Algebraic Higher index theorem

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Fields Institute, Toronto, May, 2008

Outline

In this talk, we look at Connes-Moscovici's higher index theorem from deformation quantization point of view, and discuss its extension to orbifolds.

This is a joint work with Markus Pflaum and Hessel Posthuma.

Plan:

I. Cyclic cocycles on Weyl algebra

II. Cyclic cocycles on deformation quantization

III. Algebraic higher index theorem

IV. Formal and analytic index theorem

Part I: Cocycles on Weyl algebras

We illustrate explicit formulas for cyclic cocycles on Weyl algebras.

Weyl algebra $\mathbb{W}^{\mathsf{poly}}(V)$

Let (V, ω) be a finitely dimensional symplectic vector space. In canonical coordinates $(p_1, \ldots p_n, q_1, \ldots q_n)$ the symplectic form simply reads $\omega = \sum_i dp_i \wedge dq_i$.

The Weyl algebra $\mathbb{W}^{\mathsf{poly}}(V)$ is a deformation of the algebra of polynomials $\mathsf{S}(V^*)$ on V: we have $\mathbb{W}^{\mathsf{poly}}(V) = \mathsf{S}(V^*) \otimes \mathbb{C}[\hbar, \hbar^{-1}]$ with algebra structure given by the Moyal–Weyl product

$$f \star g = (m \circ \exp(\frac{\hbar}{2}\alpha))(f \otimes g)$$

where m is the commutative multiplication and $\alpha \in \mathsf{End}\left(\mathbb{W}^{\mathsf{poly}}(V) \otimes \mathbb{W}^{\mathsf{poly}}(V)\right)$ is basically the Poisson bracket associated to ω:

$$\alpha(f \otimes g) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \otimes \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \otimes \frac{\partial g}{\partial p_i} \right).$$

Cyclic cohomology of $\mathbb{W}_{2n}^{\text{poly}}$

By spectral sequence arguments with respect to the $\hbar$-filtration, we can compute the cyclic cohomology of the Weyl algebra.

$$HC^k(\mathbb{W}_{2n}^{\text{poly}}) = \begin{cases} \mathbb{C}[\hbar, \hbar^{-1}] & \text{if } k = 2n + 2p, p \geq 0, \\ 0 & \text{else;} \end{cases}$$

$$HC_k(\mathbb{W}_{2n}^{\text{poly}}) = \begin{cases} \mathbb{C}[\hbar, \hbar^{-1}] & \text{if } k = 2n + 2p, p \geq 0, \\ 0 & \text{else.} \end{cases}$$

Nest and Tsygan proved that $1 \otimes p_1 \wedge q_1 \wedge \cdots \wedge p_n \wedge q_n$ is a normalized $b + B$ cycle which generates the cyclic homology of $\mathbb{W}_{2n}^{\text{poly}}$.

Some notations

To introduce this cocycle, we need some notations.

For $0 \leq i \neq j \leq k \leq 2n$, define $\alpha_{ij} \in \mathrm{End}\left((\mathbb{W}_{2n}^{\mathrm{poly}})^{\otimes k+1}\right)$ by

$$\begin{aligned} \alpha_{ij}(a_0 \otimes \ldots \otimes a_k) \quad &= \textstyle\sum_{s=1}^{n} \left(a_0 \otimes \ldots \otimes \frac{\partial a_i}{\partial p_s} \otimes \ldots \otimes \frac{\partial a_j}{\partial q_s} \otimes \ldots \otimes a_k \right. \\ &\left. - a_0 \ldots \otimes \frac{\partial a_i}{\partial q_s} \otimes \ldots \otimes \frac{\partial a_j}{\partial p_s} \otimes \ldots \otimes a_k \right), \end{aligned}$$

i.e., the Poisson tensor acting on i'th and j'th slot of the tensor product

And $\mu_i : (\mathbb{W}_{2n}^{\mathrm{poly}})^{\otimes(i+1)} \to \mathbb{C}[\hbar, \hbar^{-1}]$ is given by

$$\mu_i(a_0 \otimes \ldots \otimes a_i) = a_0(0) \cdots a_i(0).$$

Cyclic cocycles on $\mathbb{W}_{2n}^{\mathsf{poly}}$

For all i with $0 \le i \le 2n$ define the cochains $\tau_i \in \overline{C}^i(\mathbb{W}_{2n}^{\mathsf{poly}})$ as follows.

