Dilations on noncommutative L^p-spaces

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Noncommutative L^p-spaces

• Let M be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace τ. For any $1 \leq p < \infty$, define

$$\|x\|_p = \big(\tau(|x|^p)\big)^{\frac{1}{p}}$$

on a suitable w^*-dense subspace $\mathcal{S} \subset M$.

Then $\| \ \|_p$ is a norm on $\mathcal{S}$ and by definition

$$L^p(M) = \big(\mathcal{S}, \| \ \|_p\big)^-$$

is the resulting completion. This includes:

$\hookrightarrow$ Commutative L^p-spaces $L^p(\Omega, \mu)$, associated to $M = L^\infty(\Omega, \mu)$.

$\hookrightarrow$ Schatten spaces $S^p(H)$, associated to $B(H)$.

• Noncommutative $L^p(M)$ also have a definition in the non semifinite case (Haagerup, 1979).

Dilations

- Let $T\colon L^p(M) \to L^p(M)$ be a power bounded operator:

$$\exists\, C \geq 1, \qquad \|T^n\| \leq C, \quad n \geq 0.$$

Definition 1. We say that T admits a **loose dilation** if there exist another noncommutative L^p-space $L^p(M')$, two bounded operators

$$J\colon L^p(M) \to L^p(M') \quad \text{and} \quad Q\colon L^p(M') \to L^p(M),$$

and an invertible operator $U\colon L^p(M') \to L^p(M')$ such that the set $\{U^n \ : \ n \in \mathbb{Z}\}$ is bounded (equivalently, U and U^{-1} are power bounded) and

$$\mathbf{T^n = QU^nJ}, \qquad \mathbf{n \geq 0}.$$

$$\begin{array}{ccc} L^p(M') & \xrightarrow{U^n} & L^p(M') \\ {\scriptstyle J}\uparrow & & \downarrow{\scriptstyle Q} \\ L^p(M) & \xrightarrow{T^n} & L^p(M) \end{array}$$

Definition 2. Assume that $\|T\| \leq 1$. We say that T admits a **tight dilation** if the above property holds with

$$\|J\| \leq 1, \quad \|Q\| \leq 1, \quad U \text{ is an isometry.}$$

• Let $(T_t)_{t\geq 0}$ be a bounded c_0-semigroup (= strongly continuous semigroup on $L^p(M)$.

Definition 1'. We say that $(T_t)_{t\geq 0}$ admits a **loose dilation** if there is a space $L^p(M')$, two bounded operators

$$J\colon L^p(M) \to L^p(M') \quad \text{and} \quad Q\colon L^p(M') \to L^p(M),$$

and a bounded c_0-group $(U_t)_{t\in\mathbb{R}}$ on $L^p(M')$ such that

$$\mathbf{T_t = QU_tJ}, \qquad \mathbf{t \geq 0}.$$

Definition 2'. Assume that $\|T_t\| \leq 1$ for any $t \geq 0$. We say that T admits a **tight dilation** if the above property holds with

$$\|J\| \leq 1, \quad \|Q\| \leq 1, \quad (U_t)_t \text{ is an group of isometries.}$$

Hilbert spaces

Observations.

T admits a tight dilation iff $\|T\| \leq 1$ (Nagy's Theorem).

T admits a loose dilation iff T is similar to a contraction.

$(T_t)_{t\geq 0}$ admits a tight dilation iff $\|T_t\| \leq 1$ for any $t \geq 0$.

$(T_t)_{t\geq 0}$ admits a loose dilation iff it is similar to a contraction semigroup.

Old results.

There exist power bounded operators without a loose dilation.

(Foguel, 1964)

There exist bounded c_0-semigroups without a loose dilation.

(Packel, 1969)

Commutative L^p-spaces when $1 < p \neq 2 < \infty$.

• For $T\colon L^p(\Omega) \to L^p(\Omega)$ we have a notion of (either tight or loose) **commutative dilation** when we have a dilation property as above

$$\begin{tikzcd} L^p(\Omega') \arrow[r, "U^n"] & L^p(\Omega') \arrow[d, "Q"] \\ L^p(\Omega) \arrow[u, "J"] \arrow[r, "T^n"] & L^p(\Omega) \end{tikzcd}$$

for some <u>commutative</u> $L^p(\Omega')$.

Akcoglu's Theorem (1977). Any positive contraction T on $L^p(\Omega)$ admits a commutative tight dilation.

