Completely 1-summing maps between certain homogeneous Hilbertian operator spaces

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Joint work with Marius Junge

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Preliminaries on operator spaces

An operator space E is called

- homogeneous or λ-homogeneous if $\exists\lambda$ s.t. every bounded map u on E is c.b. and $\|u\|_{cb} \le \lambda\|u\|$;
- Hilbertian or λ-Hilbertian if E is isomorphic or λ-isomorphic to a Hilbert space (at the Banach space level).

Vector-valued Schatten classes (Pisier):

- S_p = Schatten p-class; $S_\infty = B(\ell_2)$
- $S_\infty[E] = S_\infty \otimes_{\min} E$;
- $S_1[E] = S_1 \widehat{\otimes} E$, $\widehat{\otimes}$ = the projective tensor product;

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Definition (Effros-Ruan)

Let $u : E \to F$ be a map between two operator spaces. u is called completely 1-summing if the map $\mathrm{id} \otimes u$ is bounded from $S_1 \otimes_{\min} E$ to $S_1[F]$. Then define

$$\pi_1^o(u) = \|\mathrm{id} \otimes u \,:\, S_1 \otimes_{\min} E \to S_1[F]\|$$

and

$$\Pi_1^o(E, F) = \{u : E \to F \text{ completely 1-summing}\}.$$

Remark

$\Pi_1^o(E, F)$ is an ideal in the following sense:

$$v \in CB(E_1, E),\ u \in \Pi_1^o(E, F),\ w \in CB(F, F_1) \Rightarrow$$
$$wuv \in \Pi_1^o(E_1, F_1),\ \pi_1^o(wuv) \le \|w\|_{cb}\, \pi_1^o(u)\, \|v\|_{cb}\,.$$

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Column and row spaces

C = column space; R = row space. Given a Hilbert space H let

$$H^c = B(\mathbb{C}, H) \quad \text{and} \quad H^r = B(\overline{H}, \mathbb{C}).$$

Definition

C_p = first column of S_p; R_p = first row of S_p, $1 \le p \le \infty$.

$C_\infty = C$ and $R_\infty = R$; $C_2 = R_2 = OH$.

Elementary properties

- C_p and R_p are 1-homogeneous and 1-Hilbertian. Their canonical bases will be identified with that of ℓ_2: $e_{k1} \sim e_{1k} \sim e_k$.
- $C_p^* = C_{p'} = R_p$ completely isometrically.
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Direct sum and sum

Let E and F be two operator spaces. $E \oplus_p F$ denotes the direct sum of E and F in the ℓ_p-sense.

Remark

For any $1 \leq p, q \leq \infty$

$$E \oplus_p F \simeq E \oplus_q F.$$

This allows us to use $E \oplus F$ to denote $E \oplus_p F$ for any p.

Let (E, F) be a compatible couple (i.e. $E, F \hookrightarrow \mathcal{V}$). Put

$$E \cap F = \{x : x \in E, x \in F\}, \; E + F = \{x + y : x \in E, y \in F\}.$$

We view $E \cap F$ as the diagonal subspace of $E \oplus F$ and $E + F$ as the quotient of $E \oplus F$ by the subspace $\{(x, y) \; : \; x + y = 0\}$.

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Quotients of subspaces of $C \oplus R$

Notation

$QS(E)$ denotes the family of all quotients of subspaces of E.

Properties of $QS(C \oplus R)$

- $QS(C \oplus R)$ is stable under duality.
- $OH \in QS(C \oplus R)$ (Pisier's exercise).
- $C_p \in QS(C \oplus R)$ for any p.
- Any space in $QS(C \oplus R)$ completely embeds into a noncommutative L_1 with universal constant.
- In particular, OH completely embeds into a noncommutative L_1 (Junge's embedding theorem).

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Main concern

Study various properties of homogeneous spaces in $QS(C \oplus R)$:

- their representation;
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Weighted L_2-spaces

(Ω, ν) = a measure space; σ = a weight on Ω. The norm of $L_2(\Omega, \sigma)$ is given by

$$\|f\|_{L_2(\Omega,\sigma)} = \Big(\int_\Omega |f|^2 \sigma^2 d\nu\Big)^{1/2}.$$

Similarly, we have the ℓ_2-valued weighted $L_2(\ell_2; \Omega, \sigma)$.

