Hopf-cyclic cohomology in braided monoidal categories

Abstract

We start with a Hopf algebra $(H, m, \eta, \Delta, \varepsilon, \delta, \sigma)$ **in a strict symmetric braided monoidal abelian category** $(\mathcal{C}, \otimes, I, \psi)$**, and define a Hopf cyclic theory for** H**. As a non-trivial example we develop a Hopf cyclic theory for super Hopf algebras. At the end we give some results for non-symmetric categories.**

This is joint work with M. Khalkhali.

1 Preliminaries

Definition 1.1. (monoidal category)

A monoidal category $(\mathcal{C}, \otimes, I, a, l, r)$ **consists of a category** $\mathcal{C}$**, a functor** $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$**, an object** $I \in \mathcal{C}$ **(called the unit object), and natural isomorphisms**

$$a = a_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C,$$

$$l = l_A : I \otimes A \to A, \qquad r = r_A : A \otimes I \to A,$$

such that the following diagrams commute

$$\begin{tikzcd}
& ((A \otimes B) \otimes C) \otimes D \arrow[dl] \arrow[dr] & \\
(A \otimes (B \otimes C)) \otimes D \arrow[d] & & (A \otimes B) \otimes (C \otimes D) \arrow[d] \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr] & & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr] \arrow[dr] & & A \otimes (I \otimes B) \arrow[dl] \\
& A \otimes B &
\end{tikzcd}$$

Definition 1.2. (braided monoidal category)

A braided monoidal category, is a monoidal category $\mathcal{C}$ endowed with a natural isomorphisms

$$\psi_{A,B} : A \otimes B \to B \otimes A,$$

called braiding such that the following diagrams commute

$$\begin{tikzcd}
& A \otimes (B \otimes C) \arrow[r, "\psi"] & (B \otimes C) \otimes A \arrow[dr, "a^{-1}"] & \\
(A \otimes B) \otimes C \arrow[ur, "a^{-1}"] \arrow[dr, "\psi \otimes id"'] & & & B \otimes (C \otimes A) \\
& (B \otimes A) \otimes C \arrow[r, "a^{-1}"] & B \otimes (A \otimes C) \arrow[ur, "id \otimes \psi"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& (A \otimes B) \otimes C \arrow[r, "\psi"] & C \otimes (A \otimes B) \arrow[dr, "a"] & \\
A \otimes (B \otimes C) \arrow[ur, "a"] \arrow[dr, "id \otimes \psi"'] & & & (C \otimes A) \otimes B \\
& A \otimes (C \otimes B) \arrow[r, "a"] & (A \otimes C) \otimes B \arrow[ur, "\psi \otimes id"'] &
\end{tikzcd}$$

A braided monoidal category is called symmetric if

$$\psi_{A,B} \circ \psi_{B,A} = id,$$

for all objects A **and** B **of** $\mathcal{C}$**.(Sometimes we just write** $\psi^2 = id$ **)**

It is well known that any braided monoidal category is equivalent to a *strict* **one in which** a, l **and** r **are just equalities and the above axioms are reduced to the following ones:**

$$(A \otimes B) \otimes C = A \otimes (B \otimes C),$$
$$I \otimes A = A \otimes I = A,$$
$$\psi_{A,B\otimes C} = (id_B \otimes \psi_{A,C})(\psi_{A,B} \otimes id_C),$$
$$\psi_{A\otimes B,C} = (\psi_{A,C} \otimes id_B)(id_A \otimes \psi_{B,C}),$$

for all objects A,B,C of $\mathcal{C}$**.**

Example 1.1.

Let $(H,\, R = R^{(1)} \otimes R^{(2)})$ **be a quasitriangular Hopf algebra and** $\mathcal{C}$ **be the category of all left** H**-modules. Then** $\mathcal{C}$ **is a braided monoidal abelian category, which is symmetric if and only if** $R^{-1} = R^{(2)} \otimes R^{(1)}$**. Here the tensor structure is defined by**

$$h \triangleright (v \otimes w) = h^{(1)} \triangleright v \otimes h^{(2)} \triangleright w,$$

and the braiding map $\psi_{V \otimes W}$ **by**

$$\psi_{V \otimes W}(v \otimes w) := (R^{(2)} \triangleright w \otimes R^{(1)} \triangleright v),$$

for any V **and** W **in** $\mathcal{C}$**, where** $\triangleright$ **denotes the action of** H**.**

Example 1.2. (category of super vector spaces)

