The Effros-Ruan conjecture for bilinear forms on C^*-algebras

(Joint work with Uffe Haagerup)

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Theorem (Grothendieck 1956):

Let K_1 and K_2 be compact spaces. Let $u : C(K_1) \times C(K_2) \to \mathbb{K}$ be a bounded bilinear form, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then there exist probability measures μ_1 and μ_2 on K_1 and K_2, respectively, such that

$$|u(f,g)| \le K_G^{\mathbb{K}} \|u\| \left(\int_{K_1} |f|^2 \, d\mu_1 \right)^{1/2} \left(\int_{K_2} |g|^2 \, d\mu_2 \right)^{1/2}$$

for all $f \in C(K_1)$ and $g \in C(K_2)$, where $K_G^{\mathbb{K}}$ is a universal constant.

Remarks about Grothendieck's constant $K_G^{\mathbb{K}}$:

- $K_G^{\mathbb{R}} \neq K_G^{\mathbb{C}}$.
- $\frac{\pi}{2} \le K_G^{\mathbb{R}} \le \frac{\pi}{2\log(1+\sqrt{2})} = 1.782...$
- $\frac{4}{\pi} \le K_G^{\mathbb{C}} < 1.40491$.

Theorem:

Any bounded linear operator $T : C(K_1) \to C(K_2)^*$ factors through a Hilbert space H,

$$\begin{tikzcd} C(K_1) \arrow[rr, "T"] \arrow[dr, "R"'] & & C(K_2)^* \\ & H \arrow[ur, "S"'] & \end{tikzcd}$$

such that $\|R\|\|S\| \le K_G^{\mathbb{K}} \|T\|$.

Remark: As an interesting application, it follows that the Fourier transform $\mathcal{F} : L^1(\mathbb{R}) \to C_0(\mathbb{R})$ is <u>not</u> onto.

Conjecture (Grothendieck):

Let A be a C*-algebra and $u : A \times A \to \mathbb{C}$ a bounded bilinear form. Then there exist $f\,, g \in S(A)$ such that for all $a\,, b \in A\,,$

$$|u(a,b)| \leq k\|u\| f\left(|a|^2\right)^{1/2} g\left(|b|^2\right)^{1/2},$$

where $|x| = \left(\frac{x^*x+xx^*}{2}\right)^{1/2}$, for all $x \in A\,.$

Grothendieck Inequality (Haagerup 1985) (extension of Pisier's result from 1978):

Let A and B be C*-algebras and let $u : A \times B \to \mathbb{C}$ be a bounded bilinear form. There exist $f_1\,, f_2 \in S(A)$ and $g_1\,, g_2 \in S(B)$ such that

$$|u(a,b)| \leq \|u\|(f_1(aa^*) + f_2(a^*a))^{1/2}(g_1(b^*b) + g_2(bb^*))^{1/2}\,,$$

for all $a \in A$ and $b \in B\,.$

Corollary (Haagerup 1985):

Any bounded linear operator $T : A \to B^*$, where A and B are C*-algebras, factors through a Hilbert space H,

$$\begin{tikzcd} A \arrow[rr, "T"] \arrow[dr, "R"'] & & B^* \\ & H \arrow[ur, "S"'] & \end{tikzcd}$$

such that $\|R\|\|S\| \leq 2\|T\|\,.$

Let A and B be C*-algebras. Then A is an operator space with the C*-norm on $M_n(A)$, $n \in \mathbb{N}$, while B^* is an operator space with the isometric identification $M_n(B^*) := \mathrm{CB}(B, M_n(\mathbb{C}))$, $n \in \mathbb{N}$.

Let $u : A \times B \to \mathbb{C}$ be a bounded bilinear form. There exists a unique bounded linear operator $\widetilde{u} : A \to B^*$ such that

$$u(a,b) = \langle \widetilde{u}(a) \, , b \rangle \, , \quad a \in A \, , b \in B \, .$$

