# ‘Inner product modules’ over dual operator algebras

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## I. Introduction.

One of the most natural places of application of operator space methods: operator algebras

Indeed, operator algebras are exactly the operator spaces with a product making $M_n(A)$ a Banach algebra

Originally: a much more ambitious survey of brand new examples of operator space methods for operator algebras

Instead I will focus on one part of the survey: 2 recent papers with Upasana Kashyap (September, December 07)

The latter particularly has some nice examples of how operator space methods can be crucial in an operator algebra investigation

**Revisit something I missed ten years ago**

- **Some definitions:**

**Dual operator algebra:** $\sigma$-weakly ($=$ weak*) closed subalgebra of $B(H)$

**Theorem** (Le Merdy/B/B-Magajna) Dual operator algebras are exactly the operator algebras with an operator space predual

- I will assume (for convenience) all dual operator algebras have an identity of norm $1$

**Normal representation:** completely contractive unital weak* continuous $\pi : M \to B(H)$

Makes $H$ a dual operator $M$-module

**Dual operator $M$-module:** weak* closed $X \subset B(K, H)$ with $\pi(M)X \subset X$, $\pi$ as above

**or** $=$ Dual operator space and $M$-module such that module action is separately weak* continuous

(Studied first by Effros-Ruan in the $W^*$-algebra case)

## II: A generalization of $W^*$-modules (December 07)

Recall...strong Morita equivalence/$C^*$-modules for $C^*$-algebras are not appropriate for $W^*$-algebras. Need a $W^*$-algebra/weak* topology variant originally due to Paschke:

A **$W^*$-module** is a Hilbert $C^*$-module $Y$ over a von Neumann algebra $M$ which is selfdual, i.e. every $T \in B(Y, M)_M$ is just 'taking the inner product with a fixed element'

Or equivalently which has a predual (Zettl, Effros-Ozawa-Ruan)

**Thus we have a separately weak* continuous $M$-valued inner product on $Y$, satisfying obvious analogues of the Hilbert space inner product, e.g.**

$$\langle y|y\rangle \geq 0$$

$$\langle y|za\rangle = \langle y|z\rangle a \quad , \quad a \in M$$

**They behave just like Hilbert spaces ...**

**A fundamental notion. For example, it implements the 'induced representations' functor of Rieffel:**

$$H \rightsquigarrow Y \bar{\otimes}_\theta H$$

**the 'composition tensor product' (many other names)**

**For 'operator spacers, $W^*$-modules are much the same thing as WTROs (= weak* closed operator space $Z \subset B(K, H)$ s.t. $ZZ^*Z \subset Z$)**

**(Studied by Ruan and many others)**

**The original $W^*$-algebraic theories are a big package of important results, tools, consequences, and ways of looking at things. In the (selfadjoint) theory, 'inner product modules' are essentially the same thing as Morita equivalence**

**However in the nonselfadjoint setting, this is not true, as simple $2 \times 2$ examples show**

Why generalize it to dual operator algebra? **A few of the reasons...**

- **These are beautiful and fundamental ideas, for example 'induced representations'**

- **Imbed the $C^*$-module universe in a bigger setting (where you can eg. fuse $W^*$-modules and general dual operator modules)**

- **Great application of pretty operator space ideas**

- **There is something worthwhile about pulling on the right lever, and getting the cascade/big package of hundreds of nice theorems**

**What are these ‘hundreds of nice theorems’?**

- **This is a fundamental notion in mathematics, and thus has very many different facets and ‘pictures’, corollaries, and special features. If you try write down a comprehensive account it will be quite lengthy**

- **This is a ubiquitous notion. Eg. in the $C^*$-literature, and related fields, everybody is using it, and in different ways, and you’ll find everybody has a new theorem about it**

- **In the nonselfadjoint setting you also get a bunch of theorems relating it to the selfadjoint setting**

- **You also get a bunch of theorems reprising important results from the theory of rings and modules, which only make sense for modules satisfying our definition (because e.g. direct sums are very problematic for general operator modules)**

- **It generalizes Hilbert spaces, so it has got to go in many nice directions**

- $\cdots$

**How to generalize it** to modules over a dual operator algebra $M$

• We begin by agreeing on one example of something that certainly should be a 'generalized $W^*$-module'

• Operator space notation: $C_n(M)$ (= first column of $M_n(M)$) is a right $M$-module

• In the $W^*$-algebra case, these are your basic building block:

**Theorem** A dual Banach space and module $Y$ over a $W^*$-algebra $M$ is a $W^*$-module iff there are nets of contractive module maps $\varphi_t : Y \to Y_t$ and $\psi_t : Y_t \to Y$, where each $Y_t$ is of the form $C_n(M)$ for some $n$, such that $\psi_t(\varphi_t(y)) \to y$ weak* for all $y \in Y$.

In this case the inner product is $w^*\lim_t \varphi_t(y)^* \varphi_t(z)$

($\exists$ simpler proof? Tempting to use ultraproducts, but...)

