Goldman Sachs Asset Management

# Optimization Models for Quantitative Asset Management[1]

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Goldman Sachs Asset Management
Quantitative Equity
Joint work with D. Jeria, GS

Fields Industrial Optimization Seminar
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[1] This material is provided for educational purposes only and should not be construed as investment advice or an offer or solicitation to buy or sell securities.

# Outline

1. Multi-Portfolio Optimization

2. Dynamic Portfolio Management

# Quantitative Portfolio Management

A quantitative portfolio manager seeks to find the optimal trade-off among three competing concerns:

- Maximize expected portfolio return
- Minimized portfolio risk (in absolute or relative terms)
- Minimize trading costs (t-costs, from now on)

Trading costs can be a significant part of a large manager's utility. Different approaches to managing trading costs carefully will be the main focus of this talk.

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# Generic Portfolio Optimization Problem

- Usual framework: $n$ securities, expected returns given by $\mu$ and covariance matrix $\Sigma$. A portfolio of the available securities is denoted by the vector $x = (x_1, x_2, \ldots, x_n)$.
- Let $x^0$ denote the initial portfolio and let $t = |x - x^0|$ denote the trade vector. Representing portfolio constraints in the generic form $x \in \mathcal{X}$, we can formulate a simple optimization problem:

$$\begin{array}{ll} \max & \mu^T x - \lambda x^T \Sigma x - \phi TC(t)^T t \\ \text{s.t.} & x \in \mathcal{X}. \end{array}$$

Above $\lambda$ and $\phi$ represent the risk and t-cost aversion respectively and $TC$ represents the unit t-cost function. This is one of the three alternative formulations of Markowitz' *mean-variance optimization (MVO)* problem.

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# Typical portfolio constraints

- Compliance and client constraints–e.g., a "restricted trade list"
- Exposure constraints–e.g., limits on active bets on securities, industries, sectors, etc.
- Trade constraints–e.g., limit trades to x% of the average daily volume (ADV)
- Cardinality constraints–e.g., limits on the number of trades or holdings
- Threshold constraints–e.g., do not hold a position smaller than x% of the portfolio
- Others–e.g., limit "distance" to a model portfolio

However, "Constraints ... accumulate like useless items in a closet until their cumulative effect is too large to ignore." (R. Grinold)

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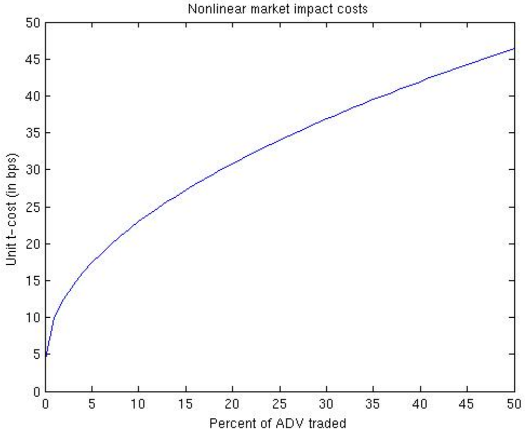
# Trading costs

Trading costs typically have two distinct components:

- Commissions and bid-ask spread (linear in trade size)
- Market impact (superlinear in trade size)



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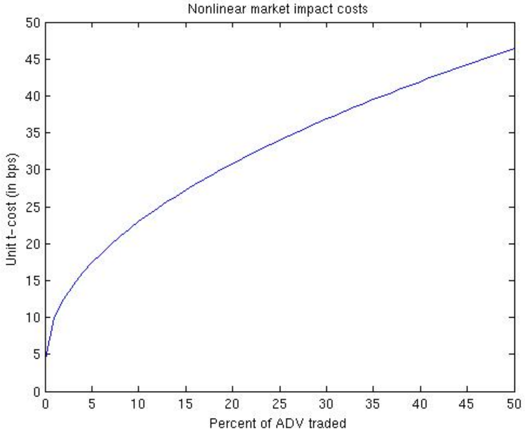

