Generalized Ornstein-Uhlenbeck semi-groups on stratified groups

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The Ornstein-Uhlenbeck semi-group on $\mathbb{R}^n$:

$$e^{-tN_0}(f)(x) = c_n \int_{\mathbb{R}^n} f(e^{-t}x + \sqrt{1-e^{-2t}}y)e^{-\frac{|y|}{2}^2}dy.$$

1. Notation:

- $G = \exp \mathcal{G}$ is a stratified group (of finite dimensional matrices)
- $\mathcal{G} = V_1 \oplus .. \oplus V_k$ is a finite dimensional Lie algebra, $[V_i, V_j] \subset V_{i+j}$
- $(Z_j)_{j=1}^N$ is a basis of $\mathcal{G}$ respecting the V_j's, so

$$g = \exp\Big(\sum_{j=1}^{N} z_j Z_j\Big) \leftrightarrow (z_1, .., z_N)$$

- $(G, dg) = (\mathbb{R}^N, dz)$ as measured spaces, dg left and right invariant
- $Z \in \mathcal{G}$ defines a left invariant derivation:

$$Zf(g) = \lim_{t \to 0} t^{-1} \Big(f(g \exp tZ) - f(g)\Big)$$

- $s > 0$, **dilation** on $\mathcal{G}$: $\delta_s Z = s^j Z$ if $Z \in V_j$, hence on G :

$$\delta_s(\exp(X + Y + ..) = \exp(sX + s^2 Y + ..) = \exp Z(s)$$

• $\delta_s(f)(g) = f \circ \delta_s(g)$ and $\delta_s(f) = s^A, A$ a first order differential operator, e.g. if G has two layers and $g = \exp(X+U)$, $A = X + 2U, X \in V_1, U \in V_2$; in general, if $\delta_s g = expZ(s)$,

$$A(f)(g) = \frac{d}{ds}\mid_{s=1} f(\delta_s g) = \Big[Z'(1) + \sum_{l=1}^{k-2} \frac{(-1)^l}{(l+1)!}(AdZ(1))^l(Z'(1))\Big](f)(g)$$

• $(X_i)_{i=1}^n$ a basis of the first layer V_1, $\quad \nabla f = (X_i f)_{i=1}^n$,

• $L = -\sum_{i=1}^n X_i^2$ the subLaplacian

$$\delta_{t^{-1}} L \delta_t = t^2 L \Longrightarrow [L, A] = 2L$$

• the heat semi-group $e^{-\frac{1}{2}tL}(f) = f * p_t$ and $p = p_1$ ($p_t dg$ = probability)

$$f * p_{t^2}(\gamma) = \int_G f(\gamma \delta_t g^{-1}) p(g) dg$$

2. The semi-group $e^{-t\nabla^*\nabla}$ on $L^q(pdg)$, $1 \leq q \leq \infty$

The sesquilinear form with domain $H^1(p) = \{f, |\nabla f| \in L^2(pdg)\}$

$$a(f,h) = \int_G (\nabla f.\overline{\nabla h})pdg = \int_G \sum_{i=1}^n X_i f \overline{X_i h} pdg;$$

defines $\nabla^*\nabla$, self-adjoint generator of a strongly continuous semi-group of contractions $e^{-t\nabla^*\nabla}$ on $L^2(pdg)$.
Hence $e^{-t\nabla^*\nabla}$ has the properties:

• it is measure preserving, i.e. $\int_G e^{-t\nabla^*\nabla} fpdg = \int_G fpdg$.

• it is positivity preserving, hence Markovian,

• it extends as a strongly continuous semi-group of contractions on $L^q(pdg)$, $1 \leq q \leq \infty$.
Obviously

$$\nabla^*\nabla = \sum_{i=1}^n X_i^* X_i = L - \frac{\nabla p}{p}\nabla.$$

Remark: Considering e^{-tL}, H.Q. Li proved a Log-Sobolev inequality for the measure pdg on the 3-dimensional Heisenberg group $G = \mathbb{H}_1$. This implies in particular the Poincaré inequality

$$\left\| f - \int_{\mathbb{H}_1} f p dg \right\|_{L^2(p)}^2 \leq C \int_{\mathbb{H}_1} |\nabla f|^2 \, p dg.$$

Is it true for all stratified G?