As a very special case of Example (1.1)**, let** $H = \mathbb{C}\mathbb{Z}_2$ **with the non-trivial quasitriangular structure** $R = R^{(1)} \otimes R^{(2)}$ **defined by**

$$R := (\frac{1}{2})(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g),$$

where g **is the generator of the cyclic group** $\mathbb{Z}_2$**. The category** $\mathcal{C} = \mathbb{Z}_2$**- Mod then is the category of super vector spaces. The braiding map** $\psi_{V \otimes W}$ **for any** $V = V_0 \oplus V_1$ **and** $W = W_0 \oplus W_1$ **in** $\mathcal{C}$ **acts as below**

$$\psi_{V \otimes W}(v \otimes w) = (-1)^{|v||w|} \ (w \otimes v).$$

Remark 1.1.

One can extend Example (1.2) **to** $\mathbb{C}\mathbb{Z}_n$ **for any** $n > 2$ **which provides a good source of non-symmetric braided monoidal categories.**

Example 1.3. (Yetter Drinfeld category)

Let H be a Hopf algebra over a field k with comultiplication $\Delta h = h^{(1)} \otimes h^{(2)}$ and the bijective antipode S.
A left-left Yetter Drinfeld (YD) H-module consist of
A vector space V, a H-module structure on V

$$H \otimes V \to V$$
$$h \otimes v \mapsto hv,$$

a H-comodule structure on V

$$V \to H \otimes V$$
$$v \mapsto v_{(-1)} \otimes v_{(0)},$$

and a compatibility (YD) condition

$$(hv)_{(-1)} \otimes (hv)_{(0)} = h^{(1)} v_{(-1)} S(h^{(3)}) \otimes h^{(2)} v_{(0)}.$$

The category ${}^H_H\mathcal{YD}$ of all YD H-modules is a braided monoidal abelian category with the braiding map

$$\psi_{V\otimes W}(v\otimes w)=v_{(-1)}w\otimes v_{(0)}.$$

This category is in general not symmetric and the inverse of the braiding is

$$\psi^{-1}_{V\otimes W}(w\otimes v)=v_{(0)}\otimes S^{-1}(v_{(-1)})w.$$

Definition 1.3. (Braided Hopf algebra)

Let $(\mathcal{C}, \otimes, I, \psi)$ **be a braided monoidal category. A Hopf algebra** $(H, m, \eta, \Delta, \varepsilon, S)$ **in** $\mathcal{C}$ **consists of:**

An object $H \in obj\mathcal{C}$ **and morphisms** $m : H \otimes H \to H$, $\eta : I \to H$, $\Delta : H \to H \otimes H$, $\varepsilon : H \to I$ *and* $S : H \to H$ **s.t.**

$$m(id \otimes m) = m(m \otimes id), \quad associativity$$
$$m(\eta \otimes id) = m(id \otimes \eta) = id, \quad unit$$
$$(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta, \quad coassociativity$$
$$(\varepsilon \otimes id)\Delta = (id \otimes \varepsilon)\Delta = id, \quad counit$$
$$\Delta m = (m \otimes m)(id \otimes \psi \otimes id)(\Delta \otimes \Delta), \ \ compatibility$$
$$\Delta\eta = \eta \otimes \eta, \quad \varepsilon m = \varepsilon \otimes \varepsilon, \quad \varepsilon\eta = id_I$$
$$m(S \otimes id)\Delta = m(id \otimes S)\Delta = \eta\varepsilon. \quad antipode$$

Example 1.4. (super Hopf algebra)

Any Hopf algebra in $\mathbb{Z}_2$-Mod is a super Hopf algebra.

Example 1.5. ($T(V)$ in ${}^H_H\mathcal{YD}$)

For any V in ${}^H_H\mathcal{YD}$ the Yetter Drinfeld category attached to a given Hopf algebra H, the tensor algebra $T(V)$ is a braided Hopf algebra in ${}^H_H\mathcal{YD}$ with the comultiplication and counit defined by $\Delta(v) = 1 \otimes v + v \otimes 1$ and $\varepsilon(v) = 0$ for all v in V.

Definition 1.4. (H-modules and comodules)

Let H be a braided Hopf algebra in $\mathcal{C}$.

