Representation of Banach algebras as sets of c.b. maps

Timur Oikhberg

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Suppose $\mathcal{A}$ is a unital Banach algebra.

- Suppose $\pi : \mathcal{A} \to B(E)$ is a unital representation. Can we equip E with an operator space structure (say X) s.t. $CB(X) = \pi(\mathcal{A}) + \text{small perturbations?}$
- Does there exist a unital isometric representation $\rho : \mathcal{A} \to CB(X)$ s.t. $CB(X) = \rho(\mathcal{A}) + \text{small perturbations?}$

There exist Banach algebras which are not isomorphic to $CB(X)$ (or $B(E)$) as Banach algebras.

Application: more examples of "pathological" operator spaces.

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$(\mathfrak{B}, \beta)$ is a Banach operator ideal if, for Banach spaces X and Y, $\mathfrak{B}(X, Y) \subset B(X, Y)$, $(\mathfrak{B}(X, Y), \beta)$ is a Banach space.

Ideal property: $\beta(BTA) \leqslant \|A\|\beta(T)\|B\|$ for any $A \in B(X_0, X)$, $B \in B(Y, Y_0)$.

Convention: $\beta(T) = \infty$ if $T \notin \mathfrak{B}$.

The ideal $\mathfrak{B}$ is maximal if, $\forall\ T \in B(X, Y)$, $\beta(T) = \sup \beta(BTA)$ ($A \in B(X_0, X)$, $B \in B(Y, Y_0)$, X_0 and Y_0 are fin. dim.).

Equivalently: $\forall\ T \in B(X, Y)$, $\beta(T) = \sup \beta(qTi_E)$, where $i_E : E \to X$ is an embedding, $q : Y \to F$ is a quotient map, $\dim E < \infty$, and $\dim Y/F < \infty$.

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Example of a maximal ideal: $T \in B(X, Y)$ is 2-summing if $\exists$ c s.t.

$$\Big(\sum_i \|Tx_i\|^2\Big)^{1/2} \leqslant c \sup_{x^* \in E^*, \|x^*\| \leqslant 1} \Big(\sum_i |x^*(x_i)|^2\Big)^{1/2}$$

$\forall$ $x_1, \ldots, x_n \in E$. $\pi_2(T)$ = the smallest c that works.

$\Pi_2(X, Y)$ is the set of all 2-summing operators between X and Y.

If H and K are Hilbert spaces, then $\Pi_2(H, K) = \mathcal{S}_2(H, K)$, with equal norms.

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$\{Tx \in \overline{\mathcal{A}x} \text{ for any } x \in X\} \iff \{T \in \mathcal{A}\}$.

$\mathcal{A} \hookrightarrow B(X, Y)$ is C-hyperreflexive if, for every $T \in B(X, Y)$,

$$\begin{aligned}\mathrm{dist}(T, \mathcal{A}) &:= \inf_{a \in \mathcal{A}} \|T - a\| \\ &\leqslant C \sup_{x \in X, \|x\|=1} \mathrm{dist}(Tx, \mathcal{A}x).\end{aligned}$$

Equivalently: $\inf_{a \in \mathcal{A}} \|T - a\| \leqslant C \sup \|q_{\mathcal{A}(E)} T i_E\|$, where $i_E : E \hookrightarrow X$ is an embedding, $q_F : Y \to Y/F$ is a quotient (can take sup with $E = \mathrm{span}[x]$).

Examples of hyperreflexive spaces: 1-dim. spaces of operators; fin. dim. reflexive spaces of operators; nest algebras (in $B(H)$); many von Neumann algebras.

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$$d_{\mathcal{A},\mathfrak{B}}(T) := \sup \beta(uTv),$$

with sup is taken over all fin. rank contractions u, v with $u\mathcal{A}v = 0$. In other words, $d_{\mathcal{A},\mathfrak{B}}(T) = \sup \beta(q_{\mathcal{A}(E)} T i_E)$, with the sup taken over all (fin. dim.) $E \hookrightarrow X$.

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Question: which spaces are Π_2-hyperreflexive?

Theorem. *Any von Neumann algebra is Π_2-hyperreflexive.*

Theorem. *For any Banach space E, $\mathrm{span}[I_E]$ is Π_2-hyperreflexive.*

Theorem. *Suppose E is a Banach space with the BAP. Then $\exists$ $T \in B(E)$ s.t. $\mathrm{span}[T]$ is not Π_2-hyperreflexive.*

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Suppose $a \in B(E,F)$. The infinite ampliation of a:

$$a^{(\infty)} = \begin{pmatrix} a & 0 & \ldots \\ 0 & a & \ldots \\ 0 & 0 & \ddots \end{pmatrix} \in B(\ell_2(E), \ell_2(F)).$$

For $\mathcal{A} \subset B(E,F)$, $\mathcal{A}^{(\infty)} = \{a^{(\infty)} \,|\, a \in \mathcal{A}\} \subset B(\ell_2(E), \ell_2(F))$.