$$\begin{aligned}
\tau_{2k}(a) &= (-1)^k \mu_{2k} \int_{\Delta^{2k}} \prod_{0\le i<j\le 2k} e^{\hbar(u_i-u_j+\frac{1}{2})\alpha_{ij}}\Big|_{u_0=0} \\
&\quad 1 \otimes (\hbar\alpha)^{\wedge k}(a) du_1 \cdots du_{2k}, \\
\tau_{2k-1}(a) &= (-1)^{k-1} \mu_{2k-1} \int_{\Delta^{2k-1}} \prod_{0\le i<j\le 2k-1} e^{\hbar(u_i-u_j+\frac{1}{2})\alpha_{ij}}\Big|_{u_0=0} \\
&\quad (\hbar\alpha)^{\wedge k}(a) du_1 \cdots du_{2k-1}.
\end{aligned}$$

The cocycle $\tau_{2n} \in \overline{C}^{2n}(\mathbb{W}_{2n}^{\mathsf{poly}})$ is the Hochschild cocycle introduced by Feigin, Felder, and Shoikhet up to a sign $(-1)^n$.

Properties of $(\tau_0, \cdots, \tau_{2n})$

Let $\iota_a : C_k(A) \to C_{k+1}(A)$ and $L_a : C_k(A) \to C_k(A)$ be defined by

$$\begin{aligned} \iota_a(a_0 \otimes \ldots \otimes a_k) &= \sum_{i=0}^{k} (-1)^{i+1} (a_0 \otimes \ldots \otimes a_i \otimes a \otimes a_{i+1} \otimes \ldots \otimes a_k), \\ L_a(a_0 \otimes \ldots \otimes a_k) &= \sum_{i=0}^{k} (a_0 \otimes \ldots \otimes [a, a_i] \otimes \ldots \otimes a_k). \end{aligned}$$

Theorem 1 *The cochains $\tau_i \in \overline{C}^i(\mathbb{W}_{2n}^{\mathrm{poly}})$ satisfy the relation*

$$-B\tau_{2k} = \tau_{2k-1} = b\tau_{2k-2}.$$

Hence, $(\tau_0, \cdots, \tau_{2n})$ is a $b + B$ cocycle on $\mathbb{W}_{2n}^{\mathrm{poly}}$. Furthermore, $\tau_{2k} \in \overline{C}^{2k}(\mathbb{W}_{2n}^{\mathrm{poly}})$, $0 \le k \le n$ are invariant and basic with respect to $\mathfrak{sp}_{2n}$, i.e.,

$$L_a \tau_{2k} = 0 \quad \text{and} \quad \iota_a \tau_{2k} = 0 \quad \text{for all } a \in \mathfrak{sp}_{2n} \subset \mathbb{W}_{2n}^{\mathrm{poly}}.$$

Part II: Cocycles on deformation quantization

We construct cyclic cocycles on deformation quantization of a symplectic manifold.

Symplectic manifold and deformation quantization

Let M be a symplectic manifold with a symplectic form ω, a nondegenerated closed 2-form on M. Define the Poisson bracket on $C^\infty(M)$ by $\{f,g\} = \omega^{-1}(df, dg) \in C^\infty(M)$ for $f, g \in C^\infty(M)$.

Let $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ be coordinates on T^*Q. Then $M = T^*Q$ and $\omega = \sum_i dp_i \wedge dq_i$ is a symplectic manifold. We can generalize symbol calculus of pseudodifferential operators on Q to the following structure on a general symplectic manifold.

A formal deformation quantization of a symplectic manifold (M, ω) is an associative product $\star$ on $C^\infty(M)[[\hbar]]$, such that
(i) $f \star g = fg + \frac{\hbar}{2}\{f,g\} + \sum_{i\geq 2} \hbar^i C_i(f,g)$,
(ii) C_i*'s are bilinear local differential operators.

Weyl algebra bundle

The Weyl algebra $\mathbb{W}_{2n}^{\mathsf{poly}}$ is a formal deformation quantization of the symplectic vector space $(\mathbb{R}^{2n}, \omega)$.

To construct a deformation quantization of a symplectic manifold (M, ω), we introduce a notion a Weyl algebra bundle.

Let FM be the symplectic frame bundle of TM. Define $\mathcal{W} = FM \times_{\mathsf{Sp}_{2n}} \mathbb{W}^+ V$, where $\mathbb{W}^+(V)$ consists of power series on V and the Moyal product extends naturally to define an associate product on $\mathbb{W}^+(V)$.

