Dynamics and perforation

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Perforation in K_0

A simple ordered Abelian group G is said to be *weakly unperforated* if, for all $g \in G$ and $n \in \mathbb{N}$, $ng > 0$ implies $g > 0$.

Theorem (Villadsen). There exists a simple C^*-algebra of the form $\varinjlim M_{n_k} \otimes C(B^{m_k})$ whose K_0 group has perforation.

Question. Can Villadsen's arguments be recast in a dynamical framework?

The embeddings in Villadsen's construction are diagonal and are composed of maps arising from block projections $B^{m_{k+1}} \longrightarrow B^{m_k}$ along with a certain number of point evaluations to ensure simplicity in the limit.

One starts with a complex line bundle ζ over B for which no tensor power of the Euler class is zero (e.g., $B = S^2$). Setting $\xi = \zeta \times \zeta$ one then applies the following lemma to show that perforation propagates across the finite stages of the inductive limit if the number of point evaluations remains relatively small.

Lemma. For a complex line bundle ζ over B and $n \in \mathbb{N}$, if $[\zeta^{\times n}] - [\theta_1] \in K^0(B^n)$ is positive then the nth tensor power of the Euler class of ζ is zero.

So by the lemma $[\xi] - [\theta_1]$ remains nonpositive under the connecting maps, while a general result shows that $\dim(B^n)$ times this element is positive.

Let Y be a compact Hausdorff space and T the shift $(x_k)_k \mapsto (x_{k+1})_k$ on $Y^{\mathbb{Z}}$.

We have a factorization

$$C(Y) \to C(Y^{\mathbb{Z}}) \rtimes \mathbb{Z} \to C(Y)$$

of the identity map on $C(Y)$ where the first map arises from the projection of $Y^{\mathbb{Z}}$ onto the zeroeth coordinate and the second arises from the universal property of the full crossed product using the map $Y \to Y^{\mathbb{Z}}$ which assigns to a point $y \in Y$ the fixed point $(\dots, y, y, y, \dots) \in Y^{\mathbb{Z}}$.

We thus see that if $K^0(Y)$ has perforation then so does $K_0(C(Y^{\mathbb{Z}}) \rtimes \mathbb{Z})$.

Although the shift is transitive, it is far from being minimal. If we desire a simple crossed product exhibiting perforation, we should look for a subshift that is “small” enough to be minimal but “large” enough to sustain a perforation-producing Euler class obstruction, which can easily be destroyed by noncommutativity.

Our subshift will be constructed as the limit of a decreasing sequence of subshifts defined by a recursive blocking procedure. In the finite blocks from which the subshifts in the sequence are built, the base space Y will appear with asymptotically nonzero density, in analogy with the asymptotically nonzero dimension-rank ratio in the examples of Villadsen and Toms.

This nonzero density is reflected in nonzero values of mean dimension, which is an entropy-like invariant for dynamical systems that provides a measure of dimension growth.

Subshifts from blocks

Consider a product $B = D_1 \times \cdots \times D_l$ of closed subsets of Y, which we call a *block*. Associated to B we have the following two closed T-invariant subsets of $Y^{\mathbb{Z}}$:

X_B: the set of all sequences in $Y^{\mathbb{Z}}$ which are concatenations of elements of B.

P_B: the set of all l-periodic sequences in X_B.

We can embed B into P_B, and hence into X_B, in l possible ways according to phase. By distributing the associated *-homomorphisms $C(X_B) \to C(B)$ down the diagonal in $l \times l$ matrices over $C(B)$ and applying the universal property of the full crossed product, we obtain a *-homomorphism

$$C(X_B) \rtimes \mathbb{Z} \xrightarrow{\varphi} M_l \otimes C(B)$$

with the dynamics on the right-hand side implemented by the shift matrix.

Let γ be the composition

$$C(Y) \longrightarrow C(X_B) \rtimes \mathbb{Z} \xrightarrow{\varphi} M_l \otimes C(B)$$

where the first map is induced from the projection of X_B onto the zeroeth coordinate. Let $\pi_i : B \to D_i \subseteq Y$ for $i = 1, \ldots, l$ be the coordinate projections.

