Classification of $\mathbb{Z}^2$-actions on the Kirchberg algebras

by

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§1. Introduction

• A: Unital C^*-algebra,
$u, v \in U(A)$: Unitaries s.t. $uv - vu$ small,

One can associate the Bott element Bott(u, v) in $K_0(A)$ to u, v (Loring).

$$e(u,v) = \begin{pmatrix} f(v) & h(v)u + g(v) \\ u^*h(v) + g(v) & 1 - f(v) \end{pmatrix}$$

$$\mathrm{Bott}(u,v) = [\chi_{[\frac{1}{2},\infty)}(e(u,v))]_0 - [1]_0 \in K_0(A).$$

• When u is homotopic to 1 in $U(A)$, Bott(u, v) is an obstruction for existence of a homotopy $\{u_t\}_{t\in[0,1]}$ in $U(A)$ s.t.
$u_0 = 1$, $u_1 = 1$, and $u_tv - vu_t$ small.

This is the only obstruction for many classes of C^*-algebras.
e.g. purely infinite simple C^*-algebras.
(Bratteli-Elliott-Evans-Kishimoto)

Automorphism case

• $\alpha, \beta \in \mathrm{Aut}(A)$: Homotopic to id s.t. $\alpha \circ \beta = \beta \circ \alpha$.

What is a relevant obstruction for existence of a homotopy $\{\alpha_t\}_{t\in[0,1]}$ in $\mathrm{Aut}(A)$ s.t. $\alpha_0 = \mathrm{id}$, $\alpha_1 = \alpha$, and $\beta \circ \alpha_t = \alpha_t \circ \beta$ for $\forall t$?

• Choose an arbitrary homotopy $\{\gamma_t\}_{t\in[0,1]}$ in $\mathrm{Aut}(A)$ s.t. $\gamma_0 = \mathrm{id}$, $\gamma_1 = \alpha$.

For $x \in A$, the function $t \mapsto \gamma_t \circ \beta \circ \gamma_t^{-1} \circ \beta^{-1}(x)$ is periodic, which gives a homomorphism $\rho : A \to C(\mathbb{T}) \otimes A$.

Let $j_A : A \ni x \mapsto 1 \otimes x \in C(\mathbb{T}) \otimes A$.
The class $[\rho] - [j_A]$ in

$$KK(A, C_0((0,1)) \otimes A) = KK^1(A, A)$$

does not depend on the choice of $\{\gamma_t\}$.

This is an obstruction!

Goal

To classify $\mathbb{Z}^2$-actions α with $KK(\alpha_g) = 1$ for $\forall g \in \mathbb{Z}^2$ by $KK^1(A, A)$.

• $KK^1(A, A)$ is enough for Kirchberg algebras (main result).

• Obviously, $KK^1(A, A)$ is not enough in stably finite case.

§2. Main Results

Γ: Discrete group,
A, B: Unital C^*-algebras,
$\alpha : \Gamma \to \mathrm{Aut}(A)$, $\beta : \Gamma \to \mathrm{Aut}(B)$: Actions.

• α and β are conjugate
$\overset{\text{def}}{\Leftrightarrow}$ $\exists\theta : A \to B$ isomorphism s.t. $\theta^{-1}\circ\beta_g\circ\theta = \alpha_g$.

• α and β are outer conjugate
$\overset{\text{def}}{\Leftrightarrow}$ $\exists\theta : A \to B$ isomorphism, $\exists u_g \in U(A)$ s.t. $\theta^{-1} \circ \beta_g \circ \theta = \mathrm{Ad}u_g \circ \alpha_g$.

• α and β are cocycle conjugate
$\overset{\text{def}}{\Leftrightarrow}$ $\exists\theta : A \to B$ isomorphism, $\exists u_g \in U(A)$ s.t. $\theta^{-1} \circ \beta_g \circ \theta = \mathrm{Ad}u_g \circ \alpha_g$ and
$u_g\alpha_g(u_h) = u_{gh}$ (1-cocycle relation).

If moreover $A = B$ and $KK(\theta) = 1$,
we say that α and β are cocycle conjugate by a KK-trivial automorphism.

• A Kirchberg algebra is a purely infinite, simple, nuclear, separable C^*-algebra.

Theorem (M.I., H. Matui)
Let A be a unital Kirchberg algebra.
Then there exists a one-to-one correspondence between the following two sets:

(1) The set of outer actions α of $\mathbb{Z}^2$ on A with $KK(\alpha_g) = 1$ for $\forall g \in \mathbb{Z}^2$, modulo cocycle conjugacy by KK-trivial automorphisms.