We fix a symplectic connection ∇ on TM, which lifts to a connection $\tilde{\nabla}$ on $\mathcal{W}$. Let $R \in \Omega^2\big(M; \mathsf{End}(TM)\big)$ be the curvature of ∇. Then $\tilde{\nabla}^2$ is equal to $\frac{1}{\hbar}[\tilde{R}, -] \in \Omega^2\big(M; \mathsf{End}(\mathcal{W})\big)$, where $\tilde{R}$ is obtained from R via the embedding $\mathfrak{sp}_{2n} \hookrightarrow \mathbb{W}_{2n}^+$.

Fedosov connection and deformation quantization

Fedosov proved that there exists a smooth section $\tilde{A} \in \Omega^1(M;\mathcal{W})$ such that $D = \tilde{\nabla} + \frac{1}{\hbar}[A,-]$ defines a flat connection on $\mathcal{W}$, i.e. $D^2 = 0 \in \Omega^2\Big(M; \mathrm{End}(\mathcal{W})\Big)$.

This implies that the Weyl curvature $\Omega = \tilde{R} + \tilde{\nabla}(A) + \frac{1}{2\hbar}[A,A]$ is in the center of $\mathcal{W}$ since $D^2 = \frac{1}{\hbar}[\Omega,-] = 0$. Since the center of $\mathbb{W}^+_{2n}$ is given by $\mathbb{C}[[\hbar]]$, $\Omega = -\omega + \hbar\omega_1 + \cdots$ is a closed 2-form in $\Omega^2\Big(M;\mathbb{C}[[\hbar]]\Big)$.

Fedosov also proved that the sheaf $\mathcal{A}^\hbar_D$ of flat sections with respect to D is isomorphic to $\mathcal{C}^\infty_M[[\hbar]]$ as a $\mathbb{C}[[\hbar]]$-module sheaf. Moreover, the induced product on $C^\infty(M)[[\hbar]]$ defines a star product on M.

Shuffle product

Let A be a graded algebra with a degree 1 derivation ∇. Recall that the shuffle product between $a_0 \otimes \cdots \otimes a_p \in \overline{C}_p(A)$ and $b_0 \otimes \cdots \otimes b_q \in \overline{C}_q(A)$ is defined to be

$$(a_0 \otimes \cdots \otimes a_p) \times (b_0 \otimes \cdots \otimes b_q) =$$
$$= (-1)^{\deg(b_0)(\sum_j \deg(a_j))}\, \mathsf{Sh}_{p,q}(a_0 b_0 \otimes a_1 \otimes \cdots \otimes a_p \otimes b_1 \otimes \cdots \otimes b_q),$$

where

$$\mathsf{Sh}_{p,q}(c_0 \otimes \cdots \otimes c_{p+q}) = \sum_{\sigma \in \mathsf{S}_{p,q}} \mathrm{sgn}(\sigma)\, c_0 \otimes c_{\sigma(1)} \otimes \cdots \otimes c_{\sigma(p+q)}$$

with sum over all (p,q)-shuffles in S_{p+q}.

Cyclic cocycles on $\mathcal{A}_D^{((\hbar))}$

Let $\mathcal{A}_D^{((\hbar))} := \mathcal{A}_D^{\hbar}[\hbar^{-1}]$ be the kernel of a Fedosov connection $D = \nabla + \frac{1}{\hbar}[A, -]$ on $\mathcal{W}[\hbar^{-1}]$.

Definition 2 *Define* $\Psi^i_{2k} \in \Omega^i(M) \otimes_{C^\infty(M)} \left(\mathcal{W}^{\otimes(2k-i+1)}\right)^*(M)$ *by putting*

$$\Psi^i_{2k}\Big(a_0 \otimes \cdots \otimes a_{2k-i}\Big) := \left(\frac{1}{\hbar}\right)^i \tau_{2k}\Big((a_0 \otimes \cdots \otimes a_{2k-i}) \times (1, A, \cdots, A)\Big).$$

Proposition 3 *For every chain* $a_0 \otimes \cdots \otimes a_{2k-i} \in C_{2k+1-i}\Big(\mathcal{A}^{((\hbar))}_{\mathrm{cpt}}\Big)$ *the above defined* Ψ^i_{2k} *satisfies the following equality:*

$$\begin{aligned} & (-1)^i d\Psi^i_{2n-2k}(a_0 \otimes \cdots \otimes a_{2k+1-i}) \\ = \; & \Psi^{i+1}_{2n-2k}(b(a_0 \otimes \cdots \otimes a_{2k+1-i})) \\ + \; & \Psi^{i+1}_{2n-2k+2}(\overline{B}(a_0 \otimes \cdots \otimes a_{2k+1-i})). \end{aligned}$$

