II_1 factors with at most one Cartan subalgebra

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What do we classify?

Γ	countable discrete group
(X, μ)	standard probability measure space
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  $\Gamma \curvearrowright X$ is *essentially-free* i.e. $\mu(\{x : sx = x\}) = 0 \ \forall s \in \Gamma \setminus \{1\}$;

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Group measure space constructions

$\Gamma \curvearrowright (X,\mu)$ p.m.p. $\longleftrightarrow$

$$\sigma\colon \Gamma \curvearrowright L^\infty(X,\mu)$$
$$\sigma_s(f)(x) = f(s^{-1}x)$$
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The unitary element $u_s = \sigma_s \otimes \lambda_s \in \mathbb{B}(L^2(X) \otimes \ell_2(\Gamma))$ satisfies

$$u_s\, f\, u_s^* = \sigma_s(f)$$

for all $f \in L^\infty(X, \mu)$, identified with $f \otimes 1 \in \mathbb{B}(L^2(X) \otimes \ell_2(\Gamma))$.

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We encode the information of $\Gamma \curvearrowright X$ into a single vN algebra

$$\mathrm{vN}(X \rtimes \Gamma) := \{\sum_{s\in\Gamma}^{\text{finite}} f_s\, u_s : f_s \in L^\infty(X)\}'' \subset \mathbb{B}(L^2(X) \otimes \ell_2(\Gamma)).$$

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$\mathrm{vN}(X \rtimes \Gamma)$ is same as the crossed product vN algebra $L^\infty(X) \rtimes \Gamma$.

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$\mathrm{vN}(X \rtimes \Gamma)$ is a vN algebra of type II_1, with the trace τ given by

$$\tau(\sum_s f_s\, u_s) = \langle \sum_s f_s\, u_s\, (\mathbf{1} \otimes \delta_1), (\mathbf{1} \otimes \delta_1) \rangle = \int f_1\, d\mu.$$

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Definition

A von Neumann subalgebra $A \subset M$ is called a *Cartan subalgebra* if it is a maximal abelian subalgebra such that the normalizer

$$\mathcal{N}(A) = \{u \in M : \text{unitary} \quad uAu^* = A\}$$

generates M.

Orbit Equivalence Relation

Theorem (Singer, Dye, Krieger, Feldman-Moore 1977)

Let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be ess-free p.m.p. actions, and

$$\theta\colon (X,\mu) \to (Y,\nu)$$

be an isomorphism. Then, the isomorphism

$$\theta^*\colon L^\infty(Y,\nu) \ni f \mapsto f\circ\theta \in L^\infty(X,\mu)$$

extends to a $*$-isomorphism

$$\pi\colon \mathrm{vN}(Y \rtimes \Lambda) \to \mathrm{vN}(X \rtimes \Gamma)$$

if and only if θ preserves the orbit equivalence relation:

$$\theta(\Gamma x) = \Lambda\theta(x) \quad \text{for } \mu\text{-a.e. } x.$$

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OE $\longrightarrow$ **vN** *is not one-to-one,*

i.e. $\exists$ *a* II_1*-factor with non-conjugate Cartan subalgebras.*

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Theorem (Dykema 1993)

$\mathrm{vN}(\Gamma_1 * \Gamma_2) \cong \mathrm{vN}(\mathbb{F}_2)$ *for any infinite amenable groups* Γ_1 *and* Γ_2.

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$\exists$ *a* II_1*-factor which is not* $*$*-isomorphic to its complex conjugate.*

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Theorem (Voiculescu 1994)

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Note: Popa (2000) proved $\mathrm{vN}(\mathbb{Z}^2) \subset \mathrm{vN}(\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z}))$ is a unique “Cartan subalgebra with the relative property (T).”

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Note: Popa (2004) proved $\mathrm{vN}([0,1]^\Gamma \rtimes \Gamma) \cong \mathrm{vN}(Y \rtimes \Lambda)$ implies $(\Gamma \curvearrowright [0,1]^\Gamma) \cong (\Lambda \curvearrowright Y)$ provided that Λ has the property (T). Further results by Popa and Vaes.