3. The semi-group e^{-tN} on $L^q(pdg)$, $1 \le q \le \infty$

$$\begin{aligned} T_t f(\gamma) &= \int_G f(\delta_{e^{-t}}\gamma\delta_{\sqrt{1-e^{-2t}}}g)p(g)dg \\ &= \delta_{\cos\theta} \circ e^{-\frac{1}{2}\sin^2\theta L}(f)(\gamma)\ if\ e^{-t} = \cos\theta. \end{aligned}$$

By a change of variables,

$$\int\int_{G^2} f(\delta_{\cos\theta}\gamma\delta_{\sin\theta}g)p(\gamma)p(g)d\gamma dg = \int_G f(\gamma)p(\gamma)d\gamma,$$

i.e. T_t is measure preserving. In other words, if γ, g are independant $G-$valued random variables with law pdg, the r.v. $\delta_{\cos\theta}\gamma\delta_{\sin\theta}g$ has the same law pdg.

Remark: Crepel and Raugi ('78) proved a central limit theorem for i.i.d centered random variables with values in G and law μ with order 2 moments.

The limit law has a density which is the kernel at time 1 of a diffusion semi-group whose generator satisfies the same dilation relation as L.

Basic properties of $(e^{-tN}) = (T_t)$
• it is a semi-group (change of variables), hence a Markovian semi-group of contractions on $L^q(G, pd\gamma)$, $1 \leq q \leq \infty$,

• it is strongly continuous if $q \neq \infty$,

• for $f \in L^q(pd\gamma)$, $1 \leq q < \infty$, : $\lim_{t\to\infty} \left\| T_t(f) - \int_G fpdg \right\|_{L^q(pd\gamma)} = 0$
• the generator $-N$ is

$$N(f)(\gamma) = L(f)(\gamma) + \frac{d}{ds} |_{s=1} f(\delta_s \gamma) = (L + A)f(\gamma).$$

In particular, since $\int_G e^{-tN}(f)pdg = \int_G fpdg$, one has

$$\int_G (Nf)pdg = 0, f \in \mathcal{S}(G)$$
$$\iff \quad (L - A - QI)p = 0,$$

where $Q = dimV_1 + 2dimV_2 + \ldots$.

The last equation is precisely the PDE whose p is the unique solution with integral 1 in $L^1(G)$.

Proposition 1 *N is not symmetric, but*

$$\int_G |\nabla f|^2 \, p dg = \mathfrak{Re} \langle Nf, f \rangle \ \ (= \langle Nf, f \rangle \ \textit{if } f \textit{ is real valued})$$

Proof:

$$(N - \nabla^* \nabla) f = A(f) + \sum_{1 \le j \le n} \frac{X_j p}{p} X_j f = \sum_{1 \le j \le N} b_j(g) Z_j f = B(f)$$

$$\Longrightarrow \langle Nf, h \rangle - \langle f, Nh \rangle = \langle Bf, h \rangle - \langle f, Bh \rangle$$
$$\langle Bf, h \rangle = - \int_G f [\sum_{1 \le j \le N} b_j (Z_j \overline{h}) p + \overline{h} Z_j (b_j p)] dg$$

$$\int_G B(f) p dg = 0 \ \ \forall f \in \mathcal{S}(G) \ \Longrightarrow \sum_{1 \le j \le N} Z_j (b_j p) = 0 \Longrightarrow \langle Bf, h \rangle = - \langle f, Bh \rangle$$

$$\Longrightarrow \langle Nf, h \rangle - \langle f, Nh \rangle = 2 \langle Bf, h \rangle .$$

More properties:

Proposition 2 $e^{-tN}, t > 0$, *has the properties:*

• *It is Hilbert-Schmidt on* $L^2(pd\gamma)$ *hence compact on* $L^q(pd\gamma)$, $1 < q < \infty$.

• *Its non zero eigenvalues and corresponding eigenspaces are the same on* $L^2(pd\gamma)$ *and* $L^q(pd\gamma)$. *Hence its spectrum is* $\{0\} \cup e^{-\sigma(N)}$, *where* $\sigma(N)$ *is the spectrum of* N.