The bilinear form u is called *jointly completely bounded* (j.c.b., for short) if $\widetilde{u} : A \to B^*$ is completely bounded, in which case we set

$$\|u\|_{jcb} := \|\widetilde{u}\|_{cb} \, .$$

Remark: It is easily checked that

$$\|u\|_{\mathrm{jcb}} = \sup_{n \in \mathbb{N}} \|u_n\| \, ,$$

where $u_n : M_n(A) \otimes M_n(B) \to M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$, $n \in \mathbb{N}$, is given by

$$u_n \left(\sum_{i=1}^{k} a_i \otimes c_i \, , \sum_{j=1}^{l} b_j \otimes d_j \right) = \sum_{i=1}^{k} \sum_{j=1}^{l} u(a_i, b_j) c_i \otimes d_j \, ,$$

for $a_i \in A$, $b_j \in B$, $c_i \, , d_j \in M_n(\mathbb{C})$, $k \, , l \in \mathbb{N}$.
Moreover, for all C*-algebras C, D, $a_i \in A$, $b_j \in B$, $c_i \in C$, $d_j \in D$,

$$\left\| \sum_{i=1}^{k} \sum_{j=1}^{l} u(a_i \, , b_j) c_i \otimes d_j \right\|_{C \otimes_{\min} D} \leq$$

$$\|u\|_{\mathrm{jcb}} \left\| \sum_{i=1}^{k} a_i \otimes c_i \right\|_{A \otimes_{\min} C} \left\| \sum_{j=1}^{l} b_j \otimes d_j \right\|_{B \otimes_{\min} D} .$$

(Cf. Pisier-Shlyakhtenko (Invent., 2002))

Conjecture (Effros-Ruan 1991):

Let A and B be C*-algebras and let $u : A \times B \to \mathbb{C}$ be a jointly completely bounded bilinear form. Then there exist $f_1, f_2 \in S(A)$ and $g_1, g_2 \in S(B)$ such that for all $a \in A$ and $b \in B$,

$$|u(a,b)| \leq K\|u\|_{jcb}\left(f_1(aa^*)^{1/2}g_1(b^*b)^{1/2} + f_2(a^*a)^{1/2}g_2(bb^*)^{1/2}\right) \quad (1)$$

where K is a universal constant.

Theorem (Pisier-Shlyakhtenko 2002):

Let $E \subseteq A$ and $F \subseteq F$ be *exact* operator spaces sitting in C*-algebras A and B. Let $u : E \times F \to \mathbb{C}$ be a j.c.b. bilinear form. Then there exist $f_1, f_2 \in S(A)$ and $g_1, g_2 \in S(B)$ such that the inequality (1) holds for all $a \in E$ and $b \in F$ with $K = 2\sqrt{2}\,\mathrm{ex}(E)\mathrm{ex}(F)$.

Theorem (Pisier-Shlyakhtenko 2002):

If either A or B is an *exact* C*-algebra and $u : A \times B \to \mathbb{C}$ is a j.c.b. bilinear form, then there exist $f_1, f_2 \in S(A)$ and $g_1, g_2 \in S(B)$ such that the inequality (1) holds for all $a \in A$ and $b \in B$ with $K = 2\sqrt{2}$.

Recall that an operator space E is called *exact* if there is $C \geq 1$ such that for every finite dimensional subspace $F \subseteq E$, there exists $n \in \mathbb{N}$ and a subspace $G \subseteq M_n(\mathbb{C})$ with $d_{\mathrm{cb}}(F, G) \leq C$. The infimum of all such constants C is denoted by $\mathrm{ex}(E)$.

Theorem (Kirchberg, Pisier): A C*-algebra is exact if and only if it is exact as an operator space. For any exact C*-algebra A, $\mathrm{ex}(A) = 1$.

Theorem (Haagerup-M. 2007)

The Effros-Ruan conjecture holds for arbitrary C*-algebras A and B with $K = 1$, and this is the best possible constant.

Corollary A:

Let A and B be C*-algebras. Any completely bounded linear map $T : A \to B^*$ admits a factorization through $H_r \oplus K_c$, where H and K are Hilbert spaces,

$$\begin{tikzcd} A \arrow[rr, "T"] \arrow[dr, "R"'] & & B^* \\ & H_r \oplus K_c \arrow[ur, "S"'] & \end{tikzcd}$$

satisfying $\|R\|_{cb}\|S\|_{cb} \leq 2\|T\|_{cb}$.

Corollary B:

Let A be a C*-algebra. If $T : A \to OH$ is a completely bounded linear map, then there exist $f_1, f_2 \in S(A)$ such that for all $a \in A$,

$$\|T(a)\| \leq \sqrt{2}\|T\|_{cb} f_1(aa^*)^{1/4} f_2(a^*a)^{1/4}.$$

(Only an improvement of constant in the corresponding result by Pisier-Shlyakhtenko; they had this with constant $2^{9/4}$.)

