Mixed-Norm Inequalities and a Transference Method

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Joint work with Marius Junge

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A variation of Rosenthal's inequality

- Let $2 \leq s < \infty$ and $g_1, g_2, \ldots \in L_s(\Omega)$ independent

$$\Big(\int_\Omega \Big[\sum_{k=1}^n |g_k|^2 \Big]^{\frac{s}{2}} d\mu \Big)^{\frac{1}{s}} \sim_{\text{Khch}} \mathbb{E} \Big\| \sum_{k=1}^n \varepsilon_k g_k \Big\|_s \sim_{\text{Rsth}} \Big(\sum_{k=1}^n \|g_k\|_s^s \Big)^{\frac{1}{s}} + \Big(\sum_{k=1}^n \|g_k\|_2^2 \Big)^{\frac{1}{2}}.$$

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- Given $1 < p \leq q \leq \infty$, we easily deduce by duality that

$$\Big(\int_\Omega \Big[\sum_{k=1}^n |f_k|^q \Big]^{\frac{p}{q}} d\mu \Big)^{\frac{1}{p}} \sim_{c_{p,q}} \inf_{f=\phi+\psi} \Big(\sum_{k=1}^n \|\phi_k\|_p^p \Big)^{\frac{1}{p}} + \Big(\sum_{k=1}^n \|\psi_k\|_q^q \Big)^{\frac{1}{q}}.$$

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- In particular, we obtain isomorphic embeddings

$$\begin{aligned} \mathcal{K}_{p,q}^n(\Omega) &= n^{\frac{1}{p}} L_p(\Omega) + n^{\frac{1}{q}} L_q(\Omega) \rightarrow L_p(\Omega; \ell_q) \qquad 1 < p \leq q \leq \infty, \\ \mathcal{J}_{p,q}^n(\Omega) &= n^{\frac{1}{p}} L_p(\Omega) \cap n^{\frac{1}{q}} L_q(\Omega) \rightarrow L_p(\Omega; \ell_q) \qquad 1 \leq q \leq p < \infty. \end{aligned}$$

Asymmetric L_p spaces

Given $1 \leq p, q \leq \infty$, let us take

$$C_p = [C, R]_{\frac{1}{p}} \quad \text{and} \quad R_q = [R, C]_{\frac{1}{q}}.$$

Then, by definition of the Haagerup tensor product and Kouba's theorem

$$C_p \otimes_h R_q \; = \; \big(C_p \otimes_h R\big) \bullet \big(C \otimes_h R_q\big)$$

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Given a general von Neumann algebra $\mathcal{M}$ and $1 \leq p, q \leq \infty$, we now define

$$L_{2p}^r(\mathcal{M}) = \big[\mathcal{M}, L_2^r(\mathcal{M})\big]_{\frac{1}{p}} \quad \text{and} \quad L_{2q}^c(\mathcal{M}) = \big[\mathcal{M}, L_2^c(\mathcal{M})\big]_{\frac{1}{q}}.$$

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We construct **asymmetric L_p spaces** by taking amalgamated Haagerup tensor products

$$L_{(2p,2q)}(\mathcal{M}) \ = \ L_{2p}^r(\mathcal{M}) \otimes_{\mathcal{M},h} L_{2q}^c(\mathcal{M}) \ = \ \Big(L_{2p}^r(\mathcal{M}) \otimes_h L_{2q}^c(\mathcal{M})\Big) / \langle x\gamma \otimes y - x \otimes \gamma y \rangle.$$

Remark. Of course, we have $L_p(\mathcal{M}) = L_{(2p,2p)}(\mathcal{M})$ cb-isometrically for $1 \leq p \leq \infty$.