A quasi-isomorphism

Definition 4 *For every i, r with $2r \leq i$ and every open $U \subset M$ define a morphism* $\chi^{i-2r}_{i,U} : \Omega^i(U)((\hbar)) \to \overline{C}^{i-2r}\left(\mathcal{A}^{((\hbar))}_{\mathrm{cpt}}\right)(U)$ *by*

$$\chi^{i-2r}_{i,U}(\alpha)(a_0 \otimes \cdots \otimes a_{i-2r}) = \int_U \alpha \wedge \Psi^{2n-i}_{2n-2r}(a_0 \otimes \cdots \otimes a_{i-2r}),$$

where $\alpha \in \Omega^i(U)((\hbar))$ *and* $a_0, \cdots, a_{i-2r} \in \mathcal{A}^{((\hbar))}_{\mathrm{cpt}}(U)$. *Using these, define morphisms* $\chi_i : \Omega^i_M((\hbar)) \to \mathrm{Tot}^i\, \mathcal{B}\overline{C}^{\bullet}(\mathcal{A}^{((\hbar))})$ *by*

$$\chi_i = \sum_{2r \leq i} \chi^{i-2r}_i .$$

The χ_i have the following crucial property.

Proposition 5 *For every $\alpha \in \Omega^{\bullet}(U)((\hbar))$ with $U \subset M$ open one has*

$$(b + B)\chi_{\bullet}(\alpha) = \chi_{\bullet}(d\alpha).$$

Cyclic cohomology of $\mathcal{A}_D^{((\hbar))}$

For every i, define a sheaf morphism

$$Q^i : \mathrm{Tot}^i\, \mathcal{B}\Omega^\bullet_M((\hbar)) := \bigoplus_{2r \le i} \Omega_M^{i-2r}((\hbar)) \to \mathrm{Tot}^i\, \mathcal{B}\overline{C}^\bullet\Big(\mathcal{A}^{((\hbar))}\Big)$$

which over $U \subset M$ open evaluated on forms $\alpha_{i-2r} \in \Omega^{i-2r}(U)((\hbar))$ gives

$$Q^i_U\Big(\sum_{2r \le i} \alpha_{i-2r}\Big) = \frac{1}{(2\pi\sqrt{-1})^n} \sum_{2r \le i} \chi_{i-2r,U}(\alpha_{i-2r}).$$

Theorem 6 *The above defined sheaf morphism*

$$Q : \Big(\mathrm{Tot}^\bullet\, \mathcal{B}\Omega^\bullet_M((\hbar)), d\Big) \to \Big(\mathrm{Tot}^\bullet\, \mathcal{B}\overline{C}^\bullet(\mathcal{A}^{((\hbar))}), b + B\Big)$$

is a quasi-isomorphism.

Part III: Higher index theorem

We prove a higher index theorem by computing the pairing between a cyclic cocycle and the Chern-Connes character of a K_0-element.

Pairing between cyclic cocycles and K-theory

Let A be a unital algebra over a field $\mathbb{k}$ and let e be an idempotent of A. The Chern character $\mathrm{Ch}_k(e)$ is a $b+B$ cycle defined by the following formulas

$$\begin{aligned} \mathrm{Ch}_k(e) &= (c_k, c_{k-1}, \cdots, c_1) \in \mathcal{B}\overline{C}_{2k}(A), \quad \text{where} \\ c_i &= (-1)^i \frac{2(2i)!}{i!}(e - \frac{1}{2}) \otimes e^{\otimes(2i)} \in A \otimes \overline{A}^{2i} \quad \text{for } 0 \le i \le k. \end{aligned}$$

For a $(b+B)$-cocycle $\phi = (\phi_{2k}, \cdots, \phi_0)$ and a projection $e \in A$, define

$$\langle \phi, e \rangle := \langle \phi, \mathrm{Ch}_k(e) \rangle = \sum_{l=0}^{k} (-1)^l \frac{2(2l)!}{l!} \phi_{2l} \left((e - \frac{1}{2}) \otimes e \otimes \cdots \otimes e \right).$$

This construction descends to cohomology and yields the desired pairing

$$HC^{2k}(A) \times K_0(A) \to \mathbb{k}.$$

Pairing on $\mathcal{A}^{((\hbar))}$

Let M be a symplectic manifold and $\mathcal{A}^{((\hbar))}(M)$ be a Fedosov deformation quantization of M. We apply the above construction to obtain a pairing between the cyclic cohomology $HC^{\bullet}(\mathcal{A}^{((\hbar))}_{\mathrm{cpt}})$ and the K_0 group of $\mathcal{A}^{((\hbar))}_{\mathrm{cpt}}(M)$.