Corollary C:

Let E be an operator space such that E and its dual E^* embed completely isomorphically into preduals M_* and N_*, respectively, of von Neumann algebras M and N. Then E is cb-isomorphic to a quotient of a subspace of $H_r \oplus K_c$, for some Hilbert spaces H and K.

Corollary D:

Let E be an operator space, and let $E \subseteq A$ and $E^* \subseteq B$ be completely isometric embeddings into C*-algebras A and B such that both subspaces are cb-complemented. Then E is cb-isomorphic to $H_r \oplus K_c$, for some Hilbert spaces H and K.

(These are non-commutative analogues of the classical result asserting that if X is a Banach space such that both X and its dual X^* embed into L_1-spaces, then X is isomorphic to a Hilbert space. Corollaries C and D above are obtained by adjusting the proof of the corresponding results by Pisier-Shlyakhtenko.)

Corollary E:

Let A_0, A, B_0 and B be C*-algebras such that $A_0 \subseteq A$ and $B_0 \subseteq B$. Then any j.c.b. bilinear form $u_0 : A_0 \times B_0 \to \mathbb{C}$ extends to a bilinear form $u : A \times B \to \mathbb{C}$ such that

$$\|u\|_{\mathrm{jcb}} \leq 2\|u_0\|_{\mathrm{jcb}} .$$

Let A and B be C*-algebras. Denote by $\mathrm{Bil}(A\,,B)$ the set of bounded bilinear forms on $A \times B$.

Lemma (Pisier-Shlyakhtenko 2002, Xu 2006):
Let $u \in \mathrm{Bil}(A\,,B)$. Assume that for all $a \in A$ and $b \in B$,

$$|u(a,b)| \leq \kappa\left(f_1(aa^*)^{1/2}g_1(b^*b)^{1/2} + f_2(a^*a)^{1/2}g_2(bb^*)^{1/2}\right) \qquad (2)$$

for some constant $\kappa \in [0,\infty]$ and some $f_1\,, f_2 \in S(A)\,, g_1\,, g_2 \in S(B)$. Then u can be decomposed as $u = u_1 + u_2$, where $u_1\,, u_2 \in \mathrm{Bil}(A\,,B)$ such that for all $a \in A$ and $b \in B$,

$$\begin{aligned} |u_1(a\,,b)| &\leq \kappa f_1(aa^*)^{1/2}g_1(b^*b)^{1/2} \\ |u_2(a\,,b)| &\leq \kappa f_2(a^*a)^{1/2}g_2(bb^*)^{1/2}. \end{aligned}$$

Definition: For $u \in \mathrm{Bil}(A\,,B)$, let $\|u\|_{ER} \in [0,\infty]$ be the infimum of all constants $\kappa \in [0,\infty]$ for which the inequality (2) holds, for some choice of $f_1\,, f_2 \in S(A)$ and $g_1\,, g_2 \in S(B)$.

Proposition:

(i) If $u \in \mathrm{Bil}(A\,,B)$, then $\|u\|_{ER} \leq \|u\|_{jcb} \leq 2\|u\|_{ER}$.

(ii) Let $c_1\,, c_2$ denote the best constants in the inequalities

$$c_1\|u\|_{ER} \leq \|u\|_{jcb} \leq c_2\|u\|_{ER}\,,$$

where A and B are arbitrary C*-algebras and $u \in \mathrm{Bil}(A\,,B)$. Then $c_1 = 1$ and $c_2 = 2$.

Proposition:

Let $u \in \mathrm{Bil}(A\,,B)$ with $\|u\|_{\mathrm{jcb}} < \infty$. Then for all C*-algebras $C\,,D\,,$ all $a_1,\ldots,a_k \in A\,,\ b_1,\ldots,b_l \in B\,,\ c_1,\ldots,c_k \in C\,,\ d_1,\ldots,d_l \in D\,,$ where $k\,,l \in \mathbb{N}\,,$

$$\left\|\sum_{i=1}^{k}\sum_{j=1}^{l} u(a_i\,,b_j)c_i\otimes d_j\right\|_{C\otimes_{\max}D} \leq \\ 2\|u\|_{\mathrm{jcb}}\left\|\sum_{i=1}^{k} a_i\otimes c_i\right\|_{A\otimes_{\min}C}\left\|\sum_{j=1}^{l} b_j\otimes d_j\right\|_{B\otimes_{\min}D}\,.$$

Proof: Follows from our main theorem, together with the splitting lemma above.