Noncommutative $\mathcal{J}_{p,q}^n$ and $\mathcal{K}_{p,q}^n$

In the classical case, we had

$$\begin{aligned} \mathcal{K}_{p,q}^n(\Omega) &= n^{\frac{1}{p}} L_p(\Omega) + n^{\frac{1}{q}} L_q(\Omega) \qquad \text{for} \qquad p \leq q, \\ \mathcal{J}_{p,q}^n(\Omega) &= n^{\frac{1}{p}} L_p(\Omega) \cap n^{\frac{1}{q}} L_q(\Omega) \qquad \text{for} \qquad p \geq q. \end{aligned}$$

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These spaces are completely isomorphic to

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Remarks.

- The cross terms disappear in the Banach space level.

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Remarks.

- The cross terms disappear in the Banach space level.
- Explicit norms for $S^m_p(\mathcal{J}^n_{p,q}(\mathcal{M}))$ and $S^m_p(\mathcal{K}^n_{p,q}(\mathcal{M})) \Rightarrow$ amalgamated/conditional L_p's.

Mixed norms of free variables

Let $(\mathcal{M}, \varphi)$ be a noncommutative probability space and

$$(\mathcal{A}, \phi) = *_{k\geq 1}\big(\mathcal{M} \oplus \mathcal{M}, \tfrac{1}{2}(\varphi \oplus \varphi)\big).$$

Let $\pi_{free}^k : \mathcal{M} \oplus \mathcal{M} \to \mathcal{A}$ be the canonical embedding into the k-th free factor of $\mathcal{A}$.

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Theorem [JP]. We have complete embeddings:

i) If $1 \leq p \leq q \leq \infty$,

$$x \in \mathcal{K}_{p,q}^n(\mathcal{M}) \mapsto \sum_{k=1}^n \pi_{free}^k(x, -x) \otimes \delta_k \in L_p(\mathcal{A}; \ell_q^n).$$

ii) If $1 \leq q \leq p \leq \infty$,

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Main tools. Free Rosenthal inequality [Junge, Parcet, Xu 2007].
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Generalizations. Amalgamation.

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Operator space L_p embeddings $\rightsquigarrow$ $\mathcal{J}_{p',2}^n$ and $\mathcal{K}_{p,2}^n$ for $1 \le p \le 2 \le p' \le \infty$.

Generalizations. Amalgamation + Vector-values.

Mixed norms of free variables

Theorem [JP]. We have complete embeddings:

i) If $1 \leq p \leq q \leq \infty$,

$$x \in \mathcal{K}_{p,q}^n(\mathcal{M}) \mapsto \sum_{k=1}^n \pi_{free}^k(x,-x) \otimes \delta_k \in L_p(\mathcal{A}; \ell_q^n).$$

ii) If $1 \leq q \leq p \leq \infty$,

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Moreover, the images are cb-complemented and the constants do not depend on n.

Main tools. Free Rosenthal inequality [Junge, Parcet, Xu 2007].
Noncommutative Burkholder inequality [Junge, Xu 2003].
Interpolation of intersections: $[\mathcal{J}_{p,1}^n(\mathcal{M}), \mathcal{J}_{p,p}^n(\mathcal{M})]_\theta \simeq_{cb} \mathcal{J}_{p,q}^n(\mathcal{M})$.

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Generalizations. Amalgamation + Vector-values + Non identically distributed variables.

Noncommutative independence

A family of von Neumann algebras

$$\mathcal{N} \subset \mathcal{M}, \mathcal{M}_k \subset \mathcal{A}$$

is a system of **increasingly independent** / **top-subsymmetric** copies over $\mathcal{N}$ when:

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$$\mathsf{E}_{\mathcal{N}}(ab) = \mathsf{E}_{\mathcal{N}}(a)\mathsf{E}_{\mathcal{N}}(b).$$

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- We have homomorphisms $\pi_k : \mathcal{M} \to \mathcal{M}_k$ with

$$\pi_{k|_{\mathcal{N}}} = id \quad \text{and} \quad \mathsf{E}_{\mathcal{N}}\big(\pi_{f(1)}(x_1) \cdots \pi_{f(m)}(x_m)\big) = \mathsf{E}_{\mathcal{N}}\big(\pi_{g(1)}(x_1) \cdots \pi_{g(m)}(x_m)\big)$$

  for all $f, g : \{1, 2, \ldots, m\} \to \mathbb{N}$ such that

  - $f_{|_{\{1,2,\ldots,m\} \setminus A}} = g_{|_{\{1,2,\ldots,m\} \setminus A}}$.
  - $|A| \leq 2$ and $A = \big\{k \mid f(k) = \max f\big\} = \{k \mid g(k) = \max g\big\}$.