Characterization (Peller, 1983). Let $T\colon L^p(\Omega) \to L^p(\Omega)$ be a contraction. Then T admits a commutative tight dilation iff there exists a positive contraction $S\colon L^p(\Omega) \to L^p(\Omega)$ such that

$$|T(x)| \leq S(|x|), \qquad x \in L^p(\Omega).$$

Remark. There exist contractions $T\colon L^p(\Omega) \to L^p(\Omega)$ without a tight dilation.

Proposition (Junge-LeM). The above characterization remains true without the word *commutative*, that is, if $T\colon L^p(\Omega) \to L^p(\Omega)$ has a tight dilation through a possibly noncommutative L^p-space, then it also has a tight dilation through a commutative L^p-space.

- **What about loose dilations?**

Theorem. There exists power bounded operators on $L^p(\Omega)$ without a loose dilation.
(Analogue of Foguel's Theorem on L^p-spaces.)

Open question. Does any contraction $T\colon L^p(\Omega) \to L^p(\Omega)$ admit a loose dilation?

This is related to:

Matsaev Conjecture. Let $\sigma_p\colon \ell^p_{\mathbb{Z}} \to \ell^p_{\mathbb{Z}}$ be the shift operator and for any polynomial F, let $N_p(F) = \|\sigma_p\|$. Let $T\colon L^p(\Omega) \to L^p(\Omega)$ be a contraction. Is there a constant $C \geq 1$ such that

$$\|F(T)\| \leq C N_p(F)$$

for any polynomial F?

Same question with $C = 1$?

• What about c_0-semigroups?

There is an analogue of Akcoglu's Theorem due to Fendler (1997).

The general picture is similar.

The (trivial) case $p = 1$

Universal construction. Let X be a Banach space.

Let $T\colon X \to X$ be a power bounded operator.

Let $J\colon X \to \ell^1_{\mathbb{Z}}(X)$ be defined by

$$J(x) = e_0 \otimes x = (\cdots 0, \cdots, 0, x, 0, \cdots, 0, \cdots)$$

and let $Q\colon \ell^1_{\mathbb{Z}}(X) \to X$ be defined by

$$Q\big[(a_n)_{n\in\mathbb{Z}}\big] \;=\; \sum_{n\geq 0} T^n(a_n)\,.$$

Then

$$\|J\| = 1 \qquad \text{and} \qquad \|Q\| \leq \sup\{\|T^n\| \,:\, n \geq 0\}.$$

Let $U\colon \ell^1_{\mathbb{Z}}(X) \to \ell^1_{\mathbb{Z}}(X)$ be the shift operator. This is an invertible isometry and

$$T^n = QU^nJ, \qquad n \geq 0.$$

Conclusion. Any power bounded operator (resp. any contraction) on a noncommutative L^1-space admits a loose (resp. tight) dilation.

- A similar construction holds for c_0-semigroups.

Noncommutative L^p-spaces when $1 < p \neq 2 < \infty$.

Complete positivity. Let M be a von Neumann algebra and for any integer $n \geq 1$, identify

$$M_n \otimes L^p(M) \approx L^p\big((M_n(M)\big).$$

An operator $T\colon L^p(M) \to L^p(M)$ is called completely positive if for any $n \geq 1$,

$$I_{M_n} \otimes T\colon L^p\big((M_n(M)\big) \longrightarrow L^p\big((M_n(M)\big) \quad \text{is positive.}$$

Theorem (Junge-LeM). There exists a completely positive contraction $T\colon S^p \to S^p$ without any tight dilation.

Dilations and rigid factorizations on noncommutative L^p-spaces, JFA 249 (2007), 220-252.

Factorizable operators. Let $\frac{1}{p} + \frac{1}{q} = 1$. We say that a contraction $T\colon L^p(M) \to L^p(M)$ is factorizable if there exist another von Neumann algebra M' and two linear isometries

$$T_1\colon L^p(M) \longrightarrow L^p(M') \quad \text{and} \quad T_2\colon L^q(M) \longrightarrow L^q(M')$$

such that $T = T_2^* \circ T_1$.

It is straightforward that

$$\text{Tight dilation} \implies \text{factorizable},$$

and we show the existence of a contraction $T\colon S^p \to S^p$ which is completely positive but not factorizable.

The proof uses *Yeadon's description of isometries (1981).*

Semigroups. We also obtain that there exists a completely positive contraction c_0-semigroup $(T_t)_{t\geq 1}$ without a tight dilation.

