisting of the compact subsets of G,
- and the dual action $\widehat{\alpha}$ of G on A induces the translation action of G on $G \cup \{\infty\}$

(The closure of the infinite point ∞ is the whole space $\mathrm{Prim}(A)$, and every clopen subset is $\mathrm{Prim}(A)$ or $\emptyset$. Therefore it can't be the primitive ideal space of an AF algebra.)

If $G = Z$ or $G = \mathbb{R}$ one finds a 1-cocycle that changes the actions into actions that fixes a full projection p of A. (The non-trivial case $G = \mathbb{R}$ follows from a Lemma in the original proof of Connes of the non-commutative Thom isomorphism.)

The question, whether a prime *unital* B with an $\mathbb{R}$-action that induces a minimal action on $\mathrm{Prim}(B)$ must be a simple algebra, appeared in a Seminar talk at Fields Institute.

Can we take $\mathcal{O}_\infty$, $\mathcal{P}_\infty$ or $\mathcal{Z}$ in place of $\mathcal{O}_2$?
(Less important than proving the UCT for tensorially self-absorbing C^*-algebras! I don't want to hinder someone from doing this first.)

Ideas of proof (as an example of "straight" applications of the above described machinery):

(a) Show that $G \cup \{\infty\}$ with the above described "minimal" topology top_{new} is in the class of spaces with properties (PN1)-(PN3):

This could be done by showing that there is a finite-dimensional l.c. Polish space F, an open and continuous map $\lambda \colon F \to G$ and a homeomorphic embedding ν of F into some cube $[0,1]^k$, such that $\lim_{g\to\infty} \mathrm{dist}(x, \nu(\lambda^{-1}(g))) = 0$ for each $x \in [0,1]^k$.

Then define for $x \in [0,1]^k$ the map $\varphi(x) := \lambda \circ \nu^{-1}(x)$ if $x \in \mu(F)$ and $\varphi(x) := \infty$ otherwise. Then φ is open and continuous. Thus $(G \cup \{\infty\}, \varphi)$ satisfies (PN1)-(PN3).

(b) There is unique separable nuclear B with $\mathrm{Prim}(B) = G \cup \{\infty\}$ (with topology top_{new}) and with $B \cong B \otimes \mathcal{O}_2 \otimes \mathbb{K}$ (by (a) and Reconstruction theorem).

(c) The homeomorphism $\ell \colon (t,s) \to (s+t, s)$ of

$(G \cup \{\infty\}) \times G \cong \mathrm{Prim}(B \otimes \mathrm{C}_0(G))$ comes from some automorphism κ of $A := B \otimes \mathrm{C}_0(G)$ because $A \cong A \otimes \mathcal{O}_2 \otimes \mathbb{K}$.

(d) Now define a G-action $\gamma \colon g \mapsto \kappa^{-1}(\mathrm{id} \otimes \rho_g) \circ \kappa$ on A and apply the Reconstruction theorem to A, γ and lattice $\Omega := \{U \times G\,;\ U \in \mathrm{top}_{new}\}$ (here top_{new} is as above):

Since $\Omega \cong \mathbb{O}(\mathrm{Prim}(B))$, the corresponding G-equivariant Hilbert A-A-module $\mathcal{H}$ defined by $H_0 \colon A \to \mathcal{M}(A)$ of the Reconstruction theorem produces the separable stable nuclear algebra $C := \mathcal{O}_{\mathcal{H}}$ with primitive ideal space $\cong \mathrm{Prim}(B)$ such that C is $\mathrm{KK}(\mathrm{Prim}(B); \cdot, \cdot)$-equivalent to A, and with a G-action that induces on $\mathbb{O}(\mathrm{Prim}(C)) \cong \Omega$ the action of G on $\mathbb{O}(\mathrm{Prim}(B))$ given by ℓ. Since A absorbs $\mathcal{O}_2$ tensorially, we get $C \cong C \otimes \mathcal{O}_2 \otimes \mathbb{K} \cong B$ and, thus, an action of G on B that induces the given action of G on $\mathrm{Prim}(B) \cong G \cup \{\infty\}$.