Generic automorphisms of approximately divisible AF algebras satisfy the Rohlin property

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Classification of automorphisms

Problem

Classify automorphisms (or group actions) up to outer conjugacy (or cocycle conjugacy).

Key tool: Rohlin property

2 Steps:

- Show "aperiodic" automorphisms have the Rohlin property
- Show automorphisms with the Rohlin property are classified up to outer conjugacy by K-theoretical invariant.

Theorem of Evans-Kishimoto

A: AF algebra
$K_0(A)$: scaled ordered group
$\mathrm{Aut}(A) \ni \alpha \mapsto \alpha_* \in \mathrm{Aut}(K_0(A))$

Theorem (Evans-Kishimoto ('97))

A: AF algebra
$\alpha, \beta \in \mathrm{Aut}(A)$ with the Rohlin property

If $\alpha_* = \beta_* \in \mathrm{Aut}(K_0(A))$, then $\alpha = \mathrm{Ad}\, u \circ \gamma \circ \beta \circ \gamma^{-1}$ for some unitary $u \in A + \mathbb{C}1$ and $\gamma \in \mathrm{Aut}(A)$.

Problem

Which $\sigma \in \mathrm{Aut}(K_0(A))$ is induced by an automorphism of A with the Rohlin property?

Definition of Rohlin property (unital case)

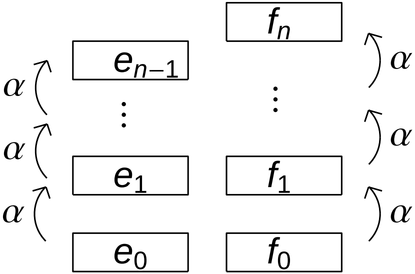
Definition

A: unital C*-algebra

$\alpha \in \mathrm{Aut}(A)$ has the Rohlin property (RP) if
$\forall n \in \mathbb{N}$, $\forall F \subset A$ finite, $\forall \varepsilon > 0$
$\exists (e_k)_{k=0}^{n-1} \cup (f_l)_{l=0}^{n}$: mut. ortho. proj. in A s.t.

1. $\sum_{k=0}^{n-1} e_k + \sum_{l=0}^{n} f_l = 1$,
2. $\|e_k x - x e_k\| < \varepsilon$, $\|f_l x - x f_l\| < \varepsilon$
   for $0 \leq k \leq n-1$, $0 \leq l \leq n$ and $x \in F$,
3. $\|(\alpha(e_k) - e_{k+1})\| < \varepsilon$, $\|(\alpha(f_l) - f_{l+1})\| < \varepsilon$
   for $0 \leq k \leq n-2$, $0 \leq l \leq n-1$.

Rohlin tower



Remark

- Assuming $\|(\alpha(e_{n-1}) - e_0)\| < \varepsilon$, $\|(\alpha(f_n) - f_0)\| < \varepsilon$ is strictly stronger than the Rohlin property.
- “Single tower version” is strictly stronger than RP.
- $\forall n \in \mathbb{N}, \exists n_1, \ldots, n_k \geq n, \cdots\cdots \iff$ RP.

Examples

X: Cantor space
$\alpha \in \mathrm{Aut}(C(X)) \leftrightsquigarrow \sigma$: homeomorphism on X

Fact

α has RP $\iff$ σ is free.

$A = \bigotimes_{k=1}^{\infty} M_{n_k}(\mathbb{C})$: UHF algebra
$\alpha = \bigotimes_{k=1}^{\infty} \mathrm{Ad}(\text{“shift unitary”})$

Fact

α has RP $\iff$ $\sup_k n_k = \infty$.

Examples

$A = \bigotimes_{k=-\infty}^{\infty} M_n(\mathbb{C})$: UHF algebra
$\alpha =$ tensor shift $\in \mathrm{Aut}(A)$

Theorem (Bratteli, Kishimoto, Rørdam, Størmer)

α *has the Rohlin property.*

If A has a unique character,
then A has no automorphism with RP.

Definition of Rohlin property (general)

Definition (K-Phillips)

A: C*-algebra
$\mathcal{M}(A)$: the multiplier algebra of A

$\alpha \in \mathrm{Aut}(A)$ has the Rohlin property (RP) if
$\forall n \in \mathbb{N}$, $\forall F \subset A$ finite, $\forall \varepsilon > 0$
$\exists (e_k)_{k=0}^{n-1} \cup (f_l)_{l=0}^{n}$: mut. ortho. proj. in $\mathcal{M}(A)$ s.t.