An element in $K_0\Big(\mathcal{A}^{((\hbar))}_{\mathrm{cpt}}\Big)$ can be represented by a pairing of projections P_0, P_1 in $\mathfrak{M}_k\Big(\mathcal{A}^{((\hbar))}\Big)$ for some $k \geq 0$ such that $P_0 - P_1$ is compactly supported.

The pairing between $\phi = (\phi_0, \cdots, \phi_{2k})$ a $b + B$ cocycle and $e = (P_1, P_2)$ a representative of a K-group element of $\mathcal{A}^{((\hbar))}_{\mathrm{cpt}}$ is defined as

$$\langle \phi, e \rangle := \langle \phi, P_1 \rangle - \langle \phi, P_2 \rangle.$$

A special element

In the following talk of this part, we will assume that M is a compact symplectic manifold, and consider the special element $1 \in K_0(\mathcal{A}^{((\hbar))}_{\mathrm{cpt}}(M))$. We show how to compute its pairing between a cyclic cocycle $Q(\alpha)$ with $\alpha \in \mathrm{Tot}^{\bullet}\mathcal{B}\Omega^{\bullet}_{M}((\hbar))$.

$$\langle Q(\alpha), 1\rangle = \sum_{l=0}^{k} \frac{1}{(2\pi\sqrt{-1})^n} \int_M \alpha_{2l} \wedge \Psi^{2n-2l}_{2n-2l}(1).$$

We are reduced to compute the expression of $\Psi^{2n-2l}_{2n-2l}(1)$, for $0 \le l \le n$.

Hochschild cohomology and Lie algebra cohomology

Let A be a unital algebra, and $\mathfrak{gl}_N(A)$ be the Lie algebra of $N \times N$-matrices with coefficients in A. There is a chain map ϕ^N from the Hochschild cochain complex $C^\bullet(A)$ to the Lie algebra cochain complex $C^\bullet\Big(\mathfrak{gl}_N(A); \mathfrak{gl}_N(A)^*\Big)$:

$$\begin{aligned} & \phi^N(c)\Big((M_1 \otimes a_1) \otimes \cdots \otimes (M_k \otimes a_k)\Big)(M_1 \otimes a_1) \\ = & \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) c(a_0 \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)}) \, \mathrm{tr}(M_0 M_{\sigma(1)} \cdots M_{\sigma(k)}). \end{aligned}$$

We define $\Theta_{N,2k}$ to be $\phi^N(\tau_{2k}) \in C^{2k}\Big(\mathfrak{gl}_N(\mathbb{W}_{2n}); \mathfrak{gl}_N(\mathbb{W}_{2n})^*\Big)$. It is easy to check that $\Psi^{2n-2k}_{2n-2k}(1) = \left(\frac{1}{\hbar}\right)^{2n-2k} \frac{1}{(2n-2k)!} \Theta_{2n-2k}(A \wedge \cdots \wedge A)(1)$.

Chern-Weil maps

Proposition 7 *For any $k \leq n$, $\Theta_{N,2k}(1)$ is a cocycle in the relative Lie algebra cohomology complex $C^{2k}\Big(\mathfrak{gl}_N(\mathbb{W}_{2n}), \mathfrak{gl}_N \oplus \mathfrak{sp}_{2n}\Big)$.*

Recall that Fedosov connection D is a flat connection on the Weyl algebra bundle $\mathcal{W}$. Accordingly, we have a flat connection on the corresponding $Aut(\mathbb{W}_{2n}^{\text{poly}})$ "principal" bundle. It is known that all derivations of $\mathbb{W}_{2n}^{\text{poly}}$ are inner, in fact there is a short exact sequence of Lie algebras

$$0 \to \mathbb{C}[\hbar, \hbar^{-1}] \to \mathbb{W}_{2n}^{\text{poly}} \to \text{Der}\left(\mathbb{W}_{2n}^{\text{poly}}\right) \to 0.$$

By Chern-Weil theory, we have a chain map

$$\rho_D : C^{\bullet}\Big(\mathfrak{gl}_N(\mathbb{W}_{2n}), \mathfrak{gl}_N \oplus \mathfrak{sp}_{2n}\Big) \to \Omega^{\bullet}(M)((\hbar)),$$

and $\rho_D(\Theta_{N,2k}(1)) = \Theta_{2n-2k}(A \wedge \cdots \wedge A)(1)$.