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This is a **quite general notion** which includes the main examples:

- Free copies + Tensor product copies + q-independent copies, etc...

A transference method

Theorem [JP]. Let $(\mathcal{M}_k)_{k\geq 1}$ be independent top-subsymmetric copies over $\mathcal{N}$:

- If $1 \leq p \leq 2$,

$$\mathbb{E}\Big\| \sum_{k=1}^{n} \varepsilon_k \pi_k(x) \Big\|_{L_p(\mathcal{A})} \sim_{c_1} \mathbb{E}\Big\| \sum_{k=1}^{n} \varepsilon_k \pi_{free}^k(x) \Big\|_{L_p(\mathcal{A}_{free})}.$$

- If $1 \leq p < q \leq \infty$,

$$\Big\| \sum_{k=1}^{n} \pi_k(x) \otimes \delta_k \Big\|_{L_p(\mathcal{A};\ell_q^n)} \sim_{c_2} \Big\| \sum_{k=1}^{n} \pi_{free}^k(x) \otimes \delta_k \Big\|_{L_p(\mathcal{A}_{free};\ell_q^n)}.$$

Moreover, in both cases the relevant constants c_1 and c_2 are independent of p, q and n.

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Remarks.

- This includes amalgamation, but we impose independent copies.
- The case $p \geq q$ requires duality $\rightsquigarrow$ complementation $\rightsquigarrow$ $c_{p,q} \xrightarrow{p\to\infty} \infty$.

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Sketch of the proof.

- By Junge's recent methods in L_1

$$\Big\| \sum_{k=1}^n \pi_k(x) \otimes \delta_k \Big\|_{L_1(\mathcal{A};\ell_\infty^n)} \sim_c \|x\|_{\mathcal{K}_{1,\infty}^n(\mathcal{M},\mathcal{E}_\mathcal{N})} \sim_c \Big\| \sum_{k=1}^n \pi_{free}^k(x) \otimes \delta_k \Big\|_{L_1(\mathcal{A}_{free};\ell_\infty^n)}.$$

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- The construction of $L_p(\mathcal{A};\ell_q^n) \hookrightarrow \prod_{j,\mathcal{U}} L_1(\mathcal{R}_j;\ell_\infty^{nm_j})$ which preserves independence.

Applications I. Removable singularities

Corollary A. If $1 \leq p \leq 2$, we have for top-subsymmetric copies

$$\mathbb{E}\Big\| \sum_{k=1}^{n} \varepsilon_k \pi_k(x) \Big\|_p \sim_c \inf_{x=a+b+c} n^{\frac{1}{p}} \|a\|_p + \sqrt{n} \big\| \mathcal{E}_{\mathcal{N}}(bb^*)^{\frac{1}{2}} \big\|_p + \sqrt{n} \big\| \mathcal{E}_{\mathcal{N}}(c^*c)^{\frac{1}{2}} \big\|_p.$$

Moreover, the involved constant c is once more independent of p and n.

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- **Weighted q-Khintchine inequalities**

  If $-1 \le q \le 1$, consider the generalized q-gaussians

$$g_{q,k} = \lambda_k \ell_q(e_k) + \mu_k \ell_q^*(e_{-k}) \quad \text{on} \quad \mathcal{F}_q(\mathcal{H}) = \Big(\mathbb{C}\Omega \oplus \bigoplus_{n \ge 1} \mathcal{H}^{\otimes n}, \langle \ , \ \rangle_q \Big).$$