(2) $\{x \in KK^1(A, A);\ [1]_0 \otimes_A x = 0\}$, where $[1]_0 \in K_0(A) = KK(\mathbb{C}, A)$.

Example
There are exactly $n-1$ cocycle conjugacy classes of outer $\mathbb{Z}^2$-actions on the Cuntz algebra $\mathcal{O}_n$ for $n < \infty$ because

$$KK^1(\mathcal{O}_n, \mathcal{O}_n) = \mathrm{Ext}(\mathbb{Z}_{n-1}, \mathbb{Z}_{n-1}) = \mathbb{Z}_{n-1}.$$

Theorem (M.I., H. Matui)
There exists only one cocycle conjugacy class of outer $\mathbb{Z}^n$-actions for $\forall n \in \mathbb{N}$ on the Cuntz algebra $\mathcal{O}_\infty$.

Remark
The same statement for $\mathcal{O}_2$ was obtained by Matui before.

Fix an outer action μ of $\mathbb{Z}^n$ on $\mathcal{O}_\infty$.

Theorem (M.I., H. Matui)
Any outer $\mathbb{Z}^n$-action α on a Kirchberg algebra A is cocycle conjugate to the diagonal action $\alpha \otimes \mu$ on $A \otimes \mathcal{O}_\infty$.
In consequence, α has the Rohlin property.

Remark
$A \cong A \otimes \mathcal{O}_\infty$ by Kirchberg-Phillips.

• (α, w) is a <u>cocycle action</u> of Γ on A
$\overset{\text{def}}{\Leftrightarrow}$ $\alpha : \Gamma \to \mathrm{Aut}(A)$, $w(g,h) \in U(A)$ s.t.
$\alpha_g \circ \alpha_h = \mathrm{Ad} w_{g,h} \circ \alpha_{gh}$,
$w_{g,h} w_{gh,k} = \alpha_g(w_{h,k}) w_{g,hk}$ (2-cocycle relation).

• Two cocycle actions (α, w) and (α', w') are <u>equivalent</u>
$\overset{\text{def}}{\Leftrightarrow}$ $\exists v_g \in U(A)$ s.t. $\alpha'_g = \mathrm{Ad} v_g \circ \alpha_g$,
$w'_{g,h} = v_g \alpha_g(v_h) w_{g,h} v^*_{gh}$ (cohomologous).

<u>Theorem</u> (M.I., H. Matui)
Any outer cocycle $\mathbb{Z}^2$-action (α, w) on a unital Kirchberg algebra A with trivial cohomology class $[[w(\cdot,\cdot)]_1] \in H^2(\mathbb{Z}^2, K_1(A))$ is equivalent to a genuine action.

In particular, when $[1]_0 = 0$ in $K_0(A)$, any outer cocycle $\mathbb{Z}^2$-action is equivalent to a genuine action.

Example

Any outer cocycle $\mathbb{Z}^2$-action on $\mathcal{O}_n$ for $n = 2, \cdots, \infty$ is equivalent to a genuine action.

Remark

There are countably many cocycle conjugacy classes of outer $\mathbb{Z}^n$-actions on $\mathcal{O}_\infty \otimes \mathbb{K}$ for $n \geq 3$.

This means that for $n \geq 3$, there are a lot of outer cocycle $\mathbb{Z}^n$-actions on $\mathcal{O}_\infty$ that are <u>not</u> equivalent to genuine actions.

§3. Strategy

• $\rho_1, \rho_2 \in \mathrm{Hom}(A,B)$ are asymptotically unitarily equivalent, $\rho_1 \overset{\text{as.u.}}{\sim} \rho_2$,
$\overset{\text{def}}{\Leftrightarrow}$ $\exists\{u(t)\}_{t\geq 0}$ continuous path in $U(B)$ s.t.

$$\lim_{t\to\infty} \|\mathrm{Ad}u(t)\circ\rho_2(x) - \rho_1(x)\| = 0,\ \forall x \in A.$$

• A, B: Unital Kirchberg algebras with $[1_A]_0 = 0$ in $K_0(A)$, and $[1_B]_0 = 0$ in $K_0(B)$.

$\widehat{H}(A,B)$:=The set of unital homomorphisms from A to B modulo asym. u. equivalence.

Theorem (Phillips)
$\widehat{H}(A,B) \cong KK(A,B)$.