1. $\sum_{k=0}^{n-1} e_k + \sum_{l=0}^{n} f_l = 1$,
2. $\|e_k x - x e_k\| < \varepsilon$, $\|f_l x - x f_l\| < \varepsilon$ for k, l, x,
3. $\|(\alpha(e_k) - e_{k+1})\alpha(x)\| < \varepsilon$,
   $\|(\alpha(f_l) - f_{l+1})\alpha(x)\| < \varepsilon$ for k, l and $x \in F$.

Permanence property

Lemma

A: C-algebra,* $\alpha \in \mathrm{Aut}(A)$ *with RP*
Then the following automorphisms also have RP;

- $\gamma \circ \alpha \circ \gamma^{-1}$ *for* $\gamma \in \mathrm{Aut}(A)$,
- $\mathrm{Ad}(u) \circ \alpha$ *for a unitary* $u \in \mathcal{M}(A)$,
- α^n *for* $n \in \mathbb{Z} \setminus \{0\}$,
- $\alpha \otimes \beta \in \mathrm{Aut}(A \otimes B)$ *for* $\beta \in \mathrm{Aut}(B)$,
- *the restriction of* α *to an* α*-invariant ideal* I,
- *the induced automorphism of the quotient* A/I,
- *the restriction of* α *to a hereditary subalgebra* $B \subset A$ *if* B *has an approx. unit of projections.*

Limit of unital C*-algebras

Lemma

A: C-algebra with an approx. unit of projections.*

$\alpha \in \mathrm{Aut}(A)$ *has the Rohlin property*
$\iff \forall n \in \mathbb{N}, \forall F \subset A$ *finite,* $\forall \varepsilon > 0$
$\exists (e_k)_{k=0}^{n-1} \cup (f_l)_{l=0}^{n}$*: mut. ortho. proj. in* A *s.t.*

1. $q = \sum_{k=0}^{n-1} e_k + \sum_{l=0}^{n} f_l$ *satisfies* $\|qx - x\| < \varepsilon$ *for* $x \in F$,
2. $\|e_k x - x e_k\| < \varepsilon$, $\|f_l x - x f_l\| < \varepsilon$ *for* k, l, x,
3. $\|(\alpha(e_k) - e_{k+1})\alpha(x)\| < \varepsilon$,
   $\|(\alpha(f_l) - f_{l+1})\alpha(x)\| < \varepsilon$ *for* k, l *and* $x \in F$.

$R(n, F, \varepsilon)$

Definition

A: C*-algebra with an approx. unit of projections.
Let $n \in \mathbb{N}$, $F \subset A$ finite, $\varepsilon > 0$.

We define $R(n, F, \varepsilon) \subset \mathrm{Aut}(A)$ to be the set of $\alpha \in \mathrm{Aut}(A)$ s.t. $\exists (e_k)_{k=0}^{n-1} \cup (f_l)_{l=0}^{n}$ as in Lemma.

$$\alpha \in \mathrm{Aut}(A)\text{: RP} \iff \alpha \in \bigcap_{n,F,\varepsilon} R(n, F, \varepsilon)$$

Intersection of countable open subsets

A: C*-algebra with an approx. unit of projections.

Lemma

$(F_k)_{k=1}^{\infty}$: increasing sequence of finite subsets of A whose union is dense in A

$$\left\{\alpha \in \mathrm{Aut}(A) : RP\right\} = \bigcap_{n,k \in \mathbb{N}} R(n, F_k, 2^{-k})$$

Lemma

$\forall n \in \mathbb{N}$, $\forall F \subset A$ finite, $\forall \varepsilon > 0$,
$R(n, F, \varepsilon) \subset \mathrm{Aut}(A)$ is open.