  Let $(\Gamma_q, \phi_q) = \big(\langle (g_{q,k})_{k \ge 1} \rangle'', \langle \Omega, \cdot \Omega \rangle \big)$ denote the q-deformed Araki-Woods factor. Then, if $1 \le p \le 2$ and $x_1, x_2, x_3, \ldots, x_n \in L_p(\mathcal{M})$, the following equivalences hold up to a constant c which is independent of p, q and n

$$\Big\| \sum_{k=1}^n x_k \otimes d_{\phi_q}^{\frac{1}{2p}} g_{q,k} d_{\phi_q}^{\frac{1}{2p}} \Big\|_p \sim_c \inf_{x_k = a_k + b_k} \Big\| \Big(\sum_k \lambda_k^{\frac{2}{p}} \mu_k^{\frac{2}{p'}} a_k a_k^* \Big)^{\frac{1}{2}} \Big\|_p + \Big\| \Big(\sum_k \lambda_k^{\frac{2}{p'}} \mu_k^{\frac{2}{p}} b_k^* b_k \Big)^{\frac{1}{2}} \Big\|_p .$$

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- **Weighted q-Khintchine inequalities**
  - Weighted free Khinchine inequalities
    Xu 2006.
  - Weighted Fermionic Khinchine inequalities
    Junge 2006 $\rightsquigarrow p = 1$ / Xu 2006 $\rightsquigarrow 1 < p \leq 2$

$$c_p \lesssim \frac{1}{p-1} \rightarrow \infty \quad \text{as} \quad p \rightarrow 1.$$

  - Weighted q-deformed Khinchine inequalities
    Junge, Parcet, Xu 2007

$$c_{p,q} \lesssim \frac{1}{\sqrt{1-|q|}} \rightarrow \infty \quad \text{as} \quad q \rightarrow \pm 1.$$

Problem. Determine the order of growth of $c_{p,q}$ as $(p,q) \rightarrow (\infty, \pm 1)$.

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- **Weighted q-Khintchine inequalities**
- **Cb-embedding $L_q \to L_p$ preserving hyperfiniteness**

  Let $\mathcal{M}$ be hyperfinite and $1 \leq p \leq q \leq 2$. Then, there exists a completely isomorphic embedding of $L_q(\mathcal{M})$ into $L_p(\mathcal{A})$ where both spaces are equipped with their natural operator space structures and satisfy:

    - $\mathcal{A}$ is hyperfinite.
    - The constants are independent of p, q.

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    Xu 2006 $\rightsquigarrow$ Weighted Fermionic Khintchine $\rightsquigarrow$ $c_p \to \infty$ as $p \to 1$.

  - Cb-embedding of $L_q(\mathcal{M})$ into hyperfinite $L_p(\mathcal{A})$

    Junge, Parcet 2007 $\rightsquigarrow$ Rosenthal inequality for copies $\rightsquigarrow$ $c_{p,q} \to \infty$ as $p \to 1$.

Applications II. There is no cb-embedding of ℓ_q into semifinite L_p

Corollary B. If $1 \leq p \leq q \leq \infty$, the mapping

$$x \in S_q^n \mapsto \frac{1}{n^{1/q}} \sum_{k=1}^{n^2} \pi_k(x) \otimes \delta_k \in L_p(M_{n^{n^2}}; \ell_q^{n^2})$$

is a complete embedding with relevant constants independent of p, q.

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Corollary C. Assume that we have a cb-embedding

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for some $1 \leq p \leq q \leq 2$. Then, the von Neumann algebra $\mathcal{A}$ cannot be semifinite.

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Sketch of the proof. By Corollary B and Rosenthal's theorem,

$$\ell_q \hookrightarrow L_p(semifinite) \Rightarrow L_q(\mathcal{R}) \hookrightarrow L_p(semifinite),$$

where $\mathcal{R}$ is the hyperfinite II_1 factor. Moreover, applying Xu's techniques we can show

$$L_q(\mathcal{R}) \hookrightarrow L_p(\mathcal{A}) \Rightarrow \mathcal{A} \text{ is of type III.}$$