Proof: Computing the kernel of $\cos^N \theta$, one must prove the convergence of

$$I(\theta) = \int\int_{G^2} p^2(\delta_{\frac{1}{\sin\theta}} z \delta_{\frac{\cos\theta}{\sin\theta}} \gamma^{-1}) \frac{p(\gamma)}{p(z)} dz d\gamma,$$

which comes from the gaussian estimates [C,SC,V]: for $\varepsilon, \varepsilon' > 0$ there exist constants C_ε and $K_{\varepsilon'}$ such that

$$C_\varepsilon e^{-\frac{1}{2-2\varepsilon} d^2(g)} \leq p(g) \leq K_{\varepsilon'} e^{-\frac{1}{2+2\varepsilon'} d^2(g)}. \quad \blacksquare$$

Remark: By similar estimates for ∇p one gets: $\frac{|\nabla p|}{p} \in L^q(pdg)$, $1 \leq q < \infty$.

Theorem 3 *The spectrum $\sigma(N)$ on $L^q(pdg), 1 \le q < \infty$, satisfies*
a) $\mathbb{N} \subset \sigma(N)$
b) $\mathbb{N} = \sigma(N)$ *if G is step 2, i.e.* $G = V_1 + V_2$.

Proof: **a) How to get eigenvectors of $\cos^N \theta$ from eigenvectors of L**

In the gaussian case, one considers $\varphi(x) = e^{i\langle \xi, x\rangle}$. Here let $\varphi \in \mathcal{C}^\infty(G)$ be an eigenvector of L and its signed dilation

$$\varphi_\alpha = \varphi \circ \delta_\alpha,\ \alpha \ge 0,\ \varphi_\alpha = \varphi \circ \tau \circ \delta_{|\alpha|},\ \alpha < 0,$$

where τ is the homomorphism of $\mathcal{G}$ such that $\tau(X_i) = -X_i$. Then

$$L\varphi = \lambda\varphi \Longrightarrow L\varphi_\alpha = \alpha^2 \lambda \varphi_\alpha$$

For

$$f_{(\alpha)} = e^{\frac{1}{2}\alpha^2\lambda}\varphi_\alpha = e^{\frac{1}{2}L}\varphi_\alpha,$$

$$\cos^N \theta(f_{(\alpha)}) = \delta_{\cos\theta} e^{\frac{1}{2}(1-\sin^2\theta)L}\varphi_\alpha = \delta_{\cos\theta} e^{\frac{1}{2}\alpha^2\lambda\cos^2\theta}\varphi_\alpha = f_{(\alpha\cos\theta)}.$$

Hence, defining

$$h_n = \frac{d^n}{d\alpha^n} \mid_{\alpha=0} f_{(\alpha)},$$

$$\begin{aligned}\frac{d^n}{d\alpha^n} \mid_{\alpha=0} f_{(\alpha\cos\theta)} &= \cos^n\theta h_n = \frac{d^n}{d\alpha^n} \mid_{\alpha=0} \cos^N\theta f_{(\alpha)} \\ &= \cos^N\theta \frac{d^n}{d\alpha^n} \mid_{\alpha=0} f_{(\alpha)} = \cos^N\theta h_n.\end{aligned}$$

These h_n's are polynomials with respect to the coordinates of g, hence they lie in $L^q(pdg)$, $1 \le q < \infty$.

How to find eigenvectors of L in $C^{\infty}(G)$?

Let Π be a non trivial irreducible unitary representation of G on some $L^2(\mathbb{R}^k, d\xi)$, inducing a representation of $\mathcal{G}$. Then

$$\Pi(e^{-tL}) = e^{-t\Pi(L)} = \Pi(p_t) = \int_G \pi(g)p_t(g)dg$$

is a semi-group of trace class operators since $p_t \in \mathcal{S}(G)$, and $\Pi(p_t)(F) \in \mathcal{C}^{\infty}(G)$ for $F \in L^2(\mathbb{R}^k, d\xi)$.