Let σ and μ be two weights on Ω the following weight condition

$$\int_\Omega \min(\sigma^2,\ \mu^2)\, d\nu < \infty.$$

Then $(L_2(\ell_2; \Omega, \sigma),\ L_2(\ell_2; \Omega, \mu))$ is a compatible couple.

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The spaces $K_{\sigma,\,\mu}$

Let

$$G_{\sigma,\,\mu} = L_2(\ell_2;\Omega,\sigma)^c + L_2(\ell_2;\Omega,\mu)^r\,.$$

Recall

$$G_{\sigma,\,\mu} = L_2(\ell_2;\Omega,\sigma)^c \oplus L_2(\ell_2;\Omega,\mu)^r/\{(a,b)\ :\ a+b=0\ a.e.\}\,.$$

Define $K_{\sigma,\,\mu}$ to be the subspace of constant functions of $G_{\sigma,\,\mu}$.
The o.s.s. of $K_{\sigma,\,\mu}$: for any finite sequence $(x_k)\subset S_\infty$

$$\big\|\sum_k x_k\otimes e_k\big\|_{S_\infty[K_{\sigma,\,\mu}]}$$

$$=\inf_{x_k=a_k+b_k\ a.e.}\Big\{\big\|\sum_k\int_\Omega a_k^*a_k\,\sigma^2 d\nu\big\|^{\frac12}+\big\|\sum_k\int_\Omega b_kb_k^*\,\mu^2 d\nu\big\|^{\frac12}\Big\},$$

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$$= \inf_{x_k = a_k + b_k\ a.e.} \Big\{\big\|\sum_k \int_\Omega a_k^* a_k\, \sigma^2 d\nu\big\|^{\frac12} + \big\|\sum_k \int_\Omega b_k b_k^*\, \mu^2 d\nu\big\|^{\frac12}\Big\},$$

$K_{\sigma,\mu}$ is a homogeneous Hilbertian space in $QS(C \oplus R)$.

The spaces $K_{\sigma,\mu}$

Let

$$G_{\sigma,\mu} = L_2(\ell_2;\Omega,\sigma)^c + L_2(\ell_2;\Omega,\mu)^r\,.$$

Recall

$$G_{\sigma,\mu} = L_2(\ell_2;\Omega,\sigma)^c \oplus L_2(\ell_2;\Omega,\mu)^r/\{(a,b)\;:\; a+b=0\ a.e.\}\,.$$

Define $K_{\sigma,\mu}$ to be the subspace of constant functions of $G_{\sigma,\mu}$.
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Representations of homogeneous spaces in $QS(C \oplus R)$

Theorem

Let $F \in QS(C \oplus R)$ be homogeneous. Then $\exists\ \alpha, \beta \in [0,\ 1]$ and $\exists\ \sigma = (\sigma(j))_{j\geq 1},\ \mu = (\mu(j))_{j\geq 1} \subset [0,\ 1]$ s.t. σ and μ satisfy the weight condition:

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and s.t. F is completely isomorphic to $\alpha C \cap \beta R \cap K_{\sigma,\,\mu}$.

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Idea of proof

- A main ingredient: Let $S \subset C \oplus R$. Then $\exists\, n_0, n_1, n \in \mathbb{N} \cup \{\infty\}$ s.t.

$$S = C^{n_0} \oplus R^{n_1} \oplus \Gamma(T),$$

where $T : \ell^n \to \ell^n$ is an injective positive operator on ℓ^n.

- Use of a completely symmetric basis of F.
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Concrete representations of C_p

The previous representation for homogeneous spaces in $QS(C \oplus R)$ is not explicit enough to do concrete calculations in some specific situations since we don't know any precise information on the two weights σ and μ.

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Let $\mathbb{R}_+$ be quipped with the measure dt/t. For $\alpha \in \mathbb{R}$ let $L_2(\ell_2; t^\alpha)$ be the weighted L_2 with weight $w_\alpha(t) = t^\alpha$.
For $1 < p < \infty$ and $\theta = 1/p$ let

$$G_\theta = L_2(\ell_2; t^{-\theta})^c + L_2(\ell_2; t^{1-\theta})^r$$

and let K_θ be the constant function subspace of G_θ.