- A right H-module M is an object in $\mathcal{C}$ equipped with a morphism $\phi_M : M \otimes H \to M$, called H action, such that

$$(\phi)(id_M \otimes m_H) = (\phi)(\phi \otimes id_H),$$
$$(\phi)(id_M \otimes \eta_H) = id_M.$$

- A left H-comodule M is an object in $\mathcal{C}$ equipped with a morphism $\rho_M : M \to H \otimes M$, called H coaction such that

$$(\Delta_H \otimes id_M)(\rho) = (id_M \otimes \rho)(\rho),$$
$$(\epsilon_H \otimes id_M)(\rho) = (id_H \otimes \epsilon_H)(\psi_{H,M})(\rho) = id_M.$$

Definition 1.5. (stable anti Yetter Drinfeld H-module)

A right-left braided stable anti Yetter Drinfeld H-module in $\mathcal{C}$ is an object M in $\mathcal{C}$ such that:

– M is a right H-module via an action $\phi_M : M \otimes H \to M$

– M is a left H-comodule via a coaction $\rho_M : M \to H \otimes M$

– M satisfies the braided anti Yetter Drinfeld condition i.e

$$\begin{aligned}(\rho)(\phi) = [(m)(S \otimes m) \otimes \phi][(\psi_{H^{\otimes 2}}, H \otimes id_M \otimes id_H)(id_{H^{\otimes 2}} \otimes \psi_{M,H} \otimes id_H) \\ (id_{H^{\otimes 2}} \otimes id_M \otimes \psi_{H,H})(id_H \otimes \psi_{M,H} \otimes id_{H^{\otimes 2}})][\rho \otimes \Delta^2].\end{aligned} \tag{1.1}$$

- M is stable i.e

$$(\phi)(\psi_{H,M})(\rho) = id_M.$$

Definition 1.6. **(H-module-coalgebra)**

A quadruple $(C,\, \Delta_C,\, \epsilon_C,\, \phi_C)$ **is called a left (braided)** H**-module-coalgebra in** $\mathcal{C}$ **if** $(C, \Delta_C, \epsilon_C)$ **is a coalgebra in** $\mathcal{C}$**, and** C **is a left** H**-module via an action** $\phi_C : H \otimes C \to C$ **such that**
ϕ_C **is a coalgebra map in** $\mathcal{C}$ **i.e.**

$$\Delta_C \phi_C = (\phi_C \otimes \phi_C)(id_H \otimes \psi_{H,C} \otimes id_C)(\Delta_H \otimes \Delta_C),$$
$$\varepsilon_C \phi_C = \varepsilon_H \otimes \varepsilon_C.$$

Now we are going to define a cocyclic module for any triples (H, C, M)**, where** H **is a Hopf algebra,** C **is a** H**-module coalgebra and** M **is a SAYD** H**-module, all in** $\mathcal{C}$**.**

Let $C^n = C^n(C, M) := M \otimes C^{n+1},\ n \geq 0$**.**

We define faces $\delta_i : C^{n-1} \to C^n$**, degeneracies** $\sigma_i : C^{n+1} \to C^n$ **and cyclic maps** $\tau_n : C^n \to C^n$ **by**

$$\delta_i = \begin{cases} (1_M, 1_{C^i}, \Delta_C, 1_{C^{n-i-1}}) & 0 \leq i < n \\ (1_M, \psi_{C,C^n})(1_M, \phi_C, 1_{C^n})(\psi_{H,M}, 1_{C^{n+1}})(\rho_M, \Delta_C, 1_{C^{n-1}}) & i = n \end{cases}$$

$$\sigma_i := (1_M, 1_{C^{i+1}}, \varepsilon_C, 1_{C^{n-i}}), \quad 0 \leq i \leq n$$

$$\tau_n := (1_M, \psi_{C,C^n})(1_M, \phi_C, 1_{C^n})(\psi_{H,M}, 1_{C^{n+1}})(\rho_M, 1_{C^{n+1}}).$$

Proposition 1.1.

If $\mathcal{C}$ **is a symmetric monoidal abelian category, then** $(C^\bullet, \delta_i, \sigma_i, \tau)$ **is a para-cocyclic module in** $\mathcal{C}$**.**

Now let $C_H^n = C_H^n(C, M) := M \otimes_H C^{n+1}, \ n \geq 0$,

with induced faces, degeneracies and cyclic maps denoted by the same letters δ_i, σ_i and τ_n.

Theorem 1.1. (main Theorem)

If $\mathcal{C}$ is a symmetric braided monoidal abelian category then, $(C_H^\bullet, \delta_i, \ \sigma_i \ , \ \tau)$ is a cocyclic module in $\mathcal{C}$.

Example 1.6.