OE to Cocycle (after Zimmer)

Suppose $(\Gamma \curvearrowright X) \cong_{\mathrm{OE}} (\Lambda \curvearrowright Y)$, i.e. $\exists\, \theta\colon X \xrightarrow{\sim} Y$ such that

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Cocycles α and β are *equivalent* if $\exists\, \phi \colon X \to \Lambda$ such that

$$\beta(x, s) = \phi(x)\alpha(x, s)\phi(s^{-1}x)^{-1}.$$

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Popa's formulation

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$\longleftrightarrow$

$\sigma \colon \Gamma \curvearrowright L^\infty(X)$

$\alpha \colon \Gamma \to L^\infty(X, \mathrm{vN}(\Lambda))$

$\cong L^\infty(X) \bar{\otimes} N$

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$\alpha\colon X \times \Gamma \to \Lambda$ ⟷ $\alpha\colon \Gamma \to L^\infty(X, \mathrm{vN}(\Lambda))$

$\cong L^\infty(X)\,\bar{\otimes}\,N$

$\alpha_s(x) = \alpha(x,s)$

$\alpha(x,s)\alpha(s^{-1}x,t) = \alpha(x,st)$ ⟷ $\alpha_s\,\sigma_s(\alpha_t) = \alpha_{st}$

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Since $\sigma_s(f) = u_s\, f\, u_s^*$ in $\mathrm{vN}(X \rtimes \Gamma)$,

$$\Gamma \ni s \mapsto \alpha_s\, u_s \in \mathrm{vN}(X \rtimes \Gamma) \bar{\otimes} N$$

is a group homomorphism which extends to an inclusion

$$\Theta\colon \mathrm{vN}(\Gamma) \hookrightarrow \mathrm{vN}(X \rtimes \Gamma) \bar{\otimes} N.$$

Profinite Action

Definition

An ergodic action $\Gamma \curvearrowright X$ is *profinite* if $X = \varprojlim \Gamma/\Gamma_n$ for some finite index subgroups $\Gamma \geq \Gamma_1 \geq \Gamma_2 \geq \cdots$;
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$\longrightarrow$ Because of the Kazhdan property (T), for a large n, $\Theta(\mathrm{vN}(\Gamma))$ is almost contained in $\mathrm{vN}((\Gamma/\Gamma_n) \rtimes \Gamma) \bar{\otimes} N$.

Complete Metric Approximation Property

Definition

A group Γ has the CMAP if $\exists\ f_n$ such that

- $f_n \colon \Gamma \to \mathbb{C}$ finitely supported,
- $f_n \to 1$ pointwise,
- $\|m_{f_n}\|_{\mathrm{cb}} \leq 1$.

Here the Herz-Schur multiplier $m_f \colon \mathrm{vN}(\Gamma) \to \mathrm{vN}(\Gamma)$ is defined by

$$m_f(\textstyle\sum_s \alpha_s\, u_s) = \sum_s \alpha_s f(s)\, u_s.$$

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Theorem (De Cannière-Haagerup 1985, Cowling-Haagerup 1989)

Besides amenable groups, free groups $\mathbb{F}_r$ have the CMAP, and so are discrete subgroups of $\mathrm{SO}(n,1)$ and $\mathrm{SU}(n,1)$.

Groups with CMAP

Theorem A (Oz-Popa)

Suppose Γ CMAP and $\exists\ \Lambda \triangleleft \Gamma$ infinite normal amenable subgroup. Then, $\exists$ a Λ-invariant mean on $\ell_\infty(\Lambda)$, which is $\mathrm{Ad}(\Gamma)$-invariant. In particular, Γ is inner-amenable.

Proof (Assuming Λ is abelian).