For a vector bundle ξ over Y, viewing projections in $M_l \otimes C(B)$ as bundles we have

$$\gamma(\xi) \cong \pi_1^*(\xi) \oplus \cdots \oplus \pi_l^*(\xi)$$

which is isomorphic to

$$\xi^{\times |L|} \oplus \theta_{\dim(\xi)(l-|L|)}$$

in the case that D_i is equal to Y for all i in a set $L \subseteq \{1, \ldots, l\}$ and is a singleton otherwise.

Suppose now that $\xi = \zeta \times \zeta$ for some complex line bundle ζ for which the Euler class of every tensor power is nonzero. Set $g = [\xi] - [\theta_1] \in K^0(Y)$. If $|L| > l/2$ then

$$\begin{aligned} K_0(\gamma)(g) &= [\xi^{\times |L|} \oplus \theta_{2(l-|L|)}] - [\theta_l] \\ &= [\xi^{\times |L|}] - [\theta_{2|L|-l}] \\ &\leq [\xi^{\times |L|}] - [\theta_1] \end{aligned}$$

so that $K_0(\gamma)(g)$ is not positive by Villadsen's lemma. Thus the image of g in $K_0(C(X_B) \rtimes \mathbb{Z})$ is not positive, while the image of $\dim(Y) g$ is positive.

The limit subshift

We now wish to build a minimal system from these building blocks in a way which respects the perforation.

Fix a metric ρ on X. Let $0 < d < 1$. A decreasing sequence $X_1 \supseteq X_2 \supseteq \ldots$ of closed T-invariant subsets of $Y^{\mathbb{Z}}$ can be constructed so that for each n the set X_n is defined by a block $B_n = D_{n,1} \times \cdots \times D_{n,l_n} \subseteq Y^{l_n}$ and

(1) for all $x, y \in X_n$ there is a k such that $\rho(T^k x, y) \leq 2^{-n+2}$,

(2) $D_{n,i}$ is equal to Y for all i in a subset of $\{1, \ldots, l_n\}$ of size greater than dl_n and $D_{n,i}$ is a singleton for all other i.

Set $X = \bigcap_{n=1}^{\infty} X_n$. Taking $d > 1/2$ ensures that perforation will propagate to the inductive limit $\varinjlim C(X_n) \rtimes \mathbb{Z} \cong C(X) \rtimes \mathbb{Z}$.

Finally, we observe that condition (1) ensures that the system (X, T) is minimal, so that $C(X) \rtimes \mathbb{Z}$ is simple.

Perforation in the Cuntz semigroup

Let A be a C^*-algebra. For elements a, b in $M_\infty(A)^+ = \bigcup_{n=1}^\infty M_n(A)^+$ we write $a \precsim b$ if there is a sequence $\{t_k\}_k$ in some $M_{m,n}(A)$ such that $\lim_{k\to\infty} t_k^* b t_k = a$, and $a \sim b$ if $a \precsim b$ and $b \precsim a$.

Set $W(A) = M_\infty(A)^+ / \sim$ and write $\langle a \rangle$ for the equivalence class of a. For $a \in M_n(A)^+$ and $b \in M_m(A)^+$ we set $\langle a \rangle + \langle b \rangle = \langle a \oplus b \rangle$ and declare that $\langle a \rangle \le \langle b \rangle$ when $a \precsim b$. We refer to $W(A)$ with this structure as the *Cuntz semigroup* of A.

In the case that $na \le mb$ implies $a \le b$ for all $a, b \in W(A)$ and positive integers $n > m$, we say that $W(A)$ is *almost unperforated*.

Theorem (Toms). There exists a simple AH algebra that has the same Elliott invariant (ordered K-theory paired with traces) as an AI algebra and a Cuntz semigroup that fails to be almost unperforated.