As an special case, if we put $C = H$ as a H-module coalgebra over itself via m_H for module structure and Δ_H for coalgebra structure, and put $M = {}^\sigma I_\delta$, then the cyclic theory in Theorem (1.1) reduces to a braided version of Connes-Moscovici's Hopf cyclic theory in any symmetric abelian braided monoidal category $\mathcal{C}$. We will explain this example in more details, in the next section.

2 Braided version of Connes-Moscovici's Hopf cyclic theory

Let $\mathcal{C}$ be a braided monoidal abelian category and H be a Hopf algebra in $\mathcal{C}$.

Definition 2.1. (character, co-character and modular pair)

A character for H is a morphism $\delta : H \to I$ in $\mathcal{C}$ which is an algebra map i.e

$$\delta m = \delta \otimes \delta \quad and \quad \delta \eta = id_I.$$

A co-character for H is a morphism $\sigma : I \to H$ which is a coalgebra map i.e

$$\Delta \sigma = \sigma \otimes \sigma \quad and \quad \varepsilon \sigma = id_I.$$

A pair (δ, σ) consisting of a character and a co-character is called a modular pair if

$$\delta \sigma = id_I.$$

Definition 2.2. **(δ-twisted antipode)**

If δ is a character for H, the δ-twisted antipode $\widetilde{S}$ is defined by

$$\widetilde{S} := (\delta \otimes S)\Delta$$

Proposition 2.1.

If $\widetilde{S}$ is a δ-twisted antipode for H then

$$\begin{gathered}
\widetilde{S}m = m\psi(\widetilde{S} \otimes \widetilde{S}), \\
\widetilde{S}\eta = \eta, \\
\Delta\widetilde{S} = \psi(\widetilde{S} \otimes S)\Delta, \\
\varepsilon\widetilde{S} = \delta, \quad \delta\widetilde{S} = \varepsilon, \quad \widetilde{S}\sigma = S\sigma, \\
m(S\sigma \otimes \sigma) = m(\widetilde{S}\sigma \otimes \sigma) = \eta.
\end{gathered}$$

Definition 2.3. (braided modular pair in involution (BMPI))

A modular pair (δ, σ) for H is called a modular pair in involution if

$$m((id \otimes m)(\widetilde{S}\sigma \otimes \widetilde{S}^2 \otimes \sigma)) = id.$$

Example 2.1. ($^{\sigma}I_{\delta}$)

One can easily check that, if I is considered as a right H-module via a character δ

$$\phi_I = \delta : I \otimes H = H \to I,$$

and as a left H-comodule via a co-character σ

$$\rho_I = \sigma : I \to H \otimes I = I,$$

then $^{\sigma}I_{\delta}$ is a braided SAYD module over H if and only if (δ, σ) is a braided MPI.

Theorem 2.1. (braided version of Connes-Moscovici's Hopf cyclic theory)

Suppose $(H,\ (\delta,\sigma),\ \widetilde{S})$ **is a braided Hopf algebra in a symmetric braided monoidal abelian category** $\mathcal{C}$**, where** (δ,σ) **is a braided MPI and** $\widetilde{S}$ **is the braided twisted antipode as above.**
If we put $(C;\ \phi_C,\ \Delta_C) = (H;\ m_H,\ \Delta_H)$**, and** $M = {}^{\sigma}I_{\delta}$ **as in example** (2.1)**, then the theory provided in Theorem** (1.1) **reduces to the following one**

$$C^0(H) = I \quad and \quad C^n(H) = H^n, \quad n \geq 1$$

with faces degeneracies and cyclic maps given by

$$\delta_i = \begin{cases} (\eta, 1, 1, ..., 1) & i = 0 \\ (1, 1, ..., 1, \underset{i^{th}position}{\Delta}, 1, 1, ...1) & 1 \leq i \leq n-1 \\ (1, 1, ..., 1, \sigma) & i = n \end{cases}$$

$$\sigma_i = (1, 1, ..., \underset{(i+1)^{th}position}{\varepsilon}, 1, 1..., 1), \quad 0 \leq i \leq n$$

$$\tau_n = \begin{cases} id_I & n = 0 \\ (m_n)(\Delta^{n-1}\widetilde{S}, 1_{H^{n-1}}, \sigma) & n \neq 0 \end{cases}$$

Here by m_n **we mean,** $m_1 := m$**, and for** $n \geq 2$

$$m_n := m_{H^n} := (\underbrace{m, m, ..., m}_{n\ \ times})\mathcal{F}_n(\psi),$$

where

$$\mathcal{F}_n(\psi) := \prod_{j=1}^{n-1} \ (1_{H^j}, \underbrace{\psi, \psi, ..., \psi}_{n-j\ \ times}, 1_{H^j}).$$