**Definition** Say that a dual operator space and module $Y$ over a $W^*$-algebra $M$ is a $w^*$-**rigged module** iff there are nets of completely contractive module maps $\varphi_t : Y \to Y_t$ and $\psi_t : Y_t \to Y$, where each $Y_t$ is of the form $C_n(M)$ for some $n$, such that $\psi_t(\varphi_t(y)) \to y$ weak* for all $y \in Y$.

- We have many alternative definitions

- In the norm topology case, a similar condition defines the rigged modules. But then you need a deep theorem to go further ([Hay], [B-Hay-Neal])

- In the weak* case we need operator space multipliers to begin cooking... did'nt know how to do it until recently...

• One can obtain a $C^*$-algebra valued inner product on such $Y$:

$$\langle y, z \rangle = w^*\lim_t \varphi_t(y)^* \varphi_t(z)$$

[Hint: tensor everything in the definition on the last slide with $I_N$ for a von Neumann algebras containing $N$. Then appeal to the last theorem]

(this is just viewing $Y$ inside its canonical '$W^*$-module envelope', the $W^*$-dilation)

• However usually it is preferable to work instead with a bilinear pairing with a canonical dual left $w^*$-rigged module:

- Define $\tilde{Y} = w^*CB(Y, M)_M$, a left $M$-module

Fact: $T : Y \to Z$ is a weak* continuous cb module map iff $T$ is adjointable:

$$\exists S : \tilde{Z} \to \tilde{Y} \text{ such that } (Sx, y) = (x, Ty) \ \ \forall x, y$$

Write $\mathbb{B}(Y, Z)$ for these adjointables ($= w^*CB(Y, Z)_M$)

**Theorem** If $Y$ is a $w^*$-rigged module over $M$ then

(1) $\mathbb{B}(Y)$ is weak* closed in $CB(Y)$, and is a dual operator algebra

(2) $Y$ is a left dual operator module over $\mathbb{B}(Y)$

(3) $\tilde{Y}$ is a right $w^*$-rigged module over $M$

(4) $\tilde{\tilde{Y}} = Y$

**Main new ingredients: 1) left multipliers of an operator space $X$, and their deep properties (see e.g. Chapter 4 of [B+Le Merdy])**

**• I defined multipliers originally in terms of the NC Shilov boundary, but characterized them in many ways (e.g. with Effros and Zarikian, as the (scalar multiples of) maps $T : X \to X$ such that for all $x, y \in X$:**

$$\left\| \begin{bmatrix} Tx \\ y \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|$$

**and similarly for matrices)**

**• The collection $\mathcal{M}_\ell(X)$ of left multipliers of $X$ is an operator algebra, indeed is a dual operator algebra if $X$ is a dual operator space**

• For special classes of operator spaces, $\mathcal{M}_\ell(X)$ often are exactly the maps of interest. Eg.

**Theorem** $\mathcal{M}_\ell(Y) = \mathbb{B}(Y)$ if $Y$ is $w^*$-rigged

2) We also need $\otimes_M^{\sigma h}$, the 'module version' of the Effros/Kishimoto/Ruan $\sigma$-Haagerup tensor product.

• This module version was introduced in [EP], as the quotient of the usual $\sigma$-Haagerup tensor product by the weak* closure of the span of terms $xm \otimes y - x \otimes my$ for $m \in M$. What we use from [EP] is the universal property of $\otimes_M^{\sigma h}$:

Every completely contractive separately weak* continuous map $u : E \times F \to Z$ such that $u(xm, y) = u(x, my)$ for $m \in M$, gives a completely contractive weak* continuous complete contraction $E \otimes_M^{\sigma h} F \to Z$

**Theorem** If $H$ is a Hilbert space on which $M$ is normally represented, then $K = Y \otimes^{\sigma h}_{M} H^c$ is a Hilbert space

- After establishing a few such basic theorems, one can fairly easily transfer everything in the older (norm topology) theory of rigged modules, to the weak* setting of $w^*$-rigged modules

- Get the whole big package...

- Here are some of the most important ideas:

## The linking algebra

$$\mathcal{L}^w = \left\{ \begin{bmatrix} a & x \\ y & b \end{bmatrix} : \; a \in M, b \in \mathbb{B}(Y), x \in \tilde{Y}, y \in Y \right\},$$

with the obvious multiplication.

If $H$ is a Hilbert space on which $Y$ is normally represented, let $K = Y \otimes_M^{\sigma h} H^c$. This is a Hilbert space ("the induced representation in Rieffel's language"), and $\mathcal{L}^w$ has a natural representation on $H \oplus K$, making $\mathcal{L}^w$ a dual operator algebra

The linking algebra is always a major tool, since now we can treat $Y$ as a concrete subspace of $B(H, K)$. Can treat everything as happening on $H \oplus K$

(Need operator space multipliers $\mathcal{M}_\ell$ for this!)