Let $E \subseteq A$ and $F \subseteq B$ be operator spaces sitting in C*-algebras A and B. Denote by $\mathrm{Bil}(E\,,F)$ the set of bounded bilinear forms on $E\times F$. For any $u \in \mathrm{Bil}(E\,,F)\,,\ \|u\|_{ER}$ is well-defined.

By the Pisier-Shlyakhtenko operator space version of Grothendieck's inequality, if E and F are exact, then for any $u \in \mathrm{Bil}(E\,,F)\,,$

$$\|u\|_{ER} \leq 2\sqrt{2}\,\mathrm{ex}(E)\mathrm{ex}(F)\|u\|_{\mathrm{jcb}}\,.$$

The next result (essentially contained in Pisier–Shlyakhtenko 2002) gives a complete characterization of those maps $u \in \mathrm{Bil}(E\,,F)$, for which $\|u\|_{ER} < \infty$.

Theorem:

Let $u \in \mathrm{Bil}(E\,,F)$. The following statements are equivalent:

(*i*) $\|u\|_{ER} < \infty$.

(*ii*) There exists a constant $\delta \in [0,\infty]$ such that for all C*-algebras C and D, and all $a_1,\dots,a_k \in E$, $b_1,\dots,b_l \in F$, $c_1,\dots,c_k \in C$, $d_1,\dots,d_l \in D$, where $k\,,l \in \mathbb{N}$,

$$\left\|\sum_{i=1}^{k}\sum_{j=1}^{l} u(a_i\,,b_j)c_i \otimes d_j\right\|_{C\otimes_{\max}D} \leq \delta\left\|\sum_{i=1}^{k} a_i \otimes c_i\right\|_{E\otimes_{\min}C}\left\|\sum_{j=1}^{l} b_j \otimes d_j\right\|_{F\otimes_{\min}D}.$$

Moreover, if $\delta(u)$ denotes the best constant in the inequality above, then one has

$$\frac{1}{2}\|u\|_{\mathrm{ER}} \leq \delta(u) \leq 2\|u\|_{\mathrm{ER}}\,.$$

Some preliminaries on Powers factors and Tomita-Takesaki theory

Let $0 < \lambda < 1$ be fixed, and let $(\mathcal{M}\,,\phi)$ be the Powers factor of type III_λ with product state ϕ, that is,

$$(\mathcal{M}\,,\phi) = \bigotimes_{n=1}^{\infty}(M_2(\mathbb{C})\,,\omega_\lambda)\,,$$

where $\phi = \bigotimes_{n=1}^{\infty}\omega_\lambda$, $\omega_\lambda(\,\cdot\,) = \mathrm{Tr}(h_\lambda\,\cdot\,)$ and $h_\lambda = \begin{pmatrix} \frac{\lambda}{1+\lambda} & 0 \\ 0 & \frac{1}{1+\lambda} \end{pmatrix}$.

The modular automorphism group $(\sigma_t^\phi)_{t\in\mathbb{R}}$ of ϕ is given by

$$\sigma_t^\phi = \bigotimes_{n=1}^{\infty}\sigma_t^{\omega_\lambda}\,,$$

where for any matrix $x = [x_{ij}]_{1\le i,j\le 2} \in M_2(\mathbb{C})$ and any $t \in \mathbb{R}$,

$$\sigma_t^{\omega_\lambda}(x) = h_\lambda^{it} x h_\lambda^{-it} = \begin{pmatrix} x_{11} & \lambda^{it}x_{12} \\ \lambda^{-it}x_{21} & x_{22} \end{pmatrix}\,.$$

Therefore $\sigma_t^{\omega_\lambda}$ and σ_t^ϕ are periodic in $t \in \mathbb{R}$ with minimal period

$$t_0 := -\frac{2\pi}{\log\lambda}\,.$$

Let $\mathcal{M}_\phi$ denote the centralizer of ϕ, that is,

$$\mathcal{M}_\phi := \{x \in \mathcal{M} : \sigma_t^\phi(x) = x\,, \forall t \in \mathbb{R}\}\,.$$