Theorem (Nakamura)
A: Kirchberg algebra
$\alpha, \beta \in \mathrm{Aut}(A)$: Aperiodic
The following conditions are equivalent
(1) $KK(\alpha) = KK(\beta)$.
(2) $\exists\theta \in \mathrm{Aut}(A)$, $\exists u \in U(A)$ s.t. $KK(\theta) = 1$, $\theta^{-1}\circ\beta\circ\theta = \mathrm{Ad}u\circ\alpha$.

A: Unital Kirchberg algebra,
α, β: Outer $\mathbb{Z}^2$-actions on A s.t.
$KK(\alpha_g) = KK(\beta_g) = 1$ for $\forall g \in \mathbb{Z}^2$.

• $\gamma := \alpha_{(1,0)}$ is a unique aperiodic automorphism with $KK(\gamma) = 1$ up to cocycle conjugacy by Nakamura.

We may assume $\mathrm{Ad}u \circ \beta_{(1,0)} = \gamma$,
$\Rightarrow B := A \rtimes_\gamma \mathbb{Z} = A \rtimes_{\beta_{(1,0)}} \mathbb{Z}$,
$\widehat{\gamma} = \widehat{\beta_{(1,0)}}$, dual actions.

• $\alpha_{(0,1)}, \beta_{(0,1)}$ extend to $\tilde{\alpha}, \tilde{\beta} \in \mathrm{Aut}(B)$ commuting with $\widehat{\gamma}_t$.

• Conversely, if $\theta \in \mathrm{Aut}(B)$ commutes with $\widehat{\gamma}_t$,
$\Rightarrow$ the restriction $\theta_0 := \theta|_A$ commutes with γ up to inner automorphism.
$\Rightarrow (\gamma, \theta_0)$ gives a cocycle $\mathbb{Z}^2$-action.

Strategy for uniqueness

• $\mathbb{T}$-equivariant versions of the above results of Phillips and Nakamura.
(for outer conjugacy.)

• Model action splitting

$$(A, \alpha) \cong (A \otimes \mathcal{O}_\infty, \alpha \otimes \mu),$$

where μ is a quasi-free action.
Every rotation algebra embeds in $\mathcal{O}_\infty$.
(outer conjugacy $\Rightarrow$ cocycle conjugacy.)

Strategy for existence

Second cohomology vanishing theorem based on Ocneanu's idea
(with special care of K-theory).

§4. Asymptotically representable actions

A: Unital C^*-algebra,
Γ: Discrete amenable group,
γ: Action of Γ on A.

• γ is asymptotically representable
$\overset{\text{def}}{\Leftrightarrow}$ $\exists\{u_g(t)\}_{t\in[0,\infty)}$ continuous path in $U(A)$ for each $g \in \Gamma$, s.t.

$$\lim_{t\to\infty} \|u_g(t)xu_g(t)^* - \gamma_g(x)\| = 0, \ \forall x \in A,$$

$$\lim_{t\to\infty} \|u_g(t)u_h(t) - u_{gh}(t)\| = 0, \ \forall g, h \in \Gamma,$$

$$\lim_{t\to\infty} \|\gamma_g(u_h(t)) - u_{ghg^{-1}}(t)\| = 0, \ \forall g, h \in \Gamma.$$

If γ is an asym. rep. action, so is its cocycle perturbation.

• If A is a Kirchberg algebra, and $\theta \in \mathrm{Aut}(A)$ is aperiodic with $KK(\theta) = 1$, θ gives a unique outer asym. rep. action of $\mathbb{Z}$ on A up to cocycle conjugacy by KK-trivial automorphisms.

$B := A \rtimes_\gamma \Gamma$: Crossed product,
$\widehat{\gamma} : B \to B \otimes C^*(\Gamma)$: Dual coaction of γ,
$\mathrm{End}_{\widehat{\Gamma}}(B)$: The set of unital $\rho \in \mathrm{End}(B)$ s.t.

$$\widehat{\gamma} \circ \rho = (\rho \otimes \mathrm{id}) \circ \widehat{\gamma}.$$

• $\rho_1, \rho_2 \in \mathrm{End}_{\widehat{\Gamma}}(B)$ are $\widehat{\Gamma}$-asymptotically unitarily equivalent, $\rho_1 \overset{\widehat{\Gamma}-\text{as.u.}}{\sim} \rho_2$,
$\overset{\text{def}}{\Leftrightarrow}$ ρ_1 and ρ_2 are asym. u. equivalent by a continuous path of unitaries <u>in A</u>.

• $\widehat{H}_{\widehat{\Gamma}}(B, B)$:= $\mathrm{End}_{\widehat{\Gamma}}(B)$ modulo $\widehat{\Gamma}$-asym. u. equivalence.