Toms furthermore produced an infinite number of simple AH-algebas with the same Elliott invariant but different radii of comparison.

Radius of comparison

We say that A has *r-comparison* if for all $a, b \in M_\infty(A)^+$ we have $\langle a \rangle \leq \langle b \rangle$ whenever $s(\langle a \rangle) + r < s(\langle b \rangle)$ for all lower semicontinuous dimension functions s on $W(A)$. The *radius of comparison* of A is the infimum of the set of all $r \in \mathbb{R}^+$ for which A has r-comparison, or ∞ if this set is empty.

Our goal is to construct crossed products of minimal homeomorphisms that have arbitrarily large radius of comparison and the same Elliott invariant as an AT algebra. By a result of Rørdam, the Cuntz semigroup of these algebras will not be almost unperforated.

As before, our minimal subshift will be constructed by a recursive blocking procedure. This time however we want to arrange for the crossed product to have simple K-theory. More precisely, the ordered K_0 group will be isomorphic to $(\mathbb{Q}, \mathbb{Q}^+, 1)$ and the K_1 group isomorphic to $\mathbb{Z}$.

In order to produce this K-theory we will introduce a spacing factor into our base space. This will enable us at each stage to group together the concatenations of elements from the block into disjoint contractible sets determined by the phase of the blocking. These disjoint sets are permuted by the action, and in the limit we obtain an extension of the universal odometer which produces an isomorphism on K-theory at the level of the crossed product.

Let Y be a compact Hausdorff space and T the shift $(x_k)_k \mapsto (x_{k+1})_k$ on $(Y \times I)^{\mathbb{Z}}$.

For a block $B \subseteq (Y \times I)^l$ and an $i \in \{1, \ldots, l\}$ we write $X_{B,i}$ for the set of all sequences $(x_k)_k \in (Y \times I)^{\mathbb{Z}}$ which are blocked off by B with phase i, i.e.,

$$(x_{i+sl}, x_{i+sl+1}, \ldots, x_{i+sl+l-1}) \in B$$

for all $s \in \mathbb{Z}$. Note that T cyclically permutes the $X_{B,i}$.

Set

$$X_B = X_{B,1} \cup \cdots \cup X_{B,l},$$

which is a closed T-invariant subset of $(Y \times I)^{\mathbb{Z}}$.

In general the sets $X_{B,1}, \ldots, X_{B,l}$ need not be pairwise disjoint (i.e., it might be possible to block off a sequence by B in more than one way), but here we will want to arrange for this to be the case in order for the K_0 group of the crossed product to be isomorphic to the rationals. This is the reason for the second factor in $Y \times I$, which will serve as a spacing device.

Fix a metric ρ on X. Let $0 < d < 1$. We build a decreasing sequence $X_1 \supseteq X_2 \supseteq \dots$ of closed T-invariant subsets of $(Y \times I)^{\mathbb{Z}}$ such that for each n the set X_n is defined by a block $B_n = (D_{n,1} \times I_{n,1}) \times \cdots \times (D_{n,l_n} \times I_{n,l_n})$ where

(1) for all $x, w \in X_n$ there is a k such that $\rho(T^k x, w) \leq 2^{-n+2}$,

(2) $I_{n,1}, \dots, I_{n,l_n}$ are pairwise disjoint closed subintervals of I each with nonempty interior and length at most 2^{-n-2},

(3) $D_{n,i} = Y$ for all i in a subset of $\{1, \dots, l_n\}$ of cardinality greater than dl_n, and $D_{n,i}$ is a singleton for all other i,

(4) n divides l_n.

The block B_{n+1} is constructed as a subset of $B_n^{l_{n+1}/l_n}$ by taking a large number of copies of B_n, trimming these by shrinking the subintervals in the second factor at each coordinate, and then forming the product of the resulting blocks together with a bunch of sets of the form $\{y\} \times J$ where J is a small subinterval of I.

Setting $X = \bigcap_{n=1}^{\infty} X_n$ we obtain a simple crossed product $C(X) \rtimes \mathbb{Z} \cong \varinjlim C(X_n) \rtimes \mathbb{Z}$.