Theorem

Let $1 < p < \infty$ and $\theta = 1/p$. Then $C_p = K_\theta$ completely isomorphically with universal constants.

Tool of the proof

Real interpolation.

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Recall

$CR_p = C_p + R_p$ for $1 \leq p < 2$; $CR_p = C_p \cap R_p$ for $2 \leq p \leq \infty$.

The previous concrete representation of C_p implies a similar one of CR_p. Let

$$\begin{aligned} v_\theta(t) &= \min(t^{-\theta},\ t^{\theta-1}), \qquad w_\theta(t) = \min(t^{1-\theta},\ t^{\theta}),\\ V_\theta(t) &= \max(t^{-\theta},\ t^{\theta-1}), \qquad W_\theta(t) = \max(t^{1-\theta},\ t^{\theta}). \end{aligned}$$

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Let $1 < p < \infty$ and $\theta = 1/p$. Then $CR_p = K_{v_\theta, w_\theta}$ for $p < 2$ and $CR_p = K_{V_\theta, W_\theta}$ for $p \geq 2$ completely isomorphically with universal constants.

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Orlicz spaces

φ = an orlicz function on $[0, \infty)$: φ is convex and $\varphi(0) = 0$.
The Orlicz space ℓ_φ consists of sequences $x = (x_n)$ s.t.

$$\|x\|_\varphi = \inf\{\lambda > 0 \ : \ \sum_n \varphi(\frac{|x_n|}{\lambda}) < \infty\} < \infty.$$

Remark

- ℓ_φ depends, up to isomorphism, only on the values of φ in a neighborhood of 0.
- The finite sequences are dense in ℓ_φ iff φ satisfies the Δ_2-condition: $\varphi(2t) \le \lambda\,\varphi(t)$.
- Under the Δ_2-condition ℓ_φ is uniquely determined, up to isomorphism, by its fundamental sequence:

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Schatten-Orlicz classes

Let x be a compact operator on ℓ_2. Let $(s_n(x))_{n\geq 1}$ denote the sequence of singular values of x, i.e. $(s_n(x))_{n\geq 1}$ is the sequence of the eigenvalues of $|x|$ ranged in decreasing order and repeated according to multiplicity.

Definition

Let φ be an Orlicz function. Define

$$\begin{aligned} S_\varphi &= \{x \,:\, x \in B(\ell_2) \text{ compact s.t. } (s_n(x)) \in \ell_\varphi\} \\ \|x\|_\varphi &= \|(s_n(x))\|_{\ell_\varphi}\,. \end{aligned}$$

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The main novelty

Theorem

Let $E, F \in QS(C \oplus R)$ be homogenous.

- $\Pi_1^o(E, F) \subset S_2$.
- There exists an Orlicz function φ satisfying the Δ_2-condition s.t. $\Pi_1^o(E, F) = S_\varphi$ isomorphically.
- All relevant constants depend only on the homogeneity constants of E and F.

Remark

Since E and F are Hilbertian, we can choose, without loss of generality, two orthonormal bases (e_k) in E and (f_k) in F. Then by homogeneity we need only to consider diagonal operators from E to F: $u(e_k) = \lambda_k f_k$.

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- The inclusion $\Pi_1^o(E, F) \subset S_2$ is easy. Since

$$C \cap R \subset E, F \subset C + R,$$

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- First ingredient: the previous representation theorem.
- Second ingredient: the complete embedding of spaces in $QS(C \oplus R)$ into noncommutative $L_1(\mathcal{M})$. Moreover, $\mathcal{M}$ can be chosen QWEP or even injective.
- A lemma of Pisier. Let $E^*, F \subset S_1$ be two f.d. subspaces. Let $E^* \otimes_1 F \subset S_1 \widehat{\otimes} S_1 = S_1(\ell_2(\mathbb{N}^2))$. Identifying a map $u : E \to F$ with its element $\widetilde{u} \in E^* \otimes F$ we have

