Reiter's property (P_1) for locally compact quantum groups

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Amenability for locally compact groups

$G :=$ a locally compact group

$L_x f :=$ left translate of f by x: $(L_x f)(y) = f(xy)$

Definition. G is *amenable* if there is $M \in L^\infty(G)^*$ with $\|M\| = \langle 1, M \rangle = 1$ and

$$\langle \phi, M \rangle = \langle L_x \phi, M \rangle$$

for $x \in G$, $\phi \in L^\infty(G)$.

Proposition. *G is amenable if and only if there is a net $(m_\alpha)_\alpha$ of non-negative, norm one functions in $L^1(G)$ such that*

$$\|L_{x^{-1}} m_\alpha - m_\alpha\|_1 \to 0$$

for $x \in G$.

Reiter's properties (P_p)

Definition. G is said to have *Reiter's property* (P_p) for $p \in [1, \infty)$ if there is a net $(m_\alpha)_\alpha$ of non-negative, norm one functions in $L^p(G)$ such that

$$\sup_{x \in K} \|L_{x^{-1}} m_\alpha - m_\alpha\|_p \to 0$$

for all compact $K \subset G$.

Reiter's Theorem. *The following are equivalent for G:*

(i) *G is amenable;*

(ii) *G has Reiter's property (P_1);*

(iii) *G has Reiter's property (P_p) for all $p \in [1, \infty)$;*

(iv) *G has Reiter's property (P_p) for some $p \in [1, \infty)$.*

Leptin's Theorem via (P_2)

$$A(G) := \text{the Fourier algebra of } G$$

$$B(G) := \text{the Fourier–Stieltjes algebra of } G$$

Leptin's Theorem.

G is amenable $\iff$ $A(G)$ has a (norm one) BAI.

Theorem. [Granirer–Leinert, 81] *The topology of uniform convergence on compacta and the multiplier topology of $A(G)$ coincide on the unit sphere of $B(G)$.*

Proof of $\Longrightarrow$ of Leptin's Theorem.

$$G \text{ amenable} \quad \Longrightarrow \quad G \text{ has } (P_2)$$

Let (ξ_α) be a net in $L^2(G)$ as required by (P_2), and define

$$e_\alpha := \xi_\alpha * \check{\xi}_\alpha \in A(G).$$

Then

$$\sup_{x \in K} |e_\alpha(x) - 1| \to 0$$

for a compact $K \subset G$. By Granirer–Leinert, $(e_\alpha)_\alpha$ is a BAI for $A(G)$. $\square$

Hopf–von Neumann algebras, I

$\bar{\otimes}$ = the von Neumann algebra tensor product

Definition 1. A *Hopf–von Neumann algebra* is a pair (M, Γ), where M is a von Neumann algebra, and Γ is a *co-multiplication*: a unital, injective, normal *-homomorphism $\Gamma : M \to M \bar{\otimes} M$ which is co-associative, i.e.,

$$(\mathrm{id} \otimes \Gamma) \circ \Gamma = (\Gamma \otimes \mathrm{id}) \circ \Gamma.$$

Example 1. $M = L^\infty(G)$:

$$\Gamma_G \colon L^\infty(G) \to L^\infty(G \times G) \cong L^\infty(G) \bar{\otimes} L^\infty(G)$$

with

$$(\Gamma_G \phi)(x, y) := \phi(xy) \qquad (\phi \in L^\infty(G),\ x, y \in G).$$

Example 2. $M = \mathrm{VN}(G)$:

$$\hat{\Gamma}_G \colon \mathrm{VN}(G) \to \mathrm{VN}(G) \bar{\otimes} \mathrm{VN}(G),$$
$$\lambda(x) \mapsto \lambda(x) \otimes \lambda(x).$$

Hopf–von Neumann algebras, II

For any Hopf–von Neumann algebra (M, Γ):

M_* is a completely contractive Banach algebra

via

$$\langle f * g, x \rangle := \langle f \otimes g, \Gamma x \rangle \text{ for } f, g \in M_*,\ x \in M$$

Example 3. $(L^\infty(G), \Gamma_G)$: $L^1(G)$ with convolution.

Example 4. $(\mathrm{VN}(G), \hat{\Gamma}_G)$: $A(G)$ with pointwise multiplication.

Haar weights

Definition 2. Let (M, Γ) be a Hopf–von Neumann algebra.

(a) a *left Haar weight* is a n.f.s. weight ϕ on M such that

$$\phi((f \otimes \mathrm{id})(\Gamma x)) = \langle f, 1 \rangle \phi(x)$$

for $f \in M_*$ and $x \in \mathcal{M}_\phi$.