## Tensor products

**Theorem** If $Y$ is a right $w^*$-rigged module over $M$, and if $Z$ is also a right $w^*$-rigged module over $N$, and if $Z$ is a left dual operator $M$-module, then $Y \otimes_M^{\sigma h} Z$ is a right $w^*$-rigged module over $N$. Moreover, $Y \otimes_M^{\sigma h} Z \cong \mathbb{B}(\tilde{Y}, Z)$

- Latter holds even if $Z$ is not $w^*$-rigged on the right; i.e. is just a dual operator module

**Corollary** On $W^*$-modules, the normal Haagerup module tensor product equals the extended Haagerup module tensor product, and equals the classical composition tensor product of $W^*$-correspondences due to Rieffel

**The $W^*$-dilation:** Setting $Z$ in the last theorem to be a von Neumann algebra $\mathcal{R}$ containing $M$, we see that $Y \otimes_M^{\sigma h} \mathcal{R}$ is a $W^*$-module canonically containing $Y$. So we are back in the classical von Neumann algebra setting, and $Y$ now has a genuine von Neumann algebra valued inner product

Conversely, you can define $w^*$-rigged modules to be a subspace of a $W^*$-module over a von Neumann algebra $\mathcal{R}$ satisfying a natural condition

## Direct sums:

A direct sum $\oplus_{k \in I} Y_k$ of $w^*$-rigged modules over $M$ is again a $w^*$-rigged module over $M$, has a nice universal property, and works as it should, e.g.

$$(\oplus_k Y_k) \otimes_M^{\sigma h} Z \cong \oplus_k (Y_k \otimes_M^{\sigma h} Z)$$

Everything is working right...

## Examples of $w^*$-rigged modules

- $W^*$-modules/WTROs

- Weak* Morita equivalence bimodules (which in turn include all examples of Morita equivalence hitherto considered in the dual operator algebra literature, e.g. similar nest algebras)

- Second duals of strong Morita equivalence bimodules, or second duals of rigged modules

- If $C^w(M)$ is the 'infinite column sum' of copies of $M$ (= 'first column' of $M\bar{\otimes}B(H)$), and if $P \in \mathbb{B}(C^w(M))$ is an orthogonal projection, then $P(C^w(M))$ is $w^*$-rigged

- If $Z$ is a WTRO, and suppose that $Z^*Z$ is contained in a dual operator algebra $M$. Then $Y = \overline{ZM}^{w*}$ is $w^*$-rigged over $M$

(this is a one-sided variant of Eleftherakis' TRO-equivalence)

- Selfdual rigged modules over dual operator algebra $M$ (B+Magajna)

## III. Weak* Morita equivalence (September, 07)

**History:**

**[B+Magajna JFA 05]** tight Morita equivalence (self duality).

**[Eleftherakis 06]** TRO-equivalence, $\Delta$-equivalence

**[Eleftherakis + Paulsen (EP), 07]:** **Theorem** Two dual operator algebras are $\Delta$-equivalent if and only if they are weak* stably isomorphic; i.e., $M_I(M) \cong M_I(N)$

**[B+Kashyap 07] We define dual operator algebras $M$ and $N$ to be weak* Morita equivalent if if there exist a pair of dual operator bimodules $X$ and $Y$, such that $M \cong X \otimes^{\sigma h}_{N} Y$ and $N \cong Y \otimes^{\sigma h}_{M} X$**

- **Natural, if you are familiar with the B-Muhly-Paulsen strong Morita equivalence**

- **Our definition contains all examples hitherto in the literature of Morita-like equivalence in a dual (weak*) setting**

## The connection to Eleftherakis' equivalence relation

• **In [EP] it was shown that $\Delta$-equivalence implies weak* Morita equivalence in our language. That is, any of his equivalences is one of our weak* Morita equivalences ...**

**Both have (different) advantages: for example...**

• **$\Delta$-equivalence matches 'weak* stable isomorphism' perfectly, and contains other very nice new ideas.**

• **There are some examples which are important for us, which are not $\Delta$-equivalent. Also ours is a very natural variant of the entire earlier strong Morita equivalence theory, and you get the same big package of theorems, so why not... .**

**Loosely speaking, ours corresponds to a mere change in the tensor product used.**

- **I like to think about $\Delta$-equivalence as 'containing a $W^*$-Morita' and our notion as 'contained inside a $W^*$-Morita'**

Now I should write down the 'big package of theorems', but I won't

They are the usual 'package' occurring in some form in pure algebra, and most perfectly in $C^*$-algebra theory (in various sources), and reprised in the [B-Muhly-Paulsen] setting

Eg. The following is the nonselfadjoint analogue of a theorem of Rieffel.

Theorem If $Y$ is a weak* Morita equivalence $N$-$M$-bimodule, and $H$ is a universal normal representation for $M$, let $K$ be the induced representation of $N$. Then $M' \cong N'$. Writing $\mathcal{R}$ for either of these commutants, we have $Y \cong B_{\mathcal{R}}(H, K)$

- So every such $Y$ is of this nice form

- Every weak* Morita equivalence between $M$ and $N$ sits naturally inside a von Neumann algebra Morita equivalence between von Neumann algebras generated by $M$ and $N$

Still to do: ... iff an equivalence of categories of representations

That is, $M$ should be weak* Morita equivalent to $N$ iff the categories of normal Hilbert space representations of $M$ and $N$ are equivalent categories

and, probably, iff their categories of dual operator modules are equivalent