Theorem (Connes 1973):
The relative commutant of $\mathcal{M}_\phi$ in $\mathcal{M}$ is trivial, i.e.,

$$\mathcal{M}_\phi' \cap \mathcal{M} = \mathbb{C}\mathbf{1}\,.$$

Theorem (Haagerup 1989):
For all $x \in \mathcal{M}$,

$$\phi(x) \cdot \mathbf{1} \in \overline{\mathrm{conv}\{vxv^* : v \in \mathcal{U}(\mathcal{M}_\phi)\}}^{\|\cdot\|}\,,$$

where $\mathcal{U}(\mathcal{M}_\phi)$ denotes the unitary group on $\mathcal{M}_\phi$.

Corollary 1 (Strong version of the Dixmier averaging process):
There exists a net $\{\alpha_i\}_{i\in I} \subseteq \mathrm{conv}\{\mathrm{ad}(v) : v \in \mathcal{U}(\mathcal{M}_\phi)\}$ such that

$$\lim_{i\in I} \|\alpha_i(x) - \phi(x)\cdot \mathbf{1}\| = 0\,, \quad x \in \mathcal{M}\,.$$

We identify $\mathcal{M}$ with $\pi_\phi(\mathcal{M})$, where $(\pi_\phi\,, H_\phi\,, \xi_\phi)$ is the GNS representation of $\mathcal{M}$ associated to the state ϕ. Then

$$H_\phi := \overline{\mathcal{M}\xi_\phi} = L^2(\mathcal{M}\,, \phi)\,.$$

By Tomita-Takesaki theory, the operator S_0 defined by

$$S_0(x\xi_\phi) = x^*\xi_\phi\,, \quad x \in \mathcal{M}$$

is closable. Its closure $S := \overline{S_0}$ has a unique polar decomposition

$$S = J\Delta^{1/2}\,,$$

where Δ is a positive self-adjoint unbounded operator on $L^2(\mathcal{M}\,, \phi)$ and J is a conjugate-linear involution.

Moreover, for all $t \in \mathbb{R}$,

$$\sigma_t^{\phi}(x) = \Delta^{it} x \Delta^{-it}, \quad x \in \mathcal{M}$$

and $J\mathcal{M}J = \mathcal{M}'$, where $\mathcal{M}'$ denotes the commutant of $\mathcal{M}$.

Theorem (Takesaki 1973):

For all $n \in \mathbb{Z}$, set

$$\begin{aligned} \mathcal{M}_n &:= \{x \in \mathcal{M} : \sigma_t^{\phi}(x) = \lambda^{int} x, \ \forall t \in \mathbb{R}\} \\ &= \{x \in \mathcal{M} : \phi(xy) = \lambda^n \phi(yx), \ \forall y \in \mathcal{M}\}. \end{aligned}$$

In particular, $\mathcal{M}_0 = \mathcal{M}_\phi$. Moreover, for all $n \in \mathbb{Z}$,

$$\mathcal{M}_n \neq \{0\}$$

and $\Delta(\eta) = \lambda^n \eta$, for every $\eta \in \overline{\mathcal{M}_n \xi_\phi}$. Furthermore,

$$L^2(\mathcal{M}, \phi) = \bigoplus_{n=-\infty}^{\infty} \overline{\mathcal{M}_n \xi_\phi}.$$

Corollary 2:

For every $n \in \mathbb{Z}$, there exists $c_n \in \mathcal{M}$ such that

$$\phi(c_n^* c_n) = \lambda^{-n/2}, \quad \phi(c_n c_n^*) = \lambda^{n/2}$$

and, moreover, $\langle c_n J c_n J \xi_\phi, \xi_\phi \rangle_{H_\phi} = 1$.

Since $\mathcal{M}$ is an injective factor, it follows (cf. Effros-Lance and Connes 1976) that the map $c \otimes d \mapsto cd$ ($c \in \mathcal{M}, d \in \mathcal{M}'$) extends uniquely to a C*-algebra isomorphism

$$C^*(\mathcal{M}, \mathcal{M}') \simeq \mathcal{M} \otimes_{\min} \mathcal{M}'.$$

In the following, let A and B be C*-algebras and let

$$u : A \times B \to \mathbb{C}$$

be a jointly completely bounded bilinear form.