# 3/2-power market impact function

A conic representation for the convex non-linear market impact function improves solver performance. This requires a simple conversion:

$$\min \sum_{k=1}^{n} q_k t_k^{3/2} \equiv \begin{array}{llll} \min & \sum_{k=1}^{n} q_k u_k & & \\ \text{s.t.} & t_k^{3/2} & \leq & u_k, forallk. \end{array}$$

We now make the following simple observation:

$$t_k^{3/2} \leq u_k \Leftrightarrow \begin{array}{lll} \exists v_k \text{s.t.} & & \\ t_k^2 & \leq & u_k \cdot v_k, \\ v_k^2 & \leq & t_k \cdot 1. \end{array}$$

Both inequalities on the RHS are *rotated quadratic cone* inequalities. Hence the 3/2-power market impact function can be optimized using standard conic optimization software.

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# Separately Managed Accounts (SMAs)

- Most clients prefer to "own" an SMA rather than shares of a mutual fund
- This gives them flexibility to customize their portfolio according to their investment goals and concerns
- Larger asset managers manage hundreds of SMAs. As a result, on any given day, multiple accounts must be optimized/rebalanced.
- The complication arises from the fact trading these accounts together generates a nonlinear cumulative market impact.
- Accounts that are traded together can not be truly optimized in isolation. This also brings up issues of bias and fairness.

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# Optimizing Independently

For account $j$, we optimize

$$\begin{array}{ll} \max & (\mu^j)^T x^j - \lambda^j (x^j)^T \Sigma^j x^j - \phi^j TC(t^j)^T t^j \\ \text{s.t.} & x^j \in \mathcal{X}^j. \end{array}$$

However, the "true" objective value is

$$(\mu^j)^T x_*^j - \lambda^j (x_*^j)^T \Sigma^j x_*^j - \phi^j TC(\sum_i t_*^i)^T t_*^j.$$

So, the objective function above under-estimates the total market impact and results in too much trading. The effect can be severe when $t_*^j$ is much smaller than $\sum_i t_*^i$.

This approach also creates a size bias–smaller accounts are disadvantaged.

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# Collusive Approach

- The idea is to optimize all accounts jointly, using a *total welfare* objective function (O'Cinneide, Scherer, Xu, *JPM*, Summer 2006.)
- The optimization problem

$$\begin{array}{ll} \max & \sum_j (\mu^j)^T x^j - \sum_j \lambda^j (x^j)^T \Sigma^j x^j - \phi TC(\sum_j t^j)^T(\sum_j t^j) \\ \text{s.t.} & x^j \in \mathcal{X}^j,\ \forall j. \end{array}$$

- Stubbs (2007) shows that this approach is not "fair"–some accounts may have to sacrifice themselves for the benefit of the group. They can improve their utility by acting unilaterally.
- Theoretically, the unfairness issue can be overcome by "equitably distributing" the objective function improvements. However, this is practically impossible to implement.

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# An equilibrium approach

- When optimizing account $j$, assume that accounts $i \neq j$ will have trades $t_*^i$ and then optimize

$$\begin{array}{ll} \max & (\mu^j)^T x^j - \lambda^j (x^j)^T \Sigma^j x^j - \phi^j TC(t^j + \sum_{i \neq j} t_*^i)^T t^j \\ \text{s.t.} & x^j \in \mathcal{X}^j. \end{array}$$

  Let us call this problem CNP($j$).

- If there exists $t_*^j$, $j = 1, \ldots, n$ such that for each $j$, $t_*^j$ solves CNP($j$), then we have an equilibrium solution. This would be a fair solution in the sense that unilateral deviation from this solution would not benefit anybody.

- In other words, we are seeking a (Cournot-)Nash equilibrium point. It must exist because of the concavity of the objective functions. How do we find it?

- While the equilibrium solution is inferior to the collusive solution in terms of the total welfare function, it is easier to justify and implement.