Theorem. *Suppose E and F are reflexive Banach spaces, and $\mathcal{A} \hookrightarrow B(E,F)$ is $\sigma(B(E,F), E\widehat{\otimes}F^*)$-closed. Then $\mathcal{A}^{(\infty)}$ is Π_2-hyperreflexive.*

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- *$\forall$ $a \in \mathcal{A}$ and $S \in \Pi_2(X)$, $\|\pi(a)\|_{cb} \leqslant \|a\|$, and $\|S\|_{cb} \leqslant \pi_2(S)$.*
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If π is isometric, then $\|\pi(a)\|_{cb} = \|a\|$ for any $a \in \mathcal{A}$.

Remark. The same conclusion holds if π satisfies a weaker technical condition: $\pi(\mathrm{Ba}(\mathcal{A}))$ is Π_2-ASHR.

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- $\mathcal{A}$ is a dual Banach space.
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Proof. It follows from the recent results of M. Daws that, for $\mathcal{A}$ as above, $\exists$ a separable reflexive Banach space E and a unital faithful weak*-to-weak* continuous representation $\rho : \mathcal{A} \to B(E)$. Define

$$\pi : \mathcal{A} \to B(\ell_2(E)) : a \mapsto \rho(a)^{(\infty)}.$$

Then $\pi(\mathcal{A})$ is Π_2-hyperreflexive. We can construct an operator space X, isometric to $\ell_2(E)$, s.t. $CB(X) = \pi(\mathcal{A}) + \Pi_2(X)$.

Theorem [O., E. Ricard]. *Suppose $n_1, n_2, \ldots \in \mathbb{N}$. Then $\exists$ a sequence of operator spaces (E_i) s.t.:*

- *E_i is isometric to $\ell_2^{n_i}$.*
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- η is an infinite ordinal, $J(\eta)$ is the long James space (viewed as a set of diagonal operators on $\ell_2(\mathcal{I})$ for the appropriate index set $\mathcal{I}$), $\mathcal{A} = J(\eta) + \mathbb{C}\mathbf{1}$, π is the identity representation. Then $\pi(\mathrm{Ba}(\mathcal{A}))$ is Π_2-ASHR.
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- *Any Y as above fails the Operator Approximation Property.*
- *X as above is completely hereditarily indecomposable. That is, any inf. dim. $Y \hookrightarrow X$ is completely indecomposable: if $P \in CB(Y)$, and $P^2 = P$, then either $\dim \ker P < \infty$, or $\dim \operatorname{ran} P < \infty$.*
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- *X as above is completely hereditarily indecomposable. That is, any inf. dim. $Y \hookrightarrow X$ is completely indecomposable: if $P \in CB(Y)$, and $P^2 = P$, then either $\dim \ker P < \infty$, or $\dim \operatorname{ran} P < \infty$.*
- *Any compact c.b. maps on X is "completely nuclear," with the completely nuclear norm equivalent to the c.b. norm.*

Idea of the proof. Build an operator space around the identity representation of $\mathcal{A} = \mathbb{C} I_{\ell_2}$.

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Theorem. *There exists a separable operator space with a Complete Transfinite Basis, but without the Completely Bounded Approximation Property.*

No such examples of Banach spaces are known.

X has the Completely Bounded Approximation Property (CBAP) if $\exists$ a net $(u_i) \subset CB(X)$ of fin. rank operators, s.t. $\sup_i \|u_i\|_{cb} < \infty$, and $u_i \to I_X$ pointwise.

X is the Complete Transfinite Basis (CTB) of length η, where η is an ordinal, if there exist projections $(P_\alpha)_{0 \leqslant \alpha \leqslant \eta} \subset CB(X)$ s.t. $\sup_\alpha \|P_\alpha\|_{cb} < \infty$, $P_\eta = I_X$, $P_0 = 0$, $\mathrm{rank}\,(P_{\alpha+} - P_\alpha) = 1 \; \forall \, \alpha$, and the function $\alpha \mapsto P_\alpha x$ is continuous for any $x \in X$.

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Theorem. *There exists an operator space X such that, for every $\varepsilon > 0$, there exists $f \in CB(X^*)$ s.t. $\|f\| = f(I_X) = 1$, and $|f(TS) - f(T)f(S)| < \varepsilon \|T\|_{cb} \|S\|_{cb}$ for any $T, S \in CB(X)$, yet $\|f - g\| > 1/2$ whenever $g \in CB(X)^*$ is multiplicative.*

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