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$$\Pi_1^o(E,\,F) \simeq E^* \otimes_1 F \quad \text{iometrically.}$$

Idea of proof

- The inclusion $\Pi_1^o(E,\,F) \subset S_2$ is easy. Since

$$C \cap R \subset E, F \subset C + R,$$

we have

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Let $E, F \in QS(C \oplus R)$ be homogeneous. Fix orthonormal bases (e_k) of E and (f_k) of F. Define $\mathrm{id}_n : E \to F$ by $\mathrm{id}_n(e_k) = f_k$ for $k \le n$ and $\mathrm{id}_n(e_k) = 0$ for $k > n$.

Let φ be s.t. $\Pi_1^o(E, F) = S_\varphi$. Then the fundamental sequence of φ is equivalent to $(\pi_1^o(\mathrm{id}_n))$.

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The determination of the whole space $\Pi_1^o(E, F)$ is reduced to that of the sequence $(\pi_1^o(\mathrm{id}_n))$.

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A formula for $\pi_1^o(\mathrm{id}_n)$

Let $(\gamma,\ \delta)$ and $(\sigma,\ \mu)$ be two pairs of weights on (Ω, ν) satisfying the weight condition. Let $E^* = K_{\gamma,\,\delta}$ and $F = K_{\sigma,\,\mu}$. Let $\otimes_\pi$ denote the Banach space projective tensor product. Set

$$\begin{aligned} A &= L_2(\Omega^2, \min(\gamma\otimes\sigma,\ \delta\otimes\mu)),\\ B &= L_2(\Omega, \min(\gamma,\ \delta))\otimes_\pi L_2(\Omega, \min(\sigma,\ \mu)), \end{aligned}$$

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$$\begin{aligned} \pi_1^o(\mathrm{id}_n) &\sim \inf\{\sqrt{n}\|a\|_A + n\|b\|_B : a\in A, b\in B, a+b=1\ a.e.\}\\ &\sim \sup\{\Big|\int f(\omega_1,\omega_2)d\nu(\omega_1)d\nu(\omega_2)\Big| :\\ &\qquad f\in A^*\cap B^*, \|f\|_{A^*}\le\sqrt{n}, \|f\|_{B^*}\le n\}. \end{aligned}$$

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In general it is very hard or even impossible to estimate $\pi_1^o(\mathrm{id}_n)$ explicitly. But using the concrete representation of C_p, we can do this for $E = C_p$ and $F = C_q$. This is our next task.

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A special Orlicz function

Let

$$\psi(t) = t^2 \log(t + 1/t), \quad t > 0.$$

ψ satisfies the Δ_2-condition. The associated Orlicz space ℓ_ψ is traditionally denoted by $\ell^2 \log \ell$.
Note that

$$\psi^{-1}(t) \sim \sqrt{2t}\,\big(\log \frac{1}{t}\big)^{-1/2} \quad \text{as} \quad t \to 0.$$

Thus the fundamental sequence (ψ_n) is given by

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Completely 1-summing maps from C_p to C_q

Theorem

- Let $1 \le p, q \le \infty$ s.t. $q \neq p'$ in the case $1 < p < \infty$. Let r be determined by $2/r = 1/p + 1/q$. Then
$$\Pi_1^o(C_p\,,\ C_q) = S_r \cap S_{r'}\ .$$
- Let $1 < p < \infty$. Then
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Sketch of proof

Step 1: $\Pi_1^o(C_p\,,\ C_q) \subset S_r \cap S_{r'}$. Consider the diagram

$$\begin{array}{ccc} C_p & \xrightarrow{u} & \mathcal{C}_q \\ {\scriptstyle v}\uparrow & & \downarrow{\scriptstyle w} \\ C & \xrightarrow{wuv} & C \end{array}$$

Then $\pi_1^o(wuv) \le \|w\|_{cb}\, \pi_1^o(u)\, \|v\|_{cb}$. However

$$CB(C,\ C_p) = S_{2p}\,, \quad CB(C_q,\ C) = S_{2q}\,, \quad \Pi_1^o(C,\ C) = S_1\ .$$

It follows that

$$\|wuv\|_1 \le \|w\|_{2q}\, \pi_1^o(u)\, \|v\|_{2p}\,;$$

whence $\|u\|_{r'} \le \pi_1^o(u)$. This shows the assertion in the case $p' < q$. The case $p' > q$ is dealt with by considering R_p instead of C_p.