Let $q \geq 2$ and suppose that $Y = I^{3q}$ and $1 - 1/q < d < 1$. We build a positive element $b \in M_{2q}(C(Y))$ by taking the q-fold product of a line bundle on S^2 with nonzero Euler class and pulling back into Y via a two-element partition of unity with one element supported on a homeomorph of $(S^2 \times [0,1])^q$. We also take a projection a in $M_{2q}(C(Y))$ corresponding to a line bundle on Y.

For a given n, the universal property of the full crossed product yields a *-homomorphism

$$C(X_{B_n}) \rtimes \mathbb{Z} \xrightarrow{\varphi_n} M_{l_n} \otimes C(B_n)$$

with the dynamics on the right-hand side implemented by the shift matrix.

Writing E for the set of all coordinates $i = 1, \ldots, l_n$ such that $D_{n,i} = Y$, we have an embedding

$$((S^2)^q)^E \to B_n$$

which gives rise to a *-homomorphism γ_n at the C^*-algebra level.

Consider the composition

$$C(Y) \to C(X_{B_n}) \rtimes \mathbb{Z} \xrightarrow{\varphi_n} M_{l_n} \otimes C(B_n) \xrightarrow{\mathrm{id} \otimes \gamma_n} M_{l_n} \otimes C(((S^2)^q)^E)$$

where the first map arises from the embedding into the zeroeth coordinate. The images a_n and b_n of a and b under this composition are θ_{l_n} and $\xi^{\times q|E|} \oplus \theta_{q(l_n - |E|)}$.

Since ξ has nonzero Euler class and

$$q(l_n - |E|) \leq q l_n (1 - d) < l_n,$$

the trivial bundle θ_{l_n} is not subequivalent to $\xi^{\times q|E|} \oplus \theta_{q(l_n - |E|)}$. Thus

$$\|t^* (\xi^{\times q|E|} \oplus \theta_{q(l_n - |E|)}) t - \theta_{l_n}\| \geq 1/2$$

for all $t \in M_{2q} \otimes M_{l_n} \otimes C(((S^2)^q)^E)$.

We conclude that $\|t^* b_n t - a_n\| \geq 1/2$ for all $t \in M_{2q} \otimes (C(X_n) \rtimes \mathbb{Z})$ and hence $\langle a_n \rangle \not\lesssim \langle b_n \rangle$.

Now map a and b to elements a_∞ and b_∞ in the limit $C(X) \rtimes \mathbb{Z}$. By minimality the tracial states on $C(X) \rtimes \mathbb{Z}$ all arise from invariant measures, so that $s(\langle a_\infty \rangle) = 1$ and $s(\langle b_\infty \rangle) \geq q$ for every lower semicontinuous dimension function s on $W(C(X) \rtimes \mathbb{Z})$. Consequently the radius of comparison of $C(X) \rtimes \mathbb{Z}$ is at least $q - 1$.

K-theory

Suppose that Y is contractible. Then for any n the sets $X_{n,1}, \ldots, X_{n,l_n}$ are contractible, and since they are pairwise disjoint it follows that $K^1(X_n) = 0$ and

$$K^0(X_n) \cong K^0(X_{n,1}) \oplus \cdots \oplus K^0(X_{n,l_n}) \cong \mathbb{Z}^{l_n}.$$

The map $K^0(X_n) \to K^0(X_{n+1})$ is given by

$$k \mapsto (k, \ldots, k) \in (\mathbb{Z}^{l_n})^{l_{n+1}/l_n} \cong \mathbb{Z}^{l_{n+1}},$$

and we have

$$K^0(X) \cong \varinjlim K^0(X_n) \cong \varinjlim \mathbb{Z}^{l_n}.$$

Writing α for the *-automorphism $f \mapsto f \circ T^{-1}$ of $C(X)$, the Pimsner-Voiculescu sequence reads

$$\begin{tikzcd}
K^*(X) \arrow[rr, "\mathrm{id}-\alpha_*"] & & K^*(X) \arrow[dl, "\iota_*"] \\
& K_*(C(X) \rtimes \mathbb{Z}) \arrow[ul] &
\end{tikzcd}$$