A NON-TRIVIAL EXAMPLE:

3 Hopf cyclic theory for super Hopf algebras

Definition 3.1. (super Hopf algebra)

A Hopf algebra H in $\mathbb{Z}_2$-Mod is called a super Hopf algebra. Thus:

– H is a super vector space $H = H_0 \oplus H_1$.
– H is a super algebra i.e. $|ab| = |a| + |b|$ where a and b are homogeneous elements of H. Here $|a|$ denotes the degree of a for any homogeneous element a in H.
– H is a super coalgebra i.e. $|a| = |a_{(1)}| + |a_{(2)}|$ for any homogeneous element a of H and for any term $a_{(1)} \otimes a_{(2)}$ in $\Delta(a) = a_{(1)} \otimes a_{(2)}$.
– H is a super bialgebra i.e.we have the compatibility condition:

$$\Delta(ab) = (ab)_{(1)} \otimes (ab)_{(2)} = (-1)^{|a_{(2)}||b_{(1)}|} \; (a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}).$$

– the antipode S is degree preserving i.e. $|S(a)| = |a|$ for all homogeneous elements a in H.

Definition 3.2. (BMPI for super Hopf algebras)

Consider a super Hopf algebra $H = H_0 \oplus H_1$ **as in definition** (3.1)

– A character for H **is an algebra map** $\delta : H \to \mathbb{C}$ **which is degree preserving i.e.** $\delta(a) = 0$ **for all a in** H_1 **.**

– A grouplike element (co-character) in H **is a group like element** σ **of** H **which is of degree 0, i.e.** $\sigma \in H_0$**.**

– A twisted antipode is a usual twisted antipode $\widetilde{S}$ **which is degree preserving i.e.** $|\widetilde{S}(a)| = |a|$ **for all homogeneous elements** a **in** H**.**

– The pair (δ, σ) **is called a modular pair if** $\delta(\sigma) = 1_{\mathbb{C}}$**, and is called a modular pair in involution (MPI) if in addition** $\sigma^{-1}(\widetilde{S})^2\sigma = id$**.**

Theorem 3.1. (Hopf cyclic theory for super Hopf algebras)

Consider a super Hopf algebra $H = H_0 \oplus H_1$ **with an MPI** (δ, σ)
then the complex, faces, degeneracies and cyclic maps of theorem (2.1) **can be written as below**

$$C^0(H) = \mathbb{C} \quad and \quad C^n(H) = H^n, \quad n \geq 1$$

$$\delta_i(h_1, ..., h_{n-1}) = \begin{cases} (1, h_1, h_2, ..., h_{n-1}) & i = 0 \\ (h_1, h_2, ..., h_i^{(1)}, h_i^{(2)}, ..., h_{n-1}) & 1 \leq i \leq n-1 \\ (h_1, h_2, ..., h_{n-1}, \sigma) & i = n \end{cases}$$

$$\sigma_i(h_1, h_2, ..., h_{n+1}) = \varepsilon(h_{i+1})\,(h_1, h_2, ..., h_i, h_{i+2}, ..., h_{n+1}), \quad 0 \leq i \leq n$$

$$\tau_n(h_1, h_2, ..., h_n) = \alpha\beta\ (S(h_1^{(n)})h_2, S(h_1^{(n-1)})h_3, ..., S(h_1^{(2)})h_n, \widetilde{S}(h_1^{(1)})\sigma),$$

where h_i's are homogeneous elements and

$$\alpha = \prod_{i=1}^{n-1}(-1)^{(|h_1^{(1)}|+...+|h_1^{(i)}|)(|h_1^{(i+1)}|)},$$

$$\beta = \prod_{j=1}^{n-1}(-1)^{|h_1^{(j)}|(|h_2|+|h_3|+...+|h_{n-j+1}|)}.$$

APPLICATION:

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a super Lie algebra, let

$$\bigwedge \mathfrak{g} := \frac{T(\mathfrak{g})}{(a \otimes b + (-1)^{|a||b|} b \otimes a)},$$

be the exterior algebra of $\mathfrak{g}$ and

$$H = U(\mathfrak{g}) := \frac{T(\mathfrak{g})}{([a,b] - a \otimes b + (-1)^{|a||b|} b \otimes a)},$$

be the enveloping algebra of $\mathfrak{g}$.