$\mathrm{Aut}(A)$: topology of pointwise norm convergence

Complete metric on $\mathrm{Aut}(A)$

A: separable C*-algebra

$\mathrm{Aut}(A)$ has a complete metric d defined by

$$d(\alpha,\beta) = \sum_{k=1}^{\infty} 2^{-k}\Big(\|\alpha(x_k)-\beta(x_k)\|+\|\alpha^{-1}(x_k)-\beta^{-1}(x_k)\|\Big)$$

for $\alpha,\beta \in \mathrm{Aut}(A)$
where $\{x_k\}_{k=1}^{\infty}$: dense in the unit ball of A.

AF algebras

A: AF algebra
$K_0(A)$: scaled ordered group
$K_0(A)_+$: positive cone

$$\mathrm{Aut}(A) \ni \alpha \mapsto \alpha_* \in \mathrm{Aut}(K_0(A))$$

Lemma

For $\sigma \in \mathrm{Aut}(K_0(A))$,
$\{\alpha \in \mathrm{Aut}(A) : \alpha_* = \sigma\}$ *is non-empty and closed.*

$\{\alpha \in \mathrm{Aut}(A) : \alpha_* = \sigma\}$ has a complete metric.

Theorem of Evans-Kishimoto

Theorem (Evans-Kishimoto ('97))

A: AF algebra
$\alpha, \beta \in \mathrm{Aut}(A)$ *with the Rohlin property*

If $\alpha_* = \beta_* \in \mathrm{Aut}(K_0(A))$, *then* $\forall \varepsilon > 0$,
$\exists u \in A + \mathbb{C}1$ *unitary with* $\|u - 1\| < \varepsilon$ *and*
$\exists \gamma \in \mathrm{Aut}(A)$ *with* $\gamma_* = \mathrm{id}$ *such that*

$$\alpha = \mathrm{Ad}\, u \circ \gamma \circ \beta \circ \gamma^{-1}.$$

Problem

Which $\sigma \in \mathrm{Aut}(K_0(A))$ is induced by
an automorphism of A with the Rohlin property?

Property (R)

Definition

$\sigma \in \mathrm{Aut}(K_0(A))$ satisfies Property (R) if
$\forall n \in \mathbb{N}$, finite sets I, J, a map $x : I \to K_0(A)_+$
and maps $m, m' : I \times J \to \mathbb{Z}_+$ such that
$\sum_{i \in I} m(i,j)\sigma(x(i)) = \sum_{i \in I} m'(i,j)x(i)$ for $j \in J$,
$\exists y_k : I \to K_0(A)_+$ for $0 \leq k \leq n$ satisfying:

1. $\sum_{k=0}^{n} y_k(i) = x(i)$ for $i \in I$,
2. $\sum_{i \in I} m(i,j)\sigma(y_k(i)) = \sum_{i \in I} m'(i,j)y_{k+1}(i)$,
   $\sum_{i \in I} m(i,j)\sigma(y_{n-1}(i)) \geq \sum_{i \in I} m'(i,j)y_n(i)$
   for $k = 0, 1, \ldots, n-2$ and $j \in J$.

Main Theorem

Theorem (K-Phillips)

A: AF algebra

$\sigma \in \mathrm{Aut}(K_0(A))$ *is induced by* $\alpha \in \mathrm{Aut}(A)$ *with RP*
$\iff$ σ *satisfies Property (R).*

Moreover for such σ*,*
$\{\alpha \in \mathrm{Aut}(A) : RP, \alpha_* = \sigma\}$ *is*
a dense G_δ-set of $\{\alpha \in \mathrm{Aut}(A) : \alpha_* = \sigma\}$.

Main Corollary

Corollary

For an AF algebra A, T.F.A.E.:

1. *Every* $\sigma \in \mathrm{Aut}(K_0(A))$ *is induced by* $\alpha \in \mathrm{Aut}(A)$ *with the Rohlin property.*
2. *Every* $\sigma \in \mathrm{Aut}(K_0(A))$ *satisfies Property (R).*
3. $\exists \alpha \in \mathrm{Aut}(A)$ *approx. inner with RP.*
4. $\mathrm{id} \in \mathrm{Aut}(K_0(A))$ *satisfies Property (R).*
5. $\forall x \in K_0(A)_+$ *and* $\forall n \in \mathbb{N}$, $\exists y \in K_0(A)_+$ *with* $ny \leq x \leq (n+1)y$.