Since $K^1(X) = \varinjlim K^1(X_n) = 0$, we have the exact sequence

$$\begin{array}{ccccc} 0 & \longrightarrow & K_1(C(X) \rtimes \mathbb{Z}) & \longrightarrow & K^0(X) \\ & & & & \big\downarrow {\scriptstyle \mathrm{id}-\alpha_*} \\ 0 & \longleftarrow & K_0(C(X) \rtimes \mathbb{Z}) & \xleftarrow{\iota_*} & K^0(X). \end{array}$$

For each n the sets $X_{n,1}, \dots, X_{n,l_n}$ are cyclically permuted by T and n divides l_n. Thus (X,T) is an extension of the universal odometer S, which is defined by addition of $(1,0,0,\dots)$ on $W = \prod_{n=1}^{\infty}\{1,\dots,l_{n+1}/l_n\}$.

Moreover, the above exact sequence is identical to that associated to the universal odometer and hence of the type arising in Giordano, Putnam, and Skau's K-theoretic classification of minimal Cantor systems up to strong orbit equivalence.

Identifying $K^0(W)$ with $C(W,\mathbb{Z})$, we have

$$K_0(C(W)\rtimes\mathbb{Z}) \cong C(W,\mathbb{Z})/\{f - f\circ S^{-1} : f \in C(W,\mathbb{Z})\}$$

by the Pimsner-Voiculescu sequence. By unique ergodicity the equivalence class of a function in $C(W,\mathbb{Z})$ is determined by the value of f on the unique S-invariant state μ.

Since $\mu(C(W,\mathbb{Z})) = \mathbb{Q}$, the K_0 groups of the crossed products of both (W,S) and (X,T) are isomorphic to $\mathbb{Q}$, and so

$$(K_0(C(X)\rtimes\mathbb{Z}), K_0(C(X)\rtimes\mathbb{Z})^+, [1]) \cong (\mathbb{Q}, \mathbb{Q}^+, 1)$$

$$K_1(C(X)\rtimes\mathbb{Z}) \cong \mathbb{Z}.$$

By a range result of Villadsen, there exists a simple AT algebra with the same Elliott invariant as $C(X) \rtimes \mathbb{Z}$ but not isomorphic to $C(X) \rtimes \mathbb{Z}$.

Question. Is the radius of comparison ever finite within this class of systems?

Question. Do there exist nonisomorphic simple crossed products with the same Elliott invariant?

Proposition. Suppose that Y is uncountable. Then the tracial state space of $C(X) \rtimes \mathbb{Z}$ has uncountably many extreme points.

Corollary. Suppose that Y is infinite. Then $C(X) \rtimes \mathbb{Z}$ does not have real rank zero.

Since $C(X) \rtimes \mathbb{Z}$ is stably finite and exact, if $C(X) \rtimes \mathbb{Z}$ had real rank zero then the canonical continuous affine map

$$T(C(X) \rtimes \mathbb{Z}) \to S(K_0(C(X) \rtimes \mathbb{Z}))$$

from tracial states to states on K_0 would be a homeomorphism by a result of Blackadar and Handelman. But $K_0(C(X) \rtimes \mathbb{Z})$ has a unique state and so by the proposition this map is not injective.

When Y has nonzero topological dimension the system (X, T) has nonzero mean dimension, and in this case the uncountability of the set of extreme points follows from the fact that $\mathbb{Z}$-systems with nonzero mean dimension do not possess the small boundary property (Lindenstrauss and Weiss) and hence cannot have only countably many extreme invariant states (Shub and Weiss).

Question. What in general is the relationship between mean dimension and radius of comparison?

Question. Is there a uniquely ergodic minimal homeomophism whose crossed product has nonzero radius of comparison?