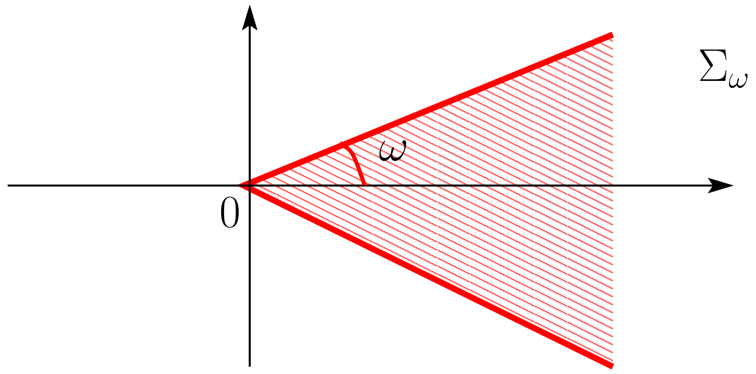
Characterization? At the moment, there is no intrinsic description of the contractions (resp. contraction c_0-semigroups) on $L^p(M)$ which admit a tight dilation.

Loose dilations. As in the commutative case, we do not know whether any contraction $T\colon L^p(M) \to L^p(M)$ admits a loose dilation, or whether any contraction c_0-semigroup on $L^p(M)$ admits a loose dilation.

H^∞ functional calculus

Sectorial operators. For any $\omega \in (0, \pi)$, let

$$\Sigma_\omega = \{z \in \mathbb{C}^* \, : \, |\mathrm{Arg}(z)| < \omega\}.$$



Let $A \colon D(A) \subset X \to X$ be a closed and densely defined operator. It is called sectorial of type ω if

$$\sigma(A) \subset \overline{\Sigma_\omega}$$

and for any $\theta > \omega$, there exists a constant $K_\theta > 0$ such that

$$\|z(z-A)^{-1}\| \leq K_\theta, \qquad z \notin \overline{\Sigma_\theta}.$$

Semigroups. If $(T_t)_{t \geq 0}$ is a bounded c_0-semigroup on X, let $-A$ be its infinitesimal generator ($T_t = e^{-tA}$). Then A is sectorial of type $\frac{\pi}{2}$.

Construction (McIntosh, 1986). Let $0 < \omega < \theta < \pi$. For any bounded analytic function $F\colon \Sigma_\theta \to \mathbb{C}$, set

$$\|F\|_{\infty,\theta} = \sup\{|F(z)| \,:\, z \in \Sigma_\theta\}.$$

Assume that

$$\int_\Gamma |F(z)| \left|\tfrac{dz}{z}\right| < \infty,$$

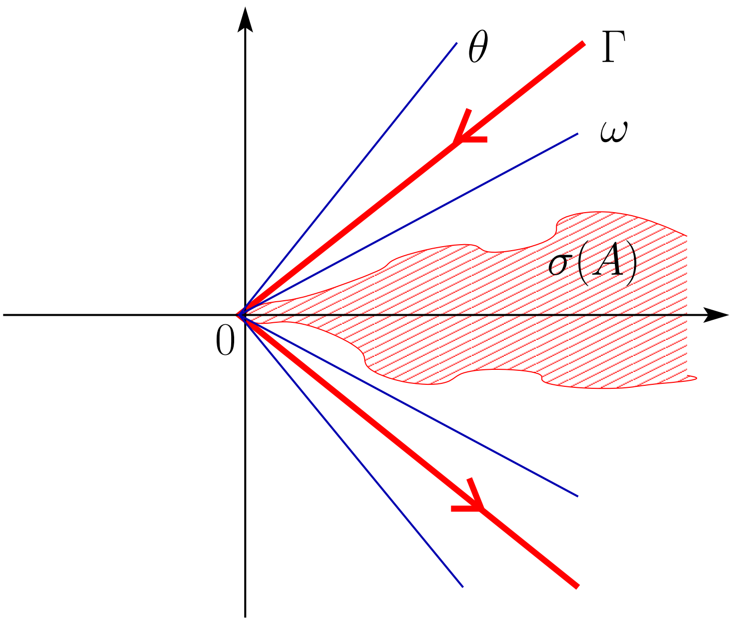
and define

$$F(A) = \frac{1}{2\pi i}\int_\Gamma F(z)\,(z-A)^{-1}\,dz$$

We say that A admits a bounded H^∞_θ-functional calculus if there is a constant $C \geq 1$ such that

$$\|F(A)\| \leq C\|F\|_{\infty,\theta}$$

for any such F.