Assumption

A: Unital Kirchberg algebra.

γ: Outer asym. rep. action.

Lemma

If a unital $\rho \in \mathrm{End}(B)$ satisfies

$$KK(\widehat{\gamma} \circ \rho) = KK((\rho \otimes \mathrm{id}) \circ \widehat{\gamma}),$$

then $\exists \rho_1 \in \mathrm{End}_{\widehat{\Gamma}}(B)$ s.t. $KK(\rho) = KK(\rho_1)$.

Lemma

If $\rho_1, \rho_2 \in \mathrm{End}_{\widehat{\Gamma}}(B)$ satisfy $KK(\rho_1) = KK(\rho_2)$, then $[\rho_1] = [\rho_2]$ in $\widehat{H}_{\widehat{\Gamma}}(B, B)$.

- The proofs use a very strong form of the "Rohlin property" of $\widehat{\gamma}$.

- These two lemmas imply

$$\widehat{H}_{\widehat{\Gamma}}(B, B) \cong \{x \in KK(B, B);\ x \otimes_B KK(\widehat{\gamma}) = KK(\widehat{\gamma}) \otimes_{B \otimes C^*(\Gamma)} (x \otimes 1_{C^*(\Gamma)})\}.$$

Theorem

If $[1]_0 = 0$ in $K_0(A)$ and $\Gamma = \mathbb{Z}$, then $\widehat{H}_{\mathbb{T}}(B, B) \cong KK(A, A) \oplus KK^1(A, A)$, with ring structure:

$$(x_0 \oplus x_1) \cdot (y_0 \oplus y_1) = (x_0 \otimes_A y_0 \oplus (x_0 \otimes_A y_1 + x_1 \otimes_A y_0)).$$

$x_0 \oplus x_1$ is invertible $\Leftrightarrow$ x_0 is invertible.

If $\rho \in \mathrm{End}_{\hat{\Gamma}}(B)$ corresponds to $x_0 \oplus x_1$, $KK(\rho|_A) = x_0$.

Outline of the proof

• Since $KK(\gamma) = 1$, (A, γ) is cocycle conjugate to $(A \otimes \mathcal{O}_\infty, \mathrm{id} \otimes \gamma')$, where γ' is a quasi-free action.
$\Rightarrow$ $(B, \widehat{\gamma}) \cong (A \otimes (\mathcal{O}_\infty \rtimes_{\gamma'} \mathbb{Z}), \mathrm{id} \otimes \widehat{\gamma}')$.

• $(\mathcal{O}_\infty, \gamma') \overset{KK_{\mathbb{Z}}}{\sim} (\mathbb{C}, \mathrm{id}) \Rightarrow$

$$\begin{aligned} (\mathcal{O}_\infty \rtimes_{\gamma'} \mathbb{Z}, \widehat{\gamma}') &\overset{KK}{\sim} (C^*(\mathbb{Z}), \delta) \\ &\overset{KK}{\sim} (\mathbb{C} \oplus C_0(\mathbb{R}), \delta'). \end{aligned}$$

To show uniqueness up to outer conjugacy, it suffices to prove

Theorem

If $\theta_1, \theta_2 \in \mathrm{Aut}_{\hat{\Gamma}}(B)$ s.t.
$\theta_1|_A$, $\theta_2|_A$ are aperiodic and
$[\theta_1] = [\theta_2]$ in $\hat{H}_{\hat{\Gamma}}(B, B)$,
then $\exists \varphi \in \mathrm{Aut}_{\hat{\Gamma}}(B)$, $\exists u \in U(A)$ s.t.
$\varphi^{-1} \circ \theta_2 \circ \varphi = \mathrm{Ad} u \circ \theta_1$ and $[\varphi] = [\mathrm{id}]$ in $\hat{H}_{\hat{\Gamma}}(B, B)$.

The proof is an $\hat{\Gamma}$-equivariant version of Nakamura's argument, which requires the Rohlin projections for θ_1 and θ_2 in A.

We need the following two lemmas.

$\omega \in \beta\mathbb{N} \setminus \mathbb{N}$: free ultrafilter,
$c_\omega(B) = \{(x_n) \in \ell^\infty(\mathbb{N}, B);\ \lim_{n\to\omega} \|x_n\| = 0\}$,
$B^\omega = \ell^\infty(\mathbb{N}, B)/c_\omega(B)$,
$A^\omega, B \subset B^\omega$.

Lemma (Equivariant Kirchberg theorem)
$A^\omega \cap B'$ is purely infinite simple.