Lie algebra Chern-Weil theory

Using Lie algebra cohomology and Chern Weil theory, we have reduced the computation of $\langle Q(\alpha), 1\rangle$ to understand the Lie algebra cocycles $\Theta_{N,2k}(1) \in C^{\bullet}\Big(\mathfrak{gl}_N(\mathbb{W}_{2n}), \mathfrak{gl}_N \oplus \mathfrak{sp}_{2n}\Big)$ for $0 \leq k \leq n$.

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a Lie subalgebra with an $\mathfrak{h}$-invariant projection $\mathrm{pr} : \mathfrak{g} \to \mathfrak{h}$. The curvature $C \in \mathrm{Hom}(\wedge^2\mathfrak{g}, \mathfrak{h})$ of pr is defined by $C(u \wedge v) := [\mathrm{pr}(u), \mathrm{pr}(v)] - \mathrm{pr}([u, v])$. Let $(S^{\bullet}\mathfrak{h}^*)^{\mathfrak{h}}$ be the algebra of $\mathfrak{h}$-invariant polynomials on $\mathfrak{h}$ graded by polynomial degree. Define a homomorphism $\sigma : (S^{\bullet}\mathfrak{h}^*)^{\mathfrak{h}} \to C^{2\bullet}(\mathfrak{g}, \mathfrak{h})$ by

$$\sigma(P)(v_1 \wedge \cdots \wedge v_{2q}) = \frac{1}{q!} \sum_{\substack{\sigma \in S_{2q}, \\ \sigma(2i-1)<\sigma(2i)}} (-1)^{\sigma} P\Big(C(v_{\sigma(1)},\ v_{\sigma(2)}), \cdots, C(v_{\sigma(2q-1)},\ v_{\sigma(2q)})\Big).$$

Characteristic classes

Recall the following invariant polynomials on the Lie algebras $\mathfrak{gl}_N$ and $\mathfrak{sp}_{2n}$: on $\mathfrak{gl}_N$ we have the Chern character

$$\mathrm{Ch}(X) := \mathrm{tr}\,(\exp X)\,, \ \text{ for } X \in \mathfrak{gl}_N.$$

On $\mathfrak{sp}_{2n}$, we have the $\hat{A}$-genus:

$$\hat{A}_\hbar(Y) := \det\left(\frac{\hbar Y/2}{\sinh(\hbar Y/2)}\right)^{1/2}, \ \text{ for } Y \in \mathfrak{sp}_{2n}.$$

With this, we can now state:

Proposition 8 *In* $H^{2k}\big(\mathfrak{gl}_N(\mathbb{W}_{2n}), \mathfrak{gl}_N \oplus \mathfrak{sp}_{2n}\big)$ *we have the identity*

$$[\Theta_{N,2k}] = \sigma((\hat{A}_\hbar\,\mathrm{Ch})_k)$$

for $k \leq n$ *and* $N \gg 0$.

Algebraic higher index theorem

In summary, using Lie algebra Chern-Weil theory, we have proved the following theorem.

Theorem 9 *(Nest-Tsygan, Pflaum-Posthuma-Tang) For a sequence of closed forms* $\alpha = (\alpha_0, \cdots, \alpha_{2k}) \in \mathrm{Tot}^{2k}\mathcal{B}\Omega^\bullet(M)((\hbar))$ *and two projects* P_1, P_2 *in* $\mathcal{A}^{((\hbar))}$ *with* $P_1 - P_2$ *compactly supported, one has*

$$\begin{aligned} &\langle Q(\alpha), P_1 - P_2\rangle \\ = \; &\sum_{l=0}^{k} \frac{1}{(2\pi\sqrt{-1})^l}\int_M \alpha_{2l} \wedge \hat{A}(M)\,\mathrm{Ch}(V_1 - V_2)\exp\Big(-\frac{\Omega}{2\pi\sqrt{-1}\hbar}\Big), \end{aligned}$$

where V_1 *and* V_2 *are vector bundles on* M *determined by the zero-th order terms of* P_1 *and* P_2.