Lemma 3:
There is a bounded bilinear form $\widehat{u} : (A \otimes_{\min} \mathcal{M}) \times (B \otimes_{\min} \mathcal{M}') \to \mathbb{C}$ such that for all $a \in A$, $b \in B$, $c \in \mathcal{M}$, $d \in \mathcal{M}'$,

$$\widehat{u}(a \otimes c\,, b \otimes d) = u(a\,, b)\langle cd\xi_\phi\,, \xi_\phi\rangle_{H_\phi}\,.$$

Moreover,

$$\|\widehat{u}\| \leq \|u\|_{\mathrm{jcb}}\,.$$

Proof: Consider $a_1, \ldots, a_k \in A$, $b_1, \ldots, b_l \in B$, $c_1, \ldots, c_k \in \mathcal{M}$ and $d_1, \ldots, d_l \in \mathcal{M}'$, where $k, l \in \mathbb{N}$. Then

$$\begin{aligned}
&\left|\sum_{i=1}^{k}\sum_{j=1}^{l} u(a_i\,, b_j)\langle c_i d_j \xi_\phi\,, \xi_\phi\rangle_{H_\phi}\right| \\
&\qquad\qquad \leq \left\|\sum_{i=1}^{k}\sum_{j=1}^{l} u(a_i\,, b_j) c_i d_j\right\|_{\mathcal{B}(L^2(\mathcal{M}\,,\phi))} \\
&\qquad\qquad = \left\|\sum_{i=1}^{k}\sum_{j=1}^{l} u(a_i\,, b_j) c_i \otimes d_j\right\|_{\mathcal{M}\otimes_{\min}\mathcal{M}'} \\
&\qquad\qquad \leq \|u\|_{\mathrm{jcb}} \left\|\sum_{i=1}^{k} a_i \otimes c_i\right\|_{A\otimes_{\min}\mathcal{M}} \left\|\sum_{j=1}^{l} b_j \otimes d_j\right\|_{B\otimes_{\min}\mathcal{M}'}\,.
\end{aligned}$$

Corollary 4:

There exist $\widehat{f_1}\,, \widehat{f_2} \in S(A \otimes_{\min} \mathcal{M})$ and $\widehat{g_1}\,, \widehat{g_2} \in S(B \otimes_{\min} \mathcal{M}')$ such that for all $x \in A \otimes_{\min} M$ and $y \in B \otimes_{\min} \mathcal{M}'$,

$$|\widehat{u}(x,y)| \leq \|u\|_{\mathrm{jcb}}(\widehat{f_1}(xx^*) + \widehat{f_2}(x^*x))^{1/2}(\widehat{g_1}(y^*y) + \widehat{g_2}(yy^*))^{1/2}\,.$$

Lemma 5:

Let $v \in \mathcal{U}(\mathcal{M}_\phi)$ and set $v' := JvJ \in \mathcal{M}'$. Then

$$\widehat{u}((\mathrm{Id}_A \otimes \mathrm{ad}(v))(x)\,, (\mathrm{Id}_B \otimes \mathrm{ad}(v'))(y)) = \widehat{u}(x\,, y)\,,$$

for all $x \in A \otimes_{\min} \mathcal{M}$ and $y \in B \otimes_{\min} \mathcal{M}'$.

Proposition 6:

There exist $f_1\,, f_2 \in S(A)$, $g_1\,, g_2 \in S(B)$ and $\phi' \in S(\mathcal{M}')$ so that

$$\begin{aligned}|\widehat{u}(x,y)| \ \leq\ \|u\|_{\mathrm{jcb}}\Big[&((f_1 \otimes \phi)(xx^*) + (f_2 \otimes \phi)(x^*x))^{1/2} \cdot \\ &\cdot ((g_1 \otimes \phi')(y^*y) + (g_2 \otimes \phi')(yy^*))^{1/2}\Big]\,,\end{aligned}$$

for all $x \in A \otimes_{\min} \mathcal{M}$ and $y \in B \otimes_{\min} \mathcal{M}'$.