Note. H^∞ calculus studied in noncommutative L^p-spaces in

[JLX] Junge-LeM-Xu, *H^∞-functional calculus and square functions on noncommutative L^p-spaces*, Astérisque 305 Soc. Math. France, 2006.

Proposition. Assume that a bounded semigroup $(T_t)_{t\geq 0}$ admits a loose dilation. Let $-A$ be its generator. Then A has a bounded H^∞_θ-functional calculus for any $\theta > \frac{\pi}{2}$.

Proof. Assume a loose dilation

$$T_t = QU_tJ, \qquad t \geq 0,$$

for some bounded c_0-group $(U_t)_t$ on some $L^p(M')$.
Let $-B$ be the infinitesimal generator of $(U_t)_t$, so that $T_t = e^{-tA}$ and $U_t = e^{-tB}$. Then for any F as above,

$$F(A) = QF(B)J,$$

$$\|F(A)\| \leq \|Q\|\|J\|\|F(B)\|.$$

Hence it suffices that B has a bounded H^∞_θ-functional calculus for any $\theta > \frac{\pi}{2}$.
This is correct (Hieber-Prüss, 1998) by transference, which allows to change B into the derivation operator $\frac{d}{dt}$ on $L^p(\mathbb{R}; L^p(M'))$, and then by multiplier theorems on UMD spaces.

Open problem. Let $(T_t)_{t\geq 1}$ be a contraction semigroup on $L^p(M)$, with generator $-A$. Does A admit a bounded H^∞_θ functional calculus for some (for any) $\theta > \frac{\pi}{2}$?

This is already open in the commutative case.

In the noncommutative case, this question is already open for a completely positive contraction semigroup.

Analytic case

Let $T_t = e^{-tA}$ be a bounded c_0-semigroup on X. This is a **bounded analytic semigroup** iff A is sectorial of type $\omega < \frac{\pi}{2}$.

For any $\theta_2 > \theta_1 > \omega$,

$$\text{bounded } H^\infty_{\theta_1} - \text{calculus} \implies \text{bounded } H^\infty_{\theta_2} - \text{calculus}.$$

The converse is not true in general.

Theorem. Let $(T_t)_{t\geq 0}$ be a bounded analytic semigroup, with generator $-A$ on some $L^p(M)$. If A has a bounded H^∞_θ-functional calculus for some $\theta < \frac{\pi}{2}$, then $(T_t)_{t\geq 0}$ admits a loose dilation.

Proof.

Follows Fröhlich-Weis (2006).

Uses square functions.

Sketch

In the commutative case. We consider $X = L^p(\Omega)$ and assume that A has a bounded H^∞_θ-functional calculus for some $\theta < \frac{\pi}{2}$. Then

$$\|x\|_p \approx \Big\| \Big(\int_0^\infty \big| A^{\frac{1}{2}} e^{-tA} x \big|^2 \, dt \Big)^{\frac{1}{2}} \Big\|_p, \qquad x \in L^p(\Omega).$$

(Cowling, Doust, McIntosh, Yagi, 1996).

Let $Y = L^p(\Omega; L^2(\mathbb{R}))$ and let $U_t \colon Y \to Y$ be the translation by t in the second variable,

$$\big[U_t \varphi(\cdot\,)\big](s) = \varphi(t+s).$$

Then $(U_t)_{t\in\mathbb{R}}$ is an isometric c_0-group.
Regard elements of Y as functions $\mathbb{R} \to L^p(\Omega)$ and let $J \colon X \to Y$ be defined by

$$[J(x)](s) = A^{\frac{1}{2}} e^{-sA} x, \qquad \text{for } s > 0$$

and $[J(x)](s) = 0$ for $s < 0$. Then for any $t \geq 0$ and $x \in X$,

$$U_t J(x) = J(T_t(x)) \quad \text{on } \mathbb{R}_+.$$

Thus if we let $P\colon Y \to Y$ be induced by the orthogonal projection $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ onto $L^2(\mathbb{R}_+)$, we have

$$PU_tJ = JT_t, \qquad t \geq 0.$$

Then using an analogous square function estimate for A^*, one shows that there exists $P'\colon Y \to X$ such that $P'J = I_X$.

Finally,

$$P'PU_tJ = T_t$$

for any $t \geq 0$.

Lastly, modify Y into an L^p-space to get the result.