(b) a *right Haar weight* is a n.f.s. weight ϕ on M such that

$$\phi((\mathrm{id} \otimes f)(\Gamma x)) = \langle f, 1 \rangle \phi(x)$$

for $f \in M_*$ and $x \in \mathcal{M}_\phi$.

Locally compact quantum groups

Definition 3. [Kustermans–Vaes, 03] A *locally compact quantum group* (LCQG) is a quadruple $\mathbb{G} = (M, \Gamma, \phi, \psi)$ where M is a Hopf–von Neumann algebra, ϕ is a left Haar weight, and ψ is a right Haar weight.

Example 5. $(L^\infty(G), \Gamma_G, \phi, \psi)$ with

$$\phi = \text{left Haar measure}$$
$$\psi = \text{right Haar measure.}$$

Example 6. $(\mathrm{VN}(G), \hat{\Gamma}, \hat{\phi}, \hat{\psi})$ with

$$\hat{\phi} = \hat{\psi} = \text{the Plancherel weight on } \mathrm{VN}(G).$$

Example 7. Woronowicz' $\mathrm{SU}_q(2)$.

The multiplicative unitary

Theorem 1. *Let* $\mathbb{G} = (M, \Gamma, \phi, \psi)$ *be a LCQG, let* H_ϕ *denote the Hilbert space obtained from* ϕ *through the GNS construction, and let* $\Lambda_\phi : \mathcal{N}_\phi \to H_\phi$ *be the GNS map. Then there is a unique unitary* $W \in \mathcal{B}(H_\phi \otimes_2 H_\phi)$*—the* multiplicative unitary *of* $\mathbb{G}$*—defined by*

$$W^*(\Lambda_\phi(x) \otimes \Lambda_\phi(y)) = (\Lambda_\phi \otimes \Lambda_\phi)((\Gamma y)(x \otimes 1))$$

for $x, y \in \mathcal{N}_\phi$*. It satisfies the* pentagonal relation

$$W_{2,3}W_{1,2} = W_{1,2}W_{1,3}W_{2,3},$$

and we have

$$\Gamma x = W^*(1 \otimes x)W$$

for $x \in M$*.*

Some notation (according to Ruan)

For any LCQG $\mathbb{G}$:

$$L^\infty(\mathbb{G}) := M$$
$$L^1(\mathbb{G}) := M_*$$
$$L^2(\mathbb{G}) := \mathfrak{H}_\phi$$
$$\mathcal{C}_0(\mathbb{G}) := \overline{\{(\mathrm{id}\otimes\omega)(W) : \omega \in \mathcal{B}(L^2(\mathbb{G}))_*\}}^{\|\cdot\|}$$
$$M(\mathbb{G}) := \mathcal{C}_0(\mathbb{G})^*$$

Example 8. For $\mathbb{G} = G = (L^\infty(G), \Gamma_G, \phi, \psi)$, these are the usual objects.

Example 9. For $\mathbb{G} = \hat{G} = (\mathrm{VN}(G), \hat{\Gamma}_G, \hat{\phi}, \hat{\psi})$:

$$L^\infty(\hat{G}) = \mathrm{VN}(G), \quad L^1(\hat{G}) = A(G),$$
$$\mathcal{C}_0(\hat{G}) = C_r^*(G), \quad M(\hat{G}) = B_r(G).$$

Duality

For every LCQG $\mathbb{G}$, there is a unique LCQG $\hat{\mathbb{G}}$ such that $\hat{\hat{\mathbb{G}}} = \mathbb{G}$

Example 10. If

$$G = (L^\infty(G), \Gamma_G, \phi, \psi),$$

then

$$\hat{G} = (\mathrm{VN}(G), \hat{\Gamma}_G, \hat{\phi}, \hat{\psi}).$$

Definition 4. $\mathbb{G}$ is

(a) *compact* if $\phi \in L^1(\mathbb{G})$ and

(b) *discrete* if $L^1(\mathbb{G})$ is unital.

Theorem 2.

$$\mathbb{G} \textit{ compact} \iff \hat{\mathbb{G}} \textit{ discrete}.$$

Amenability and co-amenability, I

Definition 5. A LCQG $\mathbb{G}$ is called

(a) *amenable* if there is a state $M \in L^\infty(\mathbb{G})^*$ such that

$$\langle (f \otimes \mathrm{id})(\Gamma x), M \rangle = \langle f, 1 \rangle \langle x, M \rangle$$

for $f \in L^1(\mathbb{G})$ and $x \in L^\infty(\mathbb{G})$, or, equivalently, if there is a net $(m_\alpha)_\alpha$ of states in $L^1(\mathbb{G})$ such that

$$\| f * m_\alpha - \langle 1, f \rangle m_\alpha \|_{L^1(\mathbb{G})} \to 0$$

for $f \in L^1(\mathbb{G})$;

(b) *co-amenable* if $L^1(\mathbb{G})$ has a BAI (consisting of states).