Proof: For all $\alpha\,, \beta \geq 0$, $\sqrt{\alpha\beta} \leq (\alpha + \beta)/2$. By Corollary 4, it follows that for all $x \in A \otimes_{\min} \mathcal{M}$ and $y \in B \otimes_{\min} \mathcal{M}'$,

$$|\widehat{u}(x,y)| \leq \frac{1}{2}\|u\|_{\mathrm{jcb}}\left(\widehat{f_1}(xx^*) + \widehat{f_2}(x^*x) + \widehat{g_1}(y^*y) + \widehat{g_2}(yy^*)\right) \tag{3}$$

Let $v \in \mathcal{U}(\mathcal{M}_\phi)$ and $v' := JvJ$. By Lemma 5 and (3),

$$\begin{aligned}|\widehat{u}(x,y)| \leq \frac{1}{2}\|u\|_{\mathrm{jcb}}\Big[&\widehat{f_1}((\mathrm{Id}_A \otimes \mathrm{ad}(v))(xx^*)) + \widehat{f_2}((\mathrm{Id}_A \otimes \mathrm{ad}(v))(x^*x)) \\ &+\widehat{g_1}((\mathrm{Id}_B \otimes \mathrm{ad}(v'))(y^*y)) + \widehat{g_2}((\mathrm{Id}_B \otimes \mathrm{ad}(v'))(yy^*))\Big]\end{aligned} \tag{4}$$

Choose a net $(\alpha_i)_{i\in I} \subseteq \mathrm{conv}\{\mathrm{ad}(v) : v \in \mathcal{U}(\mathcal{M}_\phi)\}$ such that

$$\lim_{i\in I} \|\alpha_i(c) - \phi(c)\cdot \mathbf{1}\| = 0\,, \quad c \in \mathcal{M}\,.$$

For $i \in I$, set $\alpha_i'(d) = J\alpha_i(JdJ)J$, for all $d \in \mathcal{M}'$.

By convexity we can replace $\mathrm{ad}(v)$ and $\mathrm{ad}(v')$ in the inequality (4) by α_i and α_i', respectively, to get

$$\begin{aligned}|\widehat{u}(x,y)| \le \frac{1}{2}\|u\|_{\mathrm{jcb}}\Big[&\widehat{f}_1((\mathrm{Id}_A \otimes \alpha_i)(xx^*)) + \widehat{f}_2((\mathrm{Id}_A \otimes \alpha_i)(x^*x)) + \\ &+\widehat{g}_1((\mathrm{Id}_B \otimes \alpha_i')(y^*y)) + \widehat{g}_2((\mathrm{Id}_B \otimes \alpha_i')(yy^*))\Big]\,.\end{aligned}$$

In the limit, this gives the inequality

$$\begin{aligned}|\widehat{u}(x,y)| \le \frac{1}{2}\|u\|_{\mathrm{jcb}}\Big[&(f_1 \otimes \phi)(xx^*) + (f_2 \otimes \phi)(x^*x) + \\ &+(g_1 \otimes \phi')(y^*y) + (g_2 \otimes \phi')(yy^*)\Big]\,, \end{aligned} \tag{5}$$

where $f_i(a) := \widehat{f}_i(a \otimes \mathbf{1})$, $a \in A$, $g_i := \widehat{g}_i(b \otimes \mathbf{1})$, $b \in B$, for $i = 1\,, 2$ and $\phi'(d) = \overline{\phi(JdJ)}$, for all $d \in \mathcal{M}'$.

Substituting x by $t^{1/2}x$ and y by $t^{-1/2}y$ in (5) for $t > 0$, we get

$$\begin{aligned}|\widehat{u}(x,y)| \le \frac{1}{2}\|u\|_{\mathrm{jcb}}\Big[&t\,((f_1 \otimes \phi)(xx^*) + (f_2 \otimes \phi)(x^*x)) + \\ &+t^{-1}\,((g_1 \otimes \phi')(y^*y) + (g_2 \otimes \phi')(yy^*))\Big] \end{aligned} \tag{6}$$

Since for all $\alpha\,, \beta > 0$,

$$\inf_{t>0}\,(t\alpha + t^{-1}\beta) = 2\sqrt{\alpha\beta}\,,$$

the conclusion follows by taking infimum over $t > 0$ in (6). $\square$

Lemma 7:

For any $\alpha\,,\beta \geq 0\,,$

$$\inf_{n\in\mathbb{Z}} (\lambda^n\alpha + \lambda^{-1}\beta) \leq (\lambda^{1/2} + \lambda^{-1/2})\sqrt{\alpha\beta}\,.$$