In the noncommutative case. Use the noncommutative square functions from [JLX]. Suppose that $p \geq 2$, let $X = L^p(M)$ and assume that A has a bounded H^∞_θ-functional calculus for $\theta < \frac{\pi}{2}$. Then

$$\|x\|_p \approx \max\Big\{ \Big\| \Big(\int_0^\infty \big| A^{\frac{1}{2}} e^{-tA} x \big|^2 \, dt \Big)^{\frac{1}{2}} \Big\|_p \quad \Big\| \Big(\int_0^\infty \big| \big(A^{\frac{1}{2}} e^{-tA} x\big)^* \big|^2 \, dt \Big)^{\frac{1}{2}} \Big\|_p \Big\}.$$

Etc...

R-sectoriality

R-boundedness. Let $\mathcal{F} \subset B(X)$ be a set of bounded operators on some Banach space X. We say that $\mathcal{F}$ is R-bounded if there is a constant $C > 0$ such that

$$\Big\|\sum_k \varepsilon_k \, T_k(x_k)\Big\|_{\mathrm{Rad}(X)} \leq C\Big\|\sum_k \varepsilon_k \, x_k\Big\|_{\mathrm{Rad}(X)}$$

for any $T_1, \ldots, T_n$ in $\mathcal{F}$ and any $x_1, \ldots, x_n$ in X.

Definitions. Let A be a sectorial operator of type ω on X. We say that A is R-sectorial of R-type ω if for any $\theta > \omega$, the set

$$\left\{z(z-A)^{-1} \, : \, z \notin \overline{\Sigma_\theta}\right\}$$

is R-bounded.

In this holds for some $\omega < \frac{\pi}{2}$ and if $T_t = e^{-tA}$, then we say that $(T_t)_{t\geq 0}$ is an R-bounded analytic semigroup.

Kalton-Weis, 2001. If A is R-sectorial of R-type ω and admits a bounded H^∞_θ-functional calculus for some $\theta > \omega$, then it admits a bounded H^∞_θ-functional calculus for any $\theta > \omega$.

Corollary. Let $(T_t)_{t\geq 0}$ be an R-bounded analytic semigroup, with generator $-A$ on some $L^p(M)$, and let $\theta > \omega$ (the angle of R-type). Then A has a bounded H^∞_θ-functional calculus iff $(T_t)_{t\geq 0}$ admits a loose dilation.

Analogs in the discrete case. Work in progress.

Noncommutative diffusion semigroups

Definition. A (noncommutative) diffusion semigroup is a w^* continuous semigroup $(T_t)_{t\geq 0}$ on M such that

- $\big\|T_t\colon L^p(M) \longrightarrow L^p(M)\big\| \leq 1$ for any $1 \leq p \leq \infty$ and $t \geq 0$.
- $T_t\colon L^2(M) \longrightarrow L^2(M)$ is selfadjoint for any $t \geq 0$.

For any $1 < p < \infty$, write $T_t = e^{-tA_p}$ on $L^p(M)$.
Then A_p is sectorial of some type $\omega_p = \pi|\frac{1}{p} - \frac{1}{2}| < \frac{\pi}{2}$.
(*Stein's interpolation principle*).

Corollary. Fix some $1 < p < \infty$ and let $\theta > \omega_p$. Then A_p has a bounded H^∞_θ-functional calculus iff $(T_t)_{t\geq 1}$ admits a loose dilation.

Nota Bene. In the *commutative* case, any $(T_t)_{t\geq 0}$ as above admits a tight dilation on L^p for any $1 < p < \infty$.

Open problems. For noncommutative diffusion semigroups, do we have:

- A tight dilation?
- A loose dilation? A bounded H^∞-functional calculus?

Schur multipliers

Definition. We regard elements of $B(\ell^2)$ as infinite matrices. Let $A = (a_{ij})_{i,j\geq 1}$ a bounded family of complex numbers. The Schur multiplier associated to A is the mapping

$$[t_{ij}] \in B(\ell^2) \longmapsto [a_{ij}t_{ij}].$$

If it is bounded $B(\ell^2) \to B(\ell^2)$, then it is bounded $S^p \to S^p$ for any $1 < p < \infty$.

Proposition. Let $(\alpha_k)_{k\geq 1}$ and $(\beta_k)_{k\geq 1}$ be two sequences of complex numbers. For any $t \geq 0$, the Schur multiplier T_t associated to

$$\left(e^{-t|\alpha_i - \beta_j|}\right)$$

is bounded on $B(\ell_2)$, and $(T_t)_{t\geq 0}$ is a diffusion semigroup on $B(\ell^2)$. Moreover for each $1 < p < \infty$, the realization of $(T_t)_{t\geq 0}$ on S^p admits a tight dilation.

q-Ornstein-Uhlenbeck semigroups

q-Deformation von Neumann algebras.