Main Corollary (continued)

Proposition

For an AF algebra A, T.F.A.E.:

5. $\forall x \in K_0(A)_+$ *and* $\forall n \in \mathbb{N}$, $\exists y \in K_0(A)_+$ *with* $ny \leq x \leq (n+1)y$.
6. $\forall x \in K_0(A)_+$, $\exists y \in K_0(A)_+$ *with* $2y \leq x \leq 3y$.
7. A *is approximately divisible.*
8. A *is* $\mathcal{Z}$*-absorbing.*
9. *No corner of* A *has a non-zero finite dimensional quotient.*

Approximately divisible AF algebras

Corollary

For an approx. div. (= $\mathcal{Z}$*-absorbing) AF algebra* A*, every* $\sigma \in \mathrm{Aut}(K_0(A))$ *is induced by* $\alpha \in \mathrm{Aut}(A)$ *with the Rohlin property.*

A simple, non-type-I AF algebra is approx. div.

Problem

Give a direct proof of it, and generalize it.

$\mathcal{Z}$-absorbing AF algebras

Problem

A: a $\mathcal{Z}$-absorbing AF algebra, $\alpha \in \mathrm{Aut}(A)$,
$\gamma \in \mathrm{Aut}(\mathcal{Z})$: tensor shift of $\mathcal{Z} \cong \bigotimes_{k=-\infty}^{\infty} \mathcal{Z}$.
Does $\alpha \otimes \gamma \in \mathrm{Aut}(A \otimes \mathcal{Z}) = \mathrm{Aut}(A)$ have RP?

Remark

Phillips showed that $\alpha \otimes \gamma$ has the tracial Rohlin property.

Problem

Are the tracial Rohlin property and the Rohlin property equivalent?

Sketch of the proof

I will sketch the proof of Main Theorem:

Theorem (K-Phillips)

A: AF algebra

$\sigma \in \mathrm{Aut}(K_0(A))$ is induced by $\alpha \in \mathrm{Aut}(A)$ with RP $\iff$ σ satisfies Property (R).

Moreover for such σ,

$\{\alpha \in \mathrm{Aut}(A) : RP, \alpha_* = \sigma\}$ is a dense G_δ-set of $\{\alpha \in \mathrm{Aut}(A) : \alpha_* = \sigma\}$.

Sketch of the proof

Recall $\sigma \in \mathrm{Aut}(K_0(A))$ satisfies Property (R)
$\iff \forall n \in \mathbb{N}$, a finite collection $(x(i))_i$ in $K_0(A)_+$
and a finite collection of "relations"
between $(\sigma(x(i)))_i$ and $(x(i))_i$,
$\exists$ collections $(y_k(i))_i$ in $K_0(A)_+$ for $0 \le k \le n$ with
$\sum_{k=0}^{n} y_k(i) = x(i)$ for $i \in I$ such that
$(\sigma(y_k(i)))_i$ and $(y_{k+1}(i))_i$ satisfy
the "relations" for $0 \le k \le n-2$,
and $(\sigma(y_{n-1}(i)))_i$ dominates $(y_n(i))_i$
with respect to the "relations".

Sketch of the proof

Recall $\alpha \in \mathrm{Aut}(A)$: RP $\iff \alpha \in \bigcap_{n,F,\varepsilon} R(n,F,\varepsilon)$
where for $n \in \mathbb{N}$, $F \subset A$ finite, $\varepsilon > 0$
$R(n,F,\varepsilon)$ is the set of $\alpha \in \mathrm{Aut}(A)$ such that
$\exists (e_k)_{k=0}^{n-1} \cup (f_l)_{l=0}^{n}$: mut. ortho. proj. in A satisfying

- $q = \sum_{k=0}^{n-1} e_k + \sum_{l=0}^{n} f_l$
  satisfies $\|qx - x\| < \varepsilon$ for $x \in F$,
- $\|e_k x - x e_k\| < \varepsilon$, $\|f_l x - x f_l\| < \varepsilon$ for k, l, x,
- $\|(\alpha(e_k) - e_{k+1})\alpha(x)\| < \varepsilon$,
  $\|(\alpha(f_l) - f_{l+1})\alpha(x)\| < \varepsilon$ for k, l and $x \in F$.