Part IV: Formal and analytic index theorem

We connect the algebraic higher index theorem with Connes-Moscovici's higher index theorem for elliptic differential operators.

Alexander-Spanier cohomology

Let $\mathcal{O}$ be the sheaf C^∞_Q of smooth functions on Q.

Define $\mathcal{C}^k_{\mathsf{AS}}(\mathcal{O})(U) \cong \mathcal{O}^{\hat{\boxtimes} k+1}\Big(U^{k+1}\Big)/\mathcal{J}\Big(\Delta_{k+1}(U), U^{k+1}\Big)$, where $\mathcal{J}\Big(\Delta_{k+1}(U), U^{k+1}\Big)$ denotes the ideal of sections of $\mathcal{O}^{\hat{\boxtimes} k+1}$ over U^{k+1} which vanish on the diagonal $\Delta_{k+1}(U)$.

And define $\delta : \mathcal{C}^k_{\mathsf{AS}}(\mathcal{O})(U) \to \mathcal{C}^{k+1}_{\mathsf{AS}}(\mathcal{O})(U)$ by the formula

$$\delta f = \textstyle\sum_{i=0}^k (-1)^i \, \delta^i f, \qquad \text{where}$$
$$\delta^i f(x_0, \ldots, x_{k+1}) = f(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k+1}).$$

We have a sheaf cochain complex $\Big(\mathcal{C}^\bullet_{\mathsf{AS}}(\mathcal{O}), \delta\Big)$. The *Alexander–Spanier cochain complex* of $\mathcal{O}$ is defined by $C^\bullet_{\mathsf{AS}}(\mathcal{O}) := \Gamma(Q, \mathcal{C}^\bullet_{\mathsf{AS}}(\mathcal{O}))$ the complex of global sections with the differential δ.

Cyclic cocycles on $\Psi^{-\infty}(Q)$

Let $\Psi^{-\infty}(Q)$ be the algebra of smoothing pseudo-differential operators on Q.

Given an antisymmetric Alexander–Spanier cocycle f of degree k, Connes-Moscovici introduced a cyclic cocycle on $\Psi^{-\infty}$ by the expression

$$\begin{aligned} & \tau_f(A_0, \cdots, A_k) \\ = \ & (-1)^k \int_{Q^{k+1}} \mathrm{tr}\left(R_0(x_0, x_1) \cdot \ldots \cdot R_k(x_{k-1}, x_k)\right) f(x_0, \ldots, x_k), \end{aligned}$$

where R_i is the distribution kernel of A_i.

Analytic higher index

Let D be an elliptic differential operator on a compact manifold Q with principal symbol a. a defines an element $[a]$ in $K_1(C^\infty(S^*Q))$, where S^*Q is the sphere bundle over Q.

By the exact sequence of K-groups, we have a map

$$\partial : K_1\Big(\mathcal{C}^\infty(S^*Q)\Big) \to K_0\Big(\Psi\mathrm{DO}^{-\infty}(Q)\Big),$$

which is the index map. Therefore, $R := \partial([a])$ is a difference of two pseudodifferential projections of order $-\infty$ on Q,and is homotopic to the graph projection of D .

Given an antisymmetric Alexander–Spanier cocycle f of degree $2k$, Connes-Moscovici defined

$$\mathrm{ind}_f(D) = \tau_f(R).$$

Generalized traces on deformation quantization

Let Tr be a trace functional on $\mathcal{A}^{((\hbar))}_{\mathsf{cpt}}$ (essentially unique). For $f_0, f_1, \ldots, f_k \in \mathcal{C}^\infty(U)((\hbar))$ with $U \subset M$ open and $a_0, \ldots, a_k \in \mathcal{A}^{((\hbar))}_{\mathsf{cpt}}(U)$, the formula

$$\chi_{\mathsf{Tr}}(f_0 \otimes f_1 \otimes \ldots \otimes f_k)\,(a_0 \otimes \ldots \otimes a_k) := \mathsf{Tr}\left(f_0 \star a_0 \star \ldots \star f_k \star a_k\right)$$

defines a chain map $\chi_{\mathsf{Tr}} : \mathcal{C}^\bullet_{\mathsf{AS}}\left(\mathcal{C}^\infty_M((\hbar))\right) \to \mathcal{C}^\bullet\left(\mathcal{A}^{((\hbar))}_{\mathsf{cpt}}\right)$.