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# Representing the shifted market impact function

When $TC(x) = x^{1/2}$, the market impact term in the objective function of CNP($j$) can be handled as follows. We want to:

$$\min \sum_{k=1}^{n} q_k t_k^j \cdot \sqrt{t_k^j + \sum_{i \neq j} (t_*^i)_k}$$

Let's simplify:

$$\min \sum_{k=1}^{n} t_k \sqrt{t_k + a_k} \equiv \begin{array}{lrcl} \min & \sum_{k=1}^{n} u_k & & \\ \text{s.t. } \forall k & t_k \sqrt{t_k + a_k} & \leq & u_k \end{array}$$

$$\equiv \begin{array}{ll} \min & \sum_{k=1}^{n} u_k \\ \text{s.t. } \forall k & (t_k + a_k)^{3/2} \\ & \leq u_k + a_k \sqrt{t_k + a_k} \end{array} \equiv \begin{array}{lrcl} \min & \sum_{k=1}^{n} u_k & & \\ \text{s.t. } \forall k & v_k^{3/2} & \leq & y_k \\ & v_k & = & t_k + a_k \\ & y_k & = & u_k + z_k \\ & z_k & \leq & \sqrt{t_k + a_k} \end{array}$$

All the inequalities can be written using rotated second order cone constraints.

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$$\equiv \begin{array}{ll} \min & \sum_{k=1}^{n} u_k \\ \text{s.t. } \forall k & (t_k + a_k)^{3/2} \\ & \leq u_k + a_k \sqrt{t_k + a_k} \end{array} \equiv \begin{array}{lrcl} \min & \sum_{k=1}^{n} u_k & & \\ \text{s.t. } \forall k & v_k^{3/2} & \leq & y_k \\ & v_k & = & t_k + a_k \\ & y_k & = & u_k + z_k \\ & z_k & \leq & \sqrt{t_k + a_k} \end{array}$$

All the inequalities can be written using rotated second order cone constraints.

# Solution Strategy

- A simple idea: Generate some initial trade estimates, solve each CNP($j$) with the corresponding estimates, update the estimates and iterate.
- Convergence can be difficult. An obvious problem is “zig-zagging”. Can be partly remedied by *fictitious play*:
- To generate the trade size estimate for iteration $k+1$, use a convex combination of the trade size estimate for iteration $k$ and the optimal trades computed for problem CNP($j$) in iteration $k$.

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# Other ideas

- Find "better" estimates of the equilibrium trades, e.g., from a combined account optimization. There are some difficulties with this approach–for example, different benchmarks, risk appetites, constraints among different accounts.
- Try an all-at-once approach instead of solving account-by-account and iterating. *Axioma* (Ceria, Stubbs, Schmieta, etc.) is working on this solution. This approach can easily handle *cumulative* constraints, e.g., do not trade more than x% of average daily volume in any name.
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# Challenges for the iterative approach

- Is the existence of equilibrium guaranteed given that there are non-convex constraints/costs in most problems?
- Also, can there be multiple equilibria? If so, how can we ensure we converge to the "best" one?
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# Outline

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# A Factor Model of Returns

Most quantitative portfolio construction approaches describe the return and risk characteristics of securities using factor models.

Asset and portfolio returns and risks can be decomposed into two parts: those which are due to factors prevalent throughout the market and those which are specific to asset or the securities in the porfolio. A multiple factor model tries to capture this decomposition. Its advantages are:

- A thorough breakdown of risk
- Incorporates economic logic
- Robust to outliers
- Adapts to macro movements
- Realistic, flexible, tractable and easy to understand

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# Multiple Factor Models

The decomposition of the return in asset i:

$$r_i(t) = \sum_k F_{i,k}(t) \cdot b_k(t) + u_i(t)$$

where

$$\begin{aligned}
r_i(t) &= \text{excess return of asset i in period t} \\
F_{i,k}(t) &= \text{exposure of asset i to factor k in period t} \\
b_k(t) &= \text{factor return in period t} \\
u_i(t) &= \text{specific return of asset i in period t.}
\end{aligned}$$

Matrix form:

$$\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} = \begin{bmatrix} f_{11} & \cdots & f_{1m} \\ f_{21} & \cdots & f_{2m} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nm} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

# Mean Reversion

Recall the factor decomposition of the excess returns:

$$r_t = F_t b_t + u_t$$

From this decomposition it follows that

$$\mu_t = \mathrm{E}[r_t] = F_t \mathrm{E}[b_t] + \mathrm{E}[u_t].$$

Model assumption: $\mathrm{E}[u_t] = 0$ and $\mathrm{E}[b_t]$ is stationary. As a result expected returns move with the movements in the factor exposures ($F$).