Sketch of the proof

Proposition

If $\alpha \in \mathrm{Aut}(A)$ *has RP,*
then $\sigma = \alpha_* \in \mathrm{Aut}(K_0(A))$ *satisfies Property (R).*

(Proof) May assume A is stable.
Give $n \in \mathbb{N}$ and a collection $(x(i))_i$ in $K_0(A)_+$
with "relations" between $(\sigma(x(i)))_i$ and $(x(i))_i$.
Choose mut. ortho. projs $(p(i))_{i \in I}$ with $[p(i)] = x(i)$,
and partial isometries representing "the relations".
By the RP of α, $\exists$ a Rohlin tower $(e_k)_{k=0}^{n-1} \cup (f_k)_{k=0}^{n}$.
Define $y_k(i) = [(e_k + f_k)p(i)] \in K_0(A)_+$
for $0 \le k \le n-1$, and $y_n(i) = [f_n p(i)] \in K_0(A)_+$. $\square$

Sketch of the proof

Proposition

Let $\sigma \in \mathrm{Aut}(K_0(A))$ *satisfy Property (R).*
Let $n \in \mathbb{N}$, $F \subset A$ *finite,* $\varepsilon > 0$.

Then $R(n, F, \varepsilon) \cap \{\alpha \in \mathrm{Aut}(A) : \alpha_* = \sigma\}$
is dense in $\{\alpha \in \mathrm{Aut}(A) : \alpha_* = \sigma\}$.

Main Theorem follows from two propositions and Baire's category theorem.

Sketch of the proof

This proposition follows from

Lemma

Let $\sigma \in \mathrm{Aut}(K_0(A))$ *satisfy Property (R).*
Let $\alpha \in \mathrm{Aut}(A)$ *be* $\alpha_* = \sigma$*.*
Let $n \in \mathbb{N}$, $F \subset A$ *finite* $\varepsilon > 0$,
and $B \subset A$ *finite dimensional subalgebra.*

Then $\exists\, u \in (A + \mathbb{C}1) \cap B'$ *unitary*
such that $\alpha \circ \mathrm{Ad}\, u \in R(n, F, \varepsilon)$*.*

Sketch of the proof

(Proof of Lemma)
Choose $D \subset A$: finite dimensional algebra such that $F \subset_{\varepsilon} D$, $\bigcup_{k=0}^{n} \alpha^k(B) \subset D$

Let $(x(i))_{i \in I}$ be the generators of $K_0(E)$.
Two inclusions $D \subset E$ and $\alpha(D) \subset E$ give several relations between $(\sigma(x(i)))_i$ and $(x(i))_i$.
Since σ satisfies Property (R),
we get $y_k : I \to K_0(A)_+$ for $0 \leq k \leq n$ satisfying $\sum_{k=0}^{n} y_k(i) = x(i)$ and so on.

Sketch of the proof

Choose projections $p_k(i)$ with $[p_k(i)] = y_k(i)$.
From $\{p_k(i)\}$, construct projections $p_k \in A \cap E'$
with $\sum_{k=0}^{n} p_k = 1_E$.
Using "relations" of $(y_k(i))_{k,i}$,
show $[p_k 1_D] = [\alpha^{-1}(p_{k+1})1_D]$ in $K_0((A \cap D')1_D)$.
Find a unitary $u \in (A + \mathbb{C}1) \cap D'$ such that
$\alpha \circ \mathrm{Ad}\, u(p_k 1_D) = p_{k+1}\alpha(1_D)$.
Construct a Rohlin tower from $(p_k)_{k=0}^{n}$ to conclude
$\alpha \circ \mathrm{Ad}\, u \in R(n, F, \varepsilon)$. □

Remark: $K_0((A \cap D')1_D) \to K_0(A)$ is not injective in general.

Problems

Problem

Generalize the result to AT algebras or ASH algebras or $\cdots$.

Problem

Find a condition on automorphisms (outerness?) which is equivalent to have RP, but is easier to check than RP.

Problem

What is $A \rtimes_\alpha \mathbb{Z}$ for $\alpha \in \mathrm{Aut}(A)$ with RP?

Problems

Problem

Apply the main result to AF-embedding of crossed products with controlling K-theory.

Problem

Lift $\sigma \in \mathrm{Aut}(K_0(A))$ to $\alpha \in \mathrm{Aut}(A)$ with the same order.

Problem

Generalize the main result to “corner” endomorphisms.