Quantum subgroups of a simple quantum group at roots of 1

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Extensions of finite quantum groups by finite groups

Preprint: arXiv:math/0608647v6.

I Introduction on Quantum Groups

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II Construction and characterization of the quantum subgroups of a simple quantum group

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III Consequences - Applications

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They are dual objects in some sense: $\mathcal{O}_q(G)^\circ \subseteq U_q(\mathfrak{g})$ and $U_q(\mathfrak{g})^\circ \subseteq \mathcal{O}_q(G)$ with $\mathfrak{g} = \mathrm{Lie}(G)$.

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Known example (Kassel, Jantzen, et. al.): $\mathcal{O}_q(SL_2)$ algebra generated by a, b, c and d satisfying:

$$ab=qba$$
 $ac=qca$ $bc=cb$ $bd=qdb$ $cd=qdc$ $ad-da=(q-q^{-1})bc$ $\det_q=ad-qbc=1.$

• Lusztig: if q is a primitive ℓ -th root of 1 $\mathbf{u}_q(\mathfrak{g}) \leadsto$ Frobenius-Lusztig kernel or small quantum group. It is finite-dimensional with $\dim \mathbf{u}_q(\mathfrak{g}) = \ell^{\dim \mathfrak{g}}$.

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Quantum group: "deformation of an associative algebra associated to an algebraic group"

 $G \sim \mathcal{O}(G)$ comm. Hopf alg.

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$$\Gamma_q \hookrightarrow G_q \qquad \Longleftrightarrow \qquad \mathcal{O}_q(G) \twoheadrightarrow A$$

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Strategy:

- (1) Give a general construction of the quotients.
- (2) Show that any quotient can be constructed in such a way.

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These processes may not produce really new examples.

It is very difficult (if not impossible) to write them explicitly (extensions, twistings, crossed-products).

• $\mathfrak{h} \subseteq \mathfrak{g} = Lie(G)$ fixed Cartan subalgebra, $\Pi = \{\alpha_1, \dots, \alpha_n\}$ a basis of the root system $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ and $n = \operatorname{rk} \mathfrak{g}$.

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- $\mathbf{u}_{\epsilon}(\mathfrak{g}) =$ Frobenius-Lusztig kernel; generated by $\{K_{\alpha_i}, E_i, F_i : 1 \leq i \leq n\}$. Denote $\mathbb{T} := \langle K_{\alpha_1}, \dots, K_{\alpha_n} \rangle = G(\mathbf{u}_{\epsilon}(\mathfrak{g}))$ and for $S \subseteq \Pi$, let $\mathbb{T}_S := \langle K_{\alpha_i} : \alpha_i \in S \rangle$.

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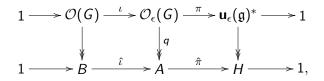
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- (d) π admits a coalgebra section φ .

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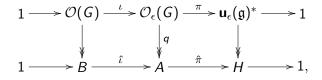
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 H^* is parameterized by (I_+, I_-, Σ) , where $I_+ \subseteq \Pi$, $I_- \subseteq -\Pi$ and $\Sigma \subseteq \mathbb{T}$ such that $K_{\alpha_i} \in \Sigma$ if $\alpha_i \in I = I_+ \cup I_-$.



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Idea: Make the construction using extensions.



Let $\mathbf{u}_{\epsilon}(\mathfrak{l}) \subseteq \mathbf{u}_{\epsilon}(\mathfrak{g})$ be the Hopf subalgebra determined by the triple (I_+, I_-, \mathbb{T}) .

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(c) π_L admits a coalgebra section ψ .

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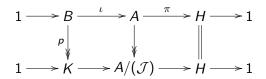
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$$\downarrow \qquad \qquad \parallel \qquad \qquad \downarrow \qquad \qquad \parallel$$

$$1 \longrightarrow K \longrightarrow A/(\mathcal{J}) \longrightarrow H \longrightarrow 1$$

Moreover, $A/(\mathcal{J}) \simeq A \otimes_B K$, the base extension through p.