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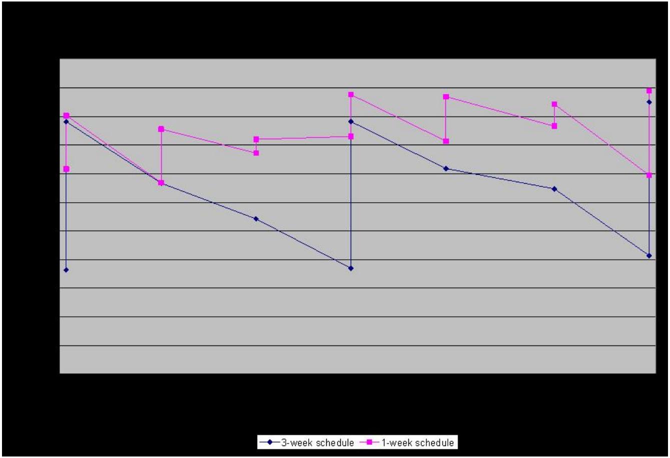
# Information Decay



# Dynamic Portfolio Selection

Consider the following infinite horizon utility maximization problem:

$$\max_{\{x_t\}_{t=0,\ldots}} \mathrm{E}\left[\sum_{t=0}^{\infty} \gamma^t u(\mu_t, x_t)\right]$$

where $\gamma$ is a discount factor, and $u(\cdot)$ is the utility function we have seen before:

$$u(\mu, x) = \mu^T x - \lambda x^T \Sigma x - \phi TC(t)^T t.$$

For simplicity, we assume $TC(t) = \Lambda t$ and ignore constraints for now.

Also assume the following mean-reverting model for expected returns:

$$\mu_t = (\mathbf{I} - \beta)\mu_{t-1} + \beta\bar{\mu} + \varepsilon_t$$

where $\beta$ is a diagonal matrix of mean reversion coefficients and $\bar{\mu}$ is the vector of average expected returns. $\varepsilon$ is white noise. $\Sigma$ and $\Lambda$ are time-invariant.

Different factors will have different mean reversion rates. This is important for trading costs.

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# Finite Horizon Formulation

The finite-horizon recursion for the problem is given by:

$$
\begin{aligned}
J_N(\mu_N, x_{N-1}) &= \max_{t_N} \quad (x_N'\mu_N - \lambda x_N'\boldsymbol{\Sigma} x_N - \phi t_N'\boldsymbol{\Lambda} t_N) \\
J_t(\mu_t, x_{t-1}) &= \max_{t_t} \quad \mathrm{E}\left[x_t'\mu_t - \lambda x_t'\boldsymbol{\Sigma} x_t - \phi t_t'\boldsymbol{\Lambda} t_t + \gamma J_{t+1}(\mu_{t+1}, x_t)\right]
\end{aligned}
$$

Period $N$ problem is a simple QP. Solving it, we obtain:

$$
t_N = \frac{1}{2}(\lambda\boldsymbol{\Sigma} + \phi\boldsymbol{\Lambda})^{-1}\mu_N - (\lambda\boldsymbol{\Sigma} + \phi\boldsymbol{\Lambda})^{-1}(\lambda\boldsymbol{\Sigma})x_{N-1}
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# The Value Function

Define the following matrices:

$$\begin{aligned}\mathbf{A}_N &= \tfrac{1}{2}\left(\lambda\boldsymbol{\Sigma}+\phi\boldsymbol{\Lambda}\right)^{-1}\\ \mathbf{B}_N &= \left(\lambda\boldsymbol{\Sigma}+\phi\boldsymbol{\Lambda}\right)^{-1}\left(\lambda\boldsymbol{\Sigma}\right)\end{aligned}$$

so that

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Then,

$$J_N\left(\mu_N, t_{N-1}\right) = \mu_N'\mathbf{M}_N\mu_N + x_{N-1}'\mathbf{N}_N x_{N-1} + x_{N-1}'\mathbf{P}_N\mu_N$$

where

$$\begin{aligned}\mathbf{M}_N &= \frac{1}{2}\mathbf{A}_N\\ \mathbf{N}_N &= -\phi\boldsymbol{\Lambda}\mathbf{B}_N\\ \mathbf{P}_N &= 2\phi\boldsymbol{\Lambda}\mathbf{A}_N\end{aligned}$$

