# Quantum subgroups of a simple quantum group at roots of 1

Gastón Andrés García

Universidad Nacional de Córdoba, Argentina
CIEM-CONICET

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*Quantum subgroups of a simple quantum group at roots of 1*
Preprint: `arXiv:0707.0070v1`.

*Extensions of finite quantum groups by finite groups*
Preprint: `arXiv:math/0608647v6`.

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- Introduction on Quantum Groups

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**I** Introduction on Quantum Groups

**II** Construction and characterization of the quantum subgroups of a simple quantum group

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**III** Consequences - Applications

# I. Introduction

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Known example (Kassel, Jantzen, *et. al.*): $\mathcal{O}_q(SL_2)$ algebra generated by $a$, $b$, $c$ and $d$ satisfying:

$$\begin{aligned} ab &= qba & ac &= qca & bc &= cb \\ bd &= qdb & cd &= qdc & ad - da &= (q - q^{-1})bc \end{aligned}$$
$$\det_q = ad - qbc = 1.$$

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$\mathbf{u}_q(\mathfrak{g}) \rightsquigarrow$ Frobenius-Lusztig kernel or small quantum group. It is finite-dimensional with $\dim \mathbf{u}_q(\mathfrak{g}) = \ell^{\dim \mathfrak{g}}$.

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Quantum group: "deformation of an associative algebra associated to an algebraic group"

$$G \rightsquigarrow \mathcal{O}(G) \text{ comm. Hopf alg.}$$

$$\mathrm{Alg}(\mathcal{O}(G),\mathbb{C}) = G \longleftrightarrow \mathcal{O}(G) \text{ comm. Hopf alg.}$$

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Strategy:

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(2) Show that any quotient can be constructed in such a way.

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Warning:

These processes may not produce *really* new examples.

It is very difficult (if not impossible) to write them explicitly (extensions, twistings, crossed-products).

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*$H^*$ is parameterized by $(I_+, I_-, \Sigma)$, where $I_+ \subseteq \Pi$, $I_- \subseteq -\Pi$ and $\Sigma \subseteq \mathbb{T}$ such that $K_{\alpha_i} \in \Sigma$ if $\alpha_i \in I = I_+ \cup I_-$.*

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Idea: Make the construction using extensions.

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Let $B$, $A$, be Hopf algebras, $B$ central in $A$ and $A$ right or left faithfully flat over $B$. Let $p : B \to K$ a surjective Hopf algebra map and $H = A/AB^{+}$. Denote $\mathcal{J} = \operatorname{Ker} p$, $(\mathcal{J}) = A\mathcal{J}$.

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Moreover, $A/(\mathcal{J}) \simeq A \otimes_B K$, the base extension through $p$.

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where $A_{\epsilon,\sigma,\mathfrak{l}} := \mathcal{O}_\epsilon(L)/(\mathcal{J})$, $\mathcal{J} = \operatorname{Ker} \, {}^t\sigma$.

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**Lemma**

*$H \simeq \mathbf{u}_\epsilon(\mathfrak{l})^*/(D^z - 1|\ z \in N)$ where $N \subseteq \widehat{\mathbb{T}_{I^c}}$ is determined uniquely by $\Sigma$ (and conversely).*

Let $\delta : N \to \widehat{\Gamma}$ be a group map. Using the coalgebra section $\psi$ of $\pi_L$, we divide out by ideals generated by central elements related to $\Sigma$ and we obtain

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Using the coalgebra section $\psi$ of $\pi_L$, we divide out by ideals generated by central elements related to $\Sigma$: for $z \in N$, $\psi(D^z) \in \mathcal{Z}(A_{\epsilon,\sigma,\mathfrak{l}})$ and if $\delta : N \to \widehat{\Gamma}$ ia a group map we have

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1 \arrow[r] & \mathcal{O}(\Gamma) \arrow[r, "\tilde{\iota}"] & A_{\mathcal{D}} \arrow[r, "\tilde{\pi}"] & H \arrow[r] & 1,
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where $A_{\mathcal{D}} = A_{\epsilon,\sigma,\mathfrak{l}}/(\psi(D^z) - \delta(z)|\ z \in N)$.

# Characterization

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## Theorem

*There is a bijection between*

(a) *Hopf algebra quotients* $q : \mathcal{O}_\epsilon(G) \twoheadrightarrow A$.

(b) *Subgroup data (up to equivalence).*

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## Corollary

*Let $\mathcal{D} = (I_+, I_-, N, \Gamma, \sigma, \delta)$ be a finite subgroup datum such that $I_+ \cap -I_- \neq \emptyset$ and $\sigma(\Gamma) \nsubseteq \mathbf{T}$. Then $A_{\mathcal{D}}$ is non-semisimple, non-pointed and its dual is also non-pointed.*

# Invariants

Let $\widetilde{\Gamma} = \Gamma \times N^{\perp}$ and $\mathbf{u}_{\epsilon}(\mathfrak{l}_0)$ be the Hopf subalgebra of $\mathbf{u}_{\epsilon}(\mathfrak{g})$ determined by the triple $(I_+, I_-, \mathbb{T}_I)$.

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**Lemma**

*$A_{\mathcal{D}}$ fits into the central exact sequence*

$$1 \to \mathcal{O}(\widetilde{\Gamma}) \to A_{\mathcal{D}} \to \mathbf{u}_\epsilon(\mathfrak{l}_0)^* \to 1.$$

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**Theorem**

*Let $\mathcal{D}$ and $\mathcal{D}'$ be subgroup data. If the Hopf algebras $A_{\mathcal{D}}$ and $A_{\mathcal{D}'}$ are isomorphic then $\widetilde{\Gamma} \simeq \widetilde{\Gamma}'$ and $\mathfrak{l}_0 \simeq \mathfrak{l}_0'$.*

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## Theorem

*If $\sigma(\Gamma)$ is not central in $G$, then there exists an infinite family $\{\sigma_j\}_{j\in J} \subset \mathrm{Emb}(\Gamma, G)$ such that the Hopf algebras $\{A_{\sigma_j}\}_{j\in J}$ of dimension $|\Gamma|\ell^{\dim \mathfrak{g}}$ are pairwise non-isomorphic, non-semisimple, non-pointed and their duals are also non-pointed.*