Proof of the Effros-Ruan conjecture:

Let $0 < \lambda < 1$ and let $(\mathcal{M}\,,\phi)$ be the Powers factor of type III_λ with product state ϕ, as before. Set

$$C(\lambda) := \sqrt{\big(\lambda^{1/2} + \lambda^{-1/2}\big)/2}\,.$$

Let $u : A \times B \to \mathbb{C}$ be a j.c.b. bilinear form on C*-algebras $A\,,B$. Let $f_1\,,f_2 \in S(A)$ and $g_1\,,g_2 \in S(B)$ be states as in Proposition 6. We will prove that for all $a \in A$ and $b \in B$,

$$|u(a,b)| \leq C(\lambda)\|u\|_{\mathrm{jcb}}\left(f_1(aa^*)^{\frac{1}{2}}g_1(b^*b)^{\frac{1}{2}} + f_2(a^*a)^{\frac{1}{2}}g_2(bb^*)^{\frac{1}{2}}\right) \quad (7)$$

that is, the Effros-Ruan conjecture holds with constant $C(\lambda)$. Since $C(\lambda) \to 1$ as $\lambda \to 1$, by a simple compactness argument it follows that the conjecture also holds with constant 1.

To prove (7), let $n \in \mathbb{Z}$ and choose $c_n \in \mathcal{M}$ as in Corollary 2. Then

$$\phi(c_n^*c_n) = \lambda^{-n/2}\,, \quad \phi(c_nc_n^*) = \lambda^{n/2}$$

and $\langle c_nJc_nJ\xi_\phi\,,\xi_\phi\rangle_{H_\phi} = 1\,.$

Then, for all $a \in A$ and $b \in B$,

$$\widehat{u}(a\otimes c_n\,, b\otimes Jc_nJ) = u(a\,,b)\langle c_nJc_nJ\xi_\phi\,,\xi_\phi\rangle_{H_\phi} = u(a,b)\,.$$

By Proposition 6, it follows that

$$\begin{aligned}
|u(a,b)|^2 &= |\widehat{u}(a\otimes c_n\,, b\otimes Jc_nJ)|^2 \\
&\le \|u\|_{\rm jcb}^2\Big[\,(f_1(aa^*)\phi(c_nc_n^*)+f_2(a^*a)\phi(c_n^*c_n))\,\cdot \\
&\qquad\qquad \cdot\,(g_1(b^*b)\phi({c_n}^*c_n)+g_2(bb^*)\phi(c_n{c_n}^*))\Big] \\
&= \|u\|_{\rm jcb}^2\Big[\,\Big(\lambda^{n/2}f_1(aa^*)+\lambda^{-n/2}f_2(a^*a)\Big)\,\cdot \\
&\qquad\qquad \cdot\,\Big(\lambda^{-n/2}g_1(b^*b)+\lambda^{n/2}g_2(bb^*)\Big)\,\Big] \\
&= \|u\|_{\rm jcb}^2\Big[f_1(aa^*)g_1(b^*b)+f_2(a^*a)g_2(bb^*)+ \\
&\qquad\qquad +\lambda^n f_1(aa^*)g_2(bb^*)+\lambda^{-n}f_2(a^*a)g_1(b^*b)\Big]\,.
\end{aligned}$$

Since $\lambda^{1/2}+\lambda^{-1/2}=2C(\lambda)^2$, we deduce by Lemma 7 that

$$\begin{aligned}
|u(a,b)|^2 &\le \|u\|_{\rm jcb}^2\Big[f_1(aa^*)g_1(b^*b)+f_2(a^*a)g_2(bb^*)+ \\
&\qquad +2C(\lambda)^2 f_1(a^*a)^{\frac12}g_1(b^*b)^{\frac12}f_2(aa^*)^{\frac12}g_2(bb^*)^{\frac12}\Big] \\
&\le C(\lambda)^2\|u\|_{\rm jcb}^2\Big[f_1(aa^*)^{\frac12}g_1(b^*b)^{\frac12}+f_2(a^*a)^{\frac12}g_2(bb^*)^{\frac12}\Big]^2,
\end{aligned}$$

wherein we have used the fact that $C(\lambda)>1$.

The inequality (7) follows now by taking square roots, and the proof is complete.