Leptin's Theorem.

$$G \textit{ amenable} \iff \hat{G} \textit{ co-amenable.}$$

Conjecture.

$$\mathbb{G} \text{ amenable} \iff \hat{\mathbb{G}} \text{ co-amenable.}$$

Amenability and co-amenability, II

Theorem 3. [Voiculescu, 79; Bedos–Tuset, 03]

$$\mathbb{G} \text{ co-amenable} \quad \Longrightarrow \quad \hat{\mathbb{G}} \text{ amenable.}$$

Theorem 4. [Ruan, 96; Tomatsu, 06] *For discrete* $\mathbb{G}$*:*

$$\mathbb{G} \text{ amenable} \quad \Longleftrightarrow \quad \hat{\mathbb{G}} \text{ co-amenable.}$$

Another look at (P_1)

For $g \in L^1(G)$, let

$$L_\bullet g\colon G \to L^1(G), \quad x \mapsto L_{x^{-1}} g.$$

Then:

(a) $L_\bullet g \in \mathcal{C}_b(G, L^1(G))$ and

(b) $f L_\bullet g \in \mathcal{C}_0(G, L^1(G)) = \mathcal{C}_0(G) \otimes^\lambda L^1(G)$ for $f \in \mathcal{C}_0(G)$.

Proposition 1. *G has property (P_1) if and only if there is a net $(m_\alpha)_\alpha$ of non-negative, norm one functions in $L^1(G)$ such that*

$$\|f L_\bullet m_\alpha - f \otimes m_\alpha\|_{\mathcal{C}_0(G) \otimes^\lambda L^1(G)} \to 0$$

for $f \in \mathcal{C}_0(G)$.

(P_1) for LCQGs, I

For $g \in L^1(\mathbb{G})$, let

$$(\Gamma|g)\colon L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}), \quad x \mapsto (\mathrm{id} \otimes g)(\Gamma x).$$

Then $(\Gamma|g) \in \mathcal{CB}(L^\infty(\mathbb{G}))$.

For $a, b \in L^\infty(\mathbb{G})$, let

$$M_{a,b}\colon L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}), \quad x \mapsto axb.$$

So, $M_{a,b} \circ (\Gamma|g) \in \mathcal{CB}(L^\infty(\mathbb{G}))$.

Example 11. For $g \in L^1(G)$, $a, b \in \mathcal{C}_0(G)$:

$$M_{a,b} \circ (\Gamma|g) = abL_\bullet g.$$

(P_1) for LCQGs, II

Proposition 2. *Let* $g \in L^1(\mathbb{G})$, $a, b \in \mathcal{C}_0(\mathbb{G})$. *Then* $M_{a,b} \circ (\Gamma | g) \in \mathcal{CB}(L^\infty(\mathbb{G}), \mathcal{C}_0(\mathbb{G}))$ *lies the norm closure of the finite rank operators in* $\mathcal{CB}(L^\infty(\mathbb{G}))$ *and can be identified with an element of* $\mathcal{C}_0(\mathbb{G}) \check{\otimes} L^1(\mathbb{G})$.

Key
$W \in \mathcal{M}(\mathcal{C}_0(\mathbb{G}) \check{\otimes} \mathcal{K}(L^2(\mathbb{G})))$

Definition 6. $\mathbb{G}$ *is said to have* Reiter's property (P_1) *if there is a net* $(m_\alpha)_\alpha$ *of states in* $L^1(\mathbb{G})$ *such that*

$$\|M_{a,b} \circ (\Gamma | m_\alpha) - ab \otimes m_\alpha\|_{\mathcal{C}_0(\mathbb{G}) \check{\otimes} L^1(\mathbb{G})} \to 0$$

for all $a, b \in \mathcal{C}_0(\mathbb{G})$.

(P_1) for LCQGs, III

Theorem 5.

$$\mathbb{G} \textit{ is amenable} \quad \Longleftrightarrow \quad \mathbb{G} \textit{ has } (P_1)$$

Lemma 1. *Let* $g \in L^1(\mathbb{G})$*,* $a, b \in \mathcal{C}_0(\mathbb{G})$*. Then* $\{b\mu a * g : \mu \in M(\mathbb{G}), \|\mu\| \leq 1\}$ *is relatively norm compact in* $L^1(\mathbb{G})$*.*

What is (P_2) for a LCQG, and is it equivalent to (P_1)?