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Step 2: Let $1 < p, q < \infty$. Then

$$\pi_1^o(\mathrm{id}_n : C_p^n \to C_q^n) \le c_{p,q} \max\left(n^{\frac{1}{r}}, n^{\frac{1}{r'}}, (n\log(n+1))^{\frac{1}{2}}\right).$$

Let $\theta = 1/p'$ and $\eta = 1/q$. Then

$$\pi_1^o(\mathrm{id}_n : C_p^n \to C_q^n) \sim_c \inf_{a+b=1\ a.e.} \left(\sqrt{n}\,\|a\|_A + n\|b\|_B\right),$$

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Let $0 < s_0 < 1 < s_1$ and $0 < t_0 < 1 < t_1$. Consider

$$b = 1\!\!1_{(0,\,s_0)} \otimes 1\!\!1_{(t_1,\,\infty)} + 1\!\!1_{(s_1,\,\infty)} \otimes 1\!\!1_{(0,\,t_0)}, \qquad a = 1 - b.$$

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This is done by duality by considering the following function (with $0 < \varepsilon < 1$)

$$f(s,t) = \min((st)^{-2\theta}\,,\ (st)^{2(1-\theta)})\mathbb{1}_{[\varepsilon,\,1]}(s)\,\mathbb{1}_{[1,\,\varepsilon^{-1}]}(t)\,.$$

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$$\begin{aligned} \|f\|_{A^*}^2 &= \int_0^\infty \int_0^\infty f(s,t)\,\frac{dt}{t}\,\frac{ds}{s} \sim_{c_p} \log\frac{1}{\varepsilon} \\ \|f\|_{B^*} &\leq c_p \varepsilon^{-1}\,. \end{aligned}$$

This allows us to conclude the proof.

Sketch of proof

Step 3: For $1 < p < \infty$

$$\pi_1^o(\mathrm{id}_n : C_p^n \to C_{p'}^n) \geq c_p \sqrt{n \log(n+1)}\,.$$

This is done by duality by considering the following function (with $0 < \varepsilon < 1$)

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Completely 1-summing maps from CR_p to CR_q

Recall

$$CR_p = C_p + R_p \ \text{ for } \ p < 2; \qquad CR_p = C_p \cap R_p \ \text{ for } \ p \geq 2.$$

Using similar arguments we get the following

Theorem

- Let $1 \leq p, q \leq \infty$ s.t. $p \neq q$ when $1 < p < \infty$. Then

$$\Pi_1^o(CR_p, CR_q) = S_2 \cap S_r\,,$$

where r is determined by $2/r = 1/p + 1/q'$.

- Let $1 < p < \infty$. Then

$$\Pi_1^o(CR_p, CR_p) = S_\psi\,.$$

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Injectivity constants

E is called injective if id_E factors as $E \xrightarrow{u} B(H) \xrightarrow{v} E$ by c.b. maps u and v. The injectivity constant of E is defined to be

$$\gamma_\infty(E) = \inf_{vu=\mathrm{id}} \|v\|_{cb} \|u\|_{cb}\,.$$

Theorem

Let $1 < p < \infty$, $p \neq 2$. Then

$$\gamma_\infty(C_p^n) \sim_{c(p)} n^{\frac{1}{\max(p,\,p')}} \quad \text{and} \quad \gamma_\infty(CR_p^n) \sim_{c(p)} \frac{\sqrt{n}}{\sqrt{\log(n+1)}}\,.$$

Proof. Use the elementary fact: Let E be an n-dimensional, λ-homogeneous and μ-Hilbertian. Then

$$n \le \pi_1^o(\mathrm{id}_E)\,\gamma_\infty(E) \le \lambda\mu n.$$

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Exactness constants

Let $\dim E < \infty$. The exactness constant of E is

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The proof uses the representation theorem of C_p and

Lemma

Let $E \subset L_1(\mathcal{M})$ f.d. Then

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