- To construct the Rohlin projections for θ_1 and θ_2 in A, it suffices to show that θ_1 and θ_2 induce aperiodic automorphisms of $A^\omega \cap B'$.

- Note that $A^\omega \cap B' = (A^\omega)^\Gamma$.

Lemma (Asymptotic Galois correspondence)
Let $\theta \in \mathrm{Aut}(A)$ s.t. $\theta^\omega(x) = x$ for $\forall x \in A^\omega \cap B'$. Then $\exists u \in U(A)$ and $\exists g \in \Gamma$ s.t. $\theta = \mathrm{Ad}u \circ \gamma_g$.

Remark
This lemma holds for any outer action of any countable discrete group.

§5. Model action splitting

$\rho_l : \mathcal{O}_\infty \ni x \mapsto x \otimes 1 \in \mathcal{O}_\infty \otimes \mathcal{O}_\infty$,
$\rho_r : \mathcal{O}_\infty \ni x \mapsto 1 \otimes x \in \mathcal{O}_\infty \otimes \mathcal{O}_\infty$.

The essence of the $\mathcal{O}_\infty$-theorem $A \cong A \otimes \mathcal{O}_\infty$ is the fact that ρ_l and ρ_r are approximately unitarily equivalent, shown by Lin-Phillips.

Γ: Countable infinite discrete amenable group.
μ^Γ: Quasi-free action of Γ on $\mathcal{O}_\infty = C^*\{S_g\}_{g\in\Gamma}$ given by $\mu^\Gamma_g(S_h) = S_{gh}$.

Lemma
If there exists a sequence of unitaries $\{u_n\}$ in $\mathcal{O}_\infty \otimes \mathcal{O}_\infty$ s.t.

$$\lim_{n\to\infty} \|\mathrm{Ad}u_n \circ \rho_l(x) - \rho_r(x)\| = 0,\ \forall x \in \mathcal{O}_\infty,$$

$$\lim_{n\to\infty} \|\mu^\Gamma_g \otimes \mu^\Gamma_g(u_n) - u_n\| = 0,\ \forall g \in \Gamma,$$

then (A, α) is cocycle conjugate to $(A \otimes \mathcal{O}_\infty, \alpha \otimes \mu^\Gamma)$ for any outer action α of Γ on any unital Kirchberg algebra A.

Theorem

If the diagonal action $\mu^\Gamma \otimes \mu^\Gamma$ on $\mathcal{O}_\infty \otimes \mathcal{O}_\infty$ is approximately representable, then (A, α) is cocycle conjugate to $(A \otimes \mathcal{O}_\infty, \alpha \otimes \mu^\Gamma)$ for any outer action α of Γ on any unital Kirchberg algebra A.

In particular, if μ^Γ is approximately representable, the conclusion follows.

Remark

•Since $KK^1(\mathcal{O}_\infty, \mathcal{O}_\infty)$ is trivial, there exists only one outer conjugacy class of outer $\mathbb{Z}^2$-actions on $\mathcal{O}_\infty$.

• The facts that any rotation algebra embeds in $\mathcal{O}_\infty$, and that $\mathcal{O}_\infty$ is isomorphic to the infinite tensor product of itself imply that there exists only one cocycle conjugacy class as well.

• This shows that $\mu^{\mathbb{Z}^2}$ is asymptotically representable.

By an induction argument, we get

Theorem

$\mu^{\mathbb{Z}^n}$ is asymptotically representable for any n.
In consequence, any outer $\mathbb{Z}^n$-action α on a Kirchberg algebra A is cocycle conjugate to the diagonal action $\alpha \otimes \mu^{\mathbb{Z}^n}$ on $A \otimes \mathcal{O}_\infty$.

The proof of the theorem shows:
$\forall \omega \in Z^2(\mathbb{Z}^n, \mathbb{T})$, $\exists \theta \in \mathrm{Aut}(\mathcal{O}_\infty)$, $\exists u_g \in U(\mathcal{O}_\infty)$ s.t. $\theta^{-1} \circ \mu_g^{\mathbb{Z}^n} \circ \theta = \mathrm{Ad} u_g \circ \mu_g^{\mathbb{Z}^n}$,
$u_g \mu_g^{\mathbb{Z}^n}(u_h) = \omega(g,h) u_{gh}$.

Corollary

Let α and β be outer $\mathbb{Z}^n$-actions on a unital Kirchberg algebra A.
If α and β are outer conjugate, then they are cocycle conjugate.

Remark

Nakamura and Katsura-Matui show that for $\mathbb{Z}^2$-actions on UHF algebras, there is a gap between cocycle conjugacy and outer conjugacy.