Recall $\mathrm{vN}(\Lambda) \cong L^\infty(\widehat{\Lambda})$ via the Fourier transform $\ell_2(\Lambda) \cong L^2(\widehat{\Lambda})$.
Let $\tau_0 \colon C(\widehat{\Lambda}) \to \mathbb{C}$ be the evaluation at the trivial character 1.
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Corollary

The lamplighter group

$$(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{F}_r = \left(\bigoplus_{\mathbb{F}_r} (\mathbb{Z}/2\mathbb{Z})\right) \rtimes \mathbb{F}_r$$

does not have the CMAP.

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Theorem (de Cornulier-Stalder-Valette)

The lamplighter group $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{F}_r$ has the Haagerup property.

von Neumann algebra with CMAP

Definition

A finite vN algebra M has the CMAP if $\exists\ \phi_n$ such that

- $\phi_n \colon M \to M$ finite rank,
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- CMAP inherits to a vN subalgebra (assuming finiteness).
- Γ has CMAP and $\Gamma \curvearrowright X$ profinite $\Rightarrow \mathrm{vN}(X \rtimes \Gamma)$ has CMAP. (Note: $\mathrm{vN}(X \rtimes \Gamma)$ can be non-(Γ).)

Upgrading Theorem **A**

Use $\mu\colon P \otimes \bar{P} \ni \sum a_k \otimes \bar{b}_k \mapsto \tau(\sum a_k b_k^*) \in \mathbb{C}$ instead of τ_0.

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- *$P \subset \mathrm{vN}(\mathbb{F}_r)$ diffuse amenable $\Rightarrow \mathcal{N}(P)''$ amenable.*
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- Q *CMAP* $\Rightarrow Q \bar{\otimes} \mathrm{vN}(\mathbb{F}_r)$ *has no Cartan subalgebra.*
- $\mathbb{F}_r \curvearrowright X$ *profinite* $\Rightarrow$ $L^\infty(X) \subset \mathrm{vN}(X \rtimes \mathbb{F}_r)$ *is the unique Cartan subalgebra.*

Proof in the case of $P \subset \mathrm{vN}(\mathbb{F}_r)$ diffuse amenable

Let $a_1, \ldots, a_r \in M = \mathrm{vN}(\mathbb{F}_r)$ be the standard unitary generators, and $M_1 = \langle b_1, \ldots, b_r \rangle$ be a copy of $\mathrm{vN}(\mathbb{F}_r)$.
For $t \in \mathbb{R}$, define a $*$-homomorphism $\alpha_t \colon M \to M * M_1$ by

$$\alpha_t(a_k) = a_k \exp(t \log b_k).$$

Observe that $E_M \circ \alpha_t$ is the Haagerup multiplier on M associated with $\mathbb{F}_r \ni s \mapsto \gamma_t^{|s|} \in \mathbb{R}$, where $\gamma_t = \tau(\exp(t \log b_k)) = \frac{\sin(\pi t)}{\pi t}$.
For a given finite subset $\mathfrak{F} \subset \mathcal{N}(P)$, choose $t > 0$ small enough so that $\alpha = \alpha_t$ satisfies $\alpha(u) \approx u$ for all $u \in \mathfrak{F}$.
Since $\eta_n \in L^2(P \mathbin{\bar{\otimes}} \bar{P})$ are "almost concentrated on the diagonal," $((E_M^\perp \circ \alpha) \otimes 1)\eta_n$ is a non-null sequence, almost $\mathrm{Ad}(\mathfrak{F})$-invariant.
But $L^2(M * M_1) \ominus L^2(M) \cong \bigoplus L^2(M) \mathbin{\bar{\otimes}} L^2(M)$ as an M-bimodule, this implies amenability of $\mathcal{N}(P)''$.

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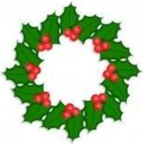
Thank you to organizers

Thank you to organizers and Fields Institute,



Thank you to organizers
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wish you all a happy holiday season!