Then, if F_0 is an eigenvector of $\Pi(L)$,

$$\varphi(g) = \langle \Pi(g)F_0, F\rangle \in \mathcal{C}^{\infty}(G)$$

is an eigenvector of L since

$$L\left(\Pi(g)\right) = \Pi(g)\Pi(L).$$

Moreover free vectors F in $\in L^2(\mathbb{R}^k, d\xi)$ give free vectors φ in $\in L^{\infty}(G)$.

The technicalities in the above computation work for these eigenvectors.

b) Do we get a total set of eigenvectors of $\cos^N\theta$ in $L^2(pdg)$?

The eigenspaces of $\cos^N\theta$ in $L^2(pdg)$ are not pairwise orthogonal!

By Plancherel formula for G, a function $\psi \in L^2(G)$ is zero as soon as $\Pi(\psi) = 0$ for all (equivalence classes of) representations supporting the Plancherel measure. Denote this set by $\mathcal{U}$. These representations act on the same $L^2(\mathbb{R}^k, d\xi)$. For $\Pi \in \mathcal{U}$, let $\mathcal{B}_\Pi$ be a basis of eigenvectors of $\Pi(L)$ in this space. Then it is easy to see that

$$\mathcal{F} = \{\varphi^{\Pi,\mu,\mu'} = \langle \Pi(.)F_\mu, F_{\mu'}\rangle \mid \Pi \in \mathcal{U}, F_\mu, F_{\mu'} \in \mathcal{B}_\Pi\}$$

is total in $L^2(pdg)$.

Let $\mathcal{H}_{\Pi,\mu,\mu'}$ be the set of eigenvectors h_n obtained from $\varphi^{\Pi,\mu,\mu'} \in \mathcal{F}$. It is enough to show that a function ψ orthogonal to $\mathcal{H}_{\Pi,\mu,\mu'}$ is orthogonal to $\varphi^{\Pi,\mu,\mu'}$. For $\alpha \in \mathbb{R}$, let

$$u(\alpha) = \int_G e^{\frac{1}{2}L}(\varphi_\alpha^{\Pi,\mu,\mu'})\psi pdg.$$

We have to show

$$\int_G \frac{d^n}{d\alpha^n}\ |_{\alpha=0}\ e^{\frac{1}{2}L}(\varphi_\alpha^{\Pi,\mu,\mu'})\psi p dg \overset{?}{=} \frac{d^n}{d\alpha^n}u(0) = 0\ \forall n \overset{?}{\Longrightarrow} u = 0.$$

This needs a holomorphic extension of u, hence a holomorphic extension of the dilation: $\omega \to \varphi_\omega^{\Pi,\mu,\mu'}$, with a control of $\varphi_\omega^{\Pi,\mu,\mu'}$ in $L^2(pdg)$.

Has $\varphi^{\Pi,\mu,\mu'}$ complex dilations in general?

When G is step 2, it is easy to see that $\varphi^{\Pi,\mu,\mu'}(g) = \varphi^{\Pi,\mu,\mu'}(x_1,..,x_n,u_1,..,u_l)$ is real analytic on $\mathbb{R}^{n+l}$, because then it is also an eigenvector of the full Laplacian on G, which is an elliptic operator with polynomial coefficients.

But we need more, because of technicalities.

In the Heisenberg case, (n is even and the central layer has dimension one), the set $\mathcal{U}$ only includes the Schrödinger representation Π_0, its dilations $\Pi_0 \circ \delta_\alpha, \alpha > 0$, and a slight modification of these.

$\Pi_0(L)$ is the harmonic oscillator on $\mathbb{R}^{\frac{n}{2}}$, so the eigenvectors $\varphi^{\Pi_0,\mu,\mu'}$ are explicitly known.

There is an explicit formula for the Fourier transform of the heat kernel p w.r. to the central variable, and this allows the computations.

In the general step 2 case, using a little more on the representations in $\mathcal{U}$ (which remain "simple"), one can again compute the eigenvectors $\varphi^{\Pi,\mu,\mu'} \in \mathcal{F}$, which are similar to those in the Heisenberg case.

One then uses the formula for p obtained by Cygan ('79).