(Bozejko-Speicher, Bozejko-Kümmerer-Speicher, 1991-97.)

Let H be a real Hilbert space and let $\mathcal{H}$ be its complexification. For any $q \in [-1, 1]$, we consider

- $\mathcal{F}_q(\mathcal{H})$, the q-Fock space over $\mathcal{H}$;
- $c(h) \colon \mathcal{F}_q(\mathcal{H}) \to \mathcal{F}_q(\mathcal{H})$, the creation operator associated with $h \in H$;
- $\Gamma_q(H) \subset B\big(\mathcal{F}_q(\mathcal{H})\big)$, the von Neumann algebra of q-deformation generated by the q-Gaussians

$$\{c(h) + c(h)^* \,:\, h \in H\}.$$

This von Neumann algebra has a natural trace,

$$\tau(x) = \langle x(\Omega), \Omega\rangle,$$

where $\Omega \in \mathcal{F}_q(\mathcal{H})$ is the vacuum vector.

$- - - - - >$ **Definition of $L^p\big(\Gamma_q(H)\big)$.**

Second quantization. For any contraction $a\colon H \to H$, we let

$$\Gamma_q(a)\colon \Gamma_q(H) \longrightarrow \Gamma_q(H)$$

be the second quantization of a, which is a normal unital completely positive map. One shows that for any $1 \leq p < \infty$,

$$\big\|\Gamma_q(a)\colon L^p\big(\Gamma_q(H)\big) \longrightarrow L^p\big(\Gamma_q(H)\big)\big\| \leq 1.$$

Proposition 1. For any contraction a and for any $1 < p < \infty$, $\Gamma_q(a)\colon L^p \to L^p$ has a tight dilation.

Proposition 2. Let $(a_t)_{t\geq 0}$ be a contraction c_0-semigroup on H and let $T_t = \Gamma_q(a_t)$ for any $t \geq 0$.

(i) Then $(T_t)_{t\geq 0}$ is a contraction c_0-semigroup on $L^p\big(\Gamma_q(H)\big)$ which admits a tight dilation ($1 < p < \infty$).

(ii) If $a_t\colon H \to H$ is selfadjoint, then $(T_t)_{t\geq 0}$ is a diffusion semigroup on $\Gamma_q(H)$.

Example. Let $a_t = e^{-t} I_H$ for $t \geq 0$. The resulting semigroup $(T_t)_{t\geq 0}$ is the so-called q-Ornstein-Uhlenbeck semigroup.

Noncommutative Poisson semigroup

Description. Let $n \geq 1$ be an integer and let $G = \mathbb{F}_n$ be the free group with n generators $c_1, \ldots, c_n$. Let $\lambda\colon G \to B(\ell^2_G)$ be the left regular representation of G and let

$$VN(G) = \lambda(G)'' = \overline{\mathrm{Span}}^{w^*}\big\{\lambda(g) \,:\, g \in G\big\}.$$

This von Neumann algebra has a natural trace,

$$\tau(x) = \langle x(\delta_e), \delta_e\rangle,$$

where $e \in G$ is the unit.

$------>$ **Definition of $L^p\big(VN(G)\big)$.**

Any $g \in G$ has a unique decomposition

$$g = c_{i_1}^{k_1} c_{i_2}^{k_2} \cdots c_{i_m}^{k_m}$$

with $m \geq 0$, $k_j \neq 0$ and $i_j \neq i_{j+1}$. Then

$$|g| = |k_1| + \cdots + |k_m|$$

is **the length of** g.

For any $t \geq 0$, set

$$T_t\big(\lambda(g)\big) = e^{-t|g|}\lambda(g)$$

for any $g \in G$. Then T_t uniquely extends to a normal unital completely positive map $T_t\colon VN(G) \to VN(G)$ (Haagerup, 1979).

It turns out that $(T_t)_{t\geq 0}$ is a diffusion semigroup on $VN(G)$.

Proposition (Lust-Piquard). For each $1 < p < \infty$, the realization of $(T_t)_{t\geq 0}$ on $L^p(VN(G))$ admits a tight dilation.

$\hookrightarrow$ This yields a simpler proof of the result that if $T_t = e^{-tA_p}$ on $L^p(VN(G))$, then

$$A_p \text{ admits a bounded } H^\infty \text{ functional calculus.}$$