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# Recursion

Given the shape of $J_N(\cdot)$, we hypothesize that $J_t(\cdot)$ will be quadratic and try to solve for the coefficients of this quadratic model. Indeed,

$$J_t(\cdot) = \mu_t' \mathbf{M}_t \mu_t + x_{t-1}' \mathbf{N}_t x_{t-1} + x_{t-1}' \mathbf{P}_t \mu_t + \mathbf{q}_t' \mu_t + \mathbf{r}_t' x_{t-1} + \mathrm{f}_t$$
$$\tau_t = \mathbf{A}_t \mu_t - \mathbf{B}_t x_{t-1} + \mathbf{c}_t$$

for certain parameters $A_t$, $B_t$, etc., derived from $\Sigma$, $\Lambda$, $\beta$, etc.

An infinite horizon extension is relatively straight-forward and requires the solution of a fixed-point problem.

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An infinite horizon extension is relatively straight-forward and requires the solution of a fixed-point problem.

# How do we use this for constrained problems?

We use our knowledge of $J(\cdot)$ to set up a quadratic program which will approximate the optimal solution to the constrained multi-period rebalancing problem. The QP is then given by:

$$\begin{aligned} \max_{t_t} \quad & \left\{ x_t' \mu_t - \lambda x_t' \Sigma x_t - \phi t_t' \Lambda t_t + \gamma \mathrm{E}\left[ J\left(\mu_{t-1}, x_{t-1} + t_t\right)\right] \right\} \\ \text{s.t.} \quad & x_t = x_{t-1} + t_t \in \mathcal{X} \end{aligned}$$

which is equivalent to

$$\begin{aligned} \max_{t_t} \quad & \Big\{ - t_t' \left[\lambda \Sigma + \phi \Lambda - \gamma \mathbf{N}\right] t_t + \Big[\mu_t + 2\left(\lambda \Sigma - \gamma \mathbf{N}\right) x_{t-1} + \\ & \qquad \gamma \mathbf{P}\left((\mathbf{I} - \beta)\mu_t + \beta \bar{\mu}\right) + \gamma \mathbf{r}\Big] t_t \Big\} \\ \text{s.t.} \quad & x_t = x_{t-1} + t_t \in \mathcal{X} \end{aligned}$$

# Alternatives

Can we construct one-period problems that partially capture the dynamics of the inputs?

For example, incorporate the decay rate into the expected returns used in optimizations.

One critical issue is the selection of the t-cost aversion parameter $\phi$. It needs to balance the expected return rates with the trading costs, so the aversion parameter must ensure that these terms are in the "same units". But this creates a chicken-and-egg problem.

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# "Consistent" t-cost aversion

$$u(\mu, x) = \mu^T x - \lambda x^T \Sigma x - \phi TC(t)^T t.$$

If $\mu$ is an annualized return estimate, and $TC(t)^T t$ is a one-time trading-cost, to bring these two terms to comparable units, $\phi$ must correspond to the expected number of trades per year for the securities in the portfolio.

The problem is, for lower $\phi$, we have higher turnover and higher expected number of trades per year–perhaps inconsistent with the $\phi$ we used. Similar problem for $\phi$ that is too high.

How do we find the "right" value of this parameter? Iterate to achieve "consistency"...

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# Recap

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- Trading multiple accounts simultaneously poses a difficult question of balancing optimality and fairness. An equilibrium approach seems best suited for this situation. Conic optimization tools are essential.
- Dynamic programming and optimal control techniques are useful in addressing the multi-period portfolio selection models. However, computational burden is still too high for a "true" solution of this problem. Instead, we focus on informed heuristics.

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