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$$1 \longrightarrow \mathcal{O}(\Gamma) \xrightarrow{\bar{\iota}} A_{\epsilon,\sigma,\mathfrak{l}} \xrightarrow{\bar{\pi}} \mathbf{u}_{\epsilon}(\mathfrak{l})^{*} \longrightarrow 1,$$

where $A_{\epsilon,\sigma,\mathfrak{l}}:=\mathcal{O}_{\epsilon}(L)/(\mathcal{J})$, $\mathcal{J}=\mathsf{Ker}^{-t}\sigma$.

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Lemma

 $H \simeq \mathbf{u}_{\epsilon}(\mathfrak{l})^*/(D^z-1|\ z\in N)$ where $N\subseteq\widehat{\mathbb{T}_{I^c}}$ is determined uniquely by Σ (and conversely).

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Using the coalgebra section ψ of π_L , we divide out by ideals generated by central elements related to Σ : for $z \in N$, $\psi(D^z) \in \mathcal{Z}(A_{\epsilon,\sigma,\mathfrak{l}})$ and if $\delta: N \to \widehat{\Gamma}$ ia a group map we have

where $A_{\mathcal{D}} = A_{\epsilon,\sigma,\mathfrak{l}}/(\psi(D^z) - \delta(z)|z \in N)$.

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Definition

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A subgroup datum is a collection $\mathcal{D} = (I_+, I_-, N, \Gamma, \sigma, \delta)$

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Theorem

There is a bijection between

- (a) Hopf algebra quotients $q: \mathcal{O}_{\epsilon}(G) \rightarrow A$.
- (b) Subgroup data (up to equivalence).

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Corollary

Let $\mathcal{D}=(I_+,I_-,N,\Gamma,\sigma,\delta)$ be a finite subgroup datum such that $I_+\cap -I_-\neq\emptyset$ and $\sigma(\Gamma)\nsubseteq \mathbf{T}$. Then $A_{\mathcal{D}}$ is non-semisimple, non-pointed and its dual is also non-pointed.

Invariants

Let $\widetilde{\Gamma} = \Gamma \times N^{\perp}$ and $\mathbf{u}_{\epsilon}(\mathfrak{l}_0)$ be the Hopf subalgebra of $\mathbf{u}_{\epsilon}(\mathfrak{g})$ determined by the triple (I_+, I_-, \mathbb{T}_I) .

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 $A_{\mathcal{D}}$ fits into the central exact sequence

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Theorem

Let \mathcal{D} and \mathcal{D}' be subgroup data. If the Hopf algebras $A_{\mathcal{D}}$ and $A_{\mathcal{D}'}$ are isomorphic then $\widetilde{\Gamma} \simeq \widetilde{\Gamma}'$ and $\mathfrak{l}_0 \simeq \mathfrak{l}_0'$.

Consider the subgroup datum $\mathcal{D}=(\Pi,-\Pi,1,\Gamma,\sigma,\varepsilon)$, where Γ is finite and $\varepsilon:1\to\widehat{\Gamma}$ is the trivial group map.

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$$1 \longrightarrow \mathcal{O}(G) \xrightarrow{\iota} \mathcal{O}_{\epsilon}(G) \xrightarrow{\pi} \mathbf{u}_{\epsilon}(\mathfrak{g})^{*} \longrightarrow 1$$

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Theorem

If $\sigma(\Gamma)$ is not central in G, then there exists an infinite family $\{\sigma_j\}_{j\in J}\subset \operatorname{Emb}(\Gamma,G)$ such that the Hopf algebras $\{A_{\sigma_j}\}_{j\in J}$ of dimension $|\Gamma|\ell^{\dim\mathfrak{g}}$ are pairwise non-isomorphic, non-semisimple, non-pointed and their duals are also non-pointed.