We have the following commutative diagram of chain maps,

$$\begin{tikzcd}
\mathsf{Tot}\,\mathcal{B}\Omega^\bullet_M((\hbar)) \arrow[r, "Q"] & \mathsf{Tot}\,\mathcal{B}C^\bullet\left(\mathcal{A}^{((\hbar))}_{\mathsf{cpt}}\right) \\
\mathcal{C}^\bullet_{\lambda\mathsf{AS}}\left(\mathcal{C}^\infty_M((\hbar))\right) \arrow[u, "\lambda"] \arrow[ur, "\chi_{\mathsf{Tr}}"'] &
\end{tikzcd} .$$

Asymptotic symbol calculus

The symbol calculus for pseudo-differential operators on Q

$$\mathsf{Op} : \mathsf{Sym}^m(U) \to \Psi\mathsf{DO}^m(U) \subset \mathsf{Hom}\Big(\mathcal{C}^\infty_{\mathsf{cpt}}(U), \mathcal{C}^\infty(U)\Big),$$

$$\Big(\mathsf{Op}(a)f\Big)(x) := \int_{T^*_xQ}\int_Q e^{-i\langle \xi, \mathsf{Exp}_x^{-1}(y)\rangle}\chi(x,y)a(x,\xi)f(y)\,dy\,d\xi,$$

naturally defines a deformation quantization $\mathcal{A}^{((\hbar))}_{\mathsf{op}}$ on T^*Q. Furthermore, the operator trace tr on $\boldsymbol{\Psi}^{-\infty}(Q)$ defines a trace functional on $\mathcal{A}^{((\hbar))}_{\mathsf{op}}$.

Under the above symbol calculus, we have

$$\mathsf{ind}_f(D) = \tau_f(R) = \chi_{\mathsf{Tr}}(f)(r) = Q \circ \lambda(r),$$

where r is the asymptotic symbol of R

Analytic higher index theorem

In summary, by the asymptotic symbol calculus and the algebraic higher index theorem, we have proved the following theorem.

Theorem 10 *(Connes-Moscovici) For an elliptic differential operator D on a riemannian manifold Q and an Alexander-Spanier cohomology class $[f]$ of degree $2k$ with compact support the localized index is given by*

$$\mathrm{ind}_{[f]}(D) = \frac{1}{(2\pi\sqrt{-1})^k} \int_{T^*Q} f_0 df_1 \wedge \ldots \wedge df_{2k} \wedge \hat{A}(T^*Q)\, \mathrm{Ch}(\sigma_{pr}(D)).$$

Generalization to orbifold

We can extend the above constructions on manifolds to orbifolds.

Theorem 11 *(Pflaum-Posthuma-Tang)Let D be an elliptic pseudodifferential operators on a reduced orbifold $X = M/\Gamma$, and $[f]$ be a $2j$ orbifold cyclic Alexander-Spanier cochomology class.*

$$\mathrm{ind}_{[f]}(D) = \sum_{<\gamma>} \sum_{r=0}^{j} \int_{T^*\widetilde{M^\gamma/C}(\gamma)} \frac{1}{(2\pi\sqrt{-1})^{j-r} m_\gamma} \frac{\tilde{\lambda}_\gamma^{2j-2r}(f) \wedge \hat{A}(T^*\widetilde{M^\gamma/C}(\gamma))\,\mathrm{Ch}_\gamma(\sigma_{\mathrm{pr}}(D))}{\mathrm{Ch}_\gamma(\lambda_{-1}N)},$$

where $C(\gamma)$ is the centralizer of γ, Ch_γ is the γ-twisted Chern character, $\mathrm{Ch}_\gamma(\lambda_{-1}N)$ is the localization data about the normal bundle of the γ-fixed point in T^*M, and m_γ is the order of local isotopy group.

Strategy for the proof of orbifold higher index theorem

The route we take to establish the orbifold index theorem is analogous to what we have done on manifolds. There are several new ingredients.

• Present an orbifold by a proper étale groupoid, and work with deformation quantization of the corresponding groupoid algebra.

• Construct cyclic cocycles on the crossed product algebra $\mathbb{W}^{\mathrm{poly}} \rtimes \Gamma$ for a finite group Γ.

• Geometry of groupoids and orbifolds, in particular the geometry of Burghelea spaces.