$U(\mathfrak{g})$ **is a super Hopf algebra and**

Theorem 3.2.

$$HP^*_{(\delta,1)}(U(\mathfrak{g})) = \bigoplus_{i = *(mod\ 2)} H_i(\mathfrak{g}; \mathbb{C}_\delta).$$

4 Connes-Moscovici's Hopf cyclic theory in non-symmetric monoidal categories

Theorem 4.1. (para-cyclic structure for non-symmetric case)

Suppose $(H, (\delta, \sigma), \widetilde{S})$ **is a braided Hopf algebra in a braided abelian monoidal category** $\mathcal{C}$**, where** (δ, σ) **is a BMPI and** $\widetilde{S}$ **is the braided twisted antipode as in definitions** (2.3) **and** (2.2)**. If we define complex** $C^n(H)$**, faces** δ_i**, degeneracies** σ_i **and cyclic maps** τ_n **as below, then we will have a para-cyclic structure.**

$$C^0(H) = I \quad and \quad C^n(H) = H^n, \quad n \geq 1$$

$$\delta_i = \begin{cases} (\eta, 1, 1, ..., 1) & i = 0 \\ (1, 1, ..., 1, \underset{i^{th} position}{\Delta}, 1, 1, ...1) & 1 \leq i \leq n-1 \\ & \\ (1, 1, ..., 1, \sigma) & i = n \end{cases}$$

$$\sigma_i = (1, 1, ..., \underset{(i+1)^{th} position}{\varepsilon}, 1, 1..., 1), \quad 0 \leq i \leq n$$

$$\tau_n = \begin{cases} id_I & n = 0 \\ (m_n)(\Delta^{n-1}\widetilde{S}, 1_{H^{n-1}}, \sigma) & n \neq 0 \end{cases}$$

Here by m_n **we mean,** $m_1 := m$**, and for** $n \geq 2$

$$m_n := m_{H^n} := (\underbrace{m, m, ..., m}_{n \ \ times})\mathcal{F}_n(\psi),$$

where

$$\mathcal{F}_n(\psi) := \prod_{j=1}^{n-1} \ (1_{H^j}, \underbrace{\psi, \psi, ..., \psi}_{n-j \ \ times}, 1_{H^j}).$$

Theorem 4.2.

Under the conditions of Theorem (4.1),

$$\tau_2^3 = \psi_{H,H}^2$$

.

Proof.

$$
\begin{aligned}
&\tau_2^3 = \tau_2\tau_2^2\\
&= \tau_2(m_2)(\Delta\widetilde{S},1,\sigma)(m_2)(\Delta\widetilde{S},1,\sigma)\\
&= \tau_2(m,m)(1,\psi,1)(\psi(\widetilde{S},S)\Delta,1,\sigma)(m,m)(1,\psi,1)(\psi(\widetilde{S},S)\Delta,1,\sigma)\\
&= \tau_2(m,m)(1,\psi,1)(\psi,2)(\widetilde{S},S,1,\sigma)(\Delta,1)(m,m)(1,\psi,1)(\psi,2)(\widetilde{S},S,1,\sigma)(\Delta,1)\\
&= \tau_2(m,m)(1,\psi,1)(\psi,2)(\widetilde{S},S,1,\sigma)(m,m,m)(1,\psi,3)(\Delta,\Delta,2)(S,1,\widetilde{S},\sigma)(1,\psi)(\psi,1)(\Delta,1)\\
&= \tau_2(m,m)(1,\psi,1)(\psi,2)(\widetilde{S}m,Sm,m,\sigma)(1,\psi,3)(\Delta S,\Delta,\widetilde{S},\sigma)(1,\psi)(\psi,1)(\Delta,1)\\
&= \tau_2(m,m)(1,\psi,1)(\psi,2)(m(\widetilde{S},\widetilde{S})\psi,m(S,S)\psi,m,\sigma)(1,\psi,3)((S,S)\psi\Delta,\Delta,\widetilde{S},\sigma)(1,\psi)(\psi,1)(\Delta,1)\\
&= \tau_2(m,m)(1,\psi,1)(\psi,2)(m,m,m,\sigma)(\widetilde{S},\widetilde{S},S,S,1,1)(\psi,4)(2,\psi,2)(1,\psi,3)(S,S,1,1,\widetilde{S},\sigma)\\
&\quad(\psi,3)(\Delta,\Delta,1)(1,\psi)(\psi,1)(\Delta,1)\\
&= \tau_2(m,m)(1,\psi,1)(\psi,2)(m,m,m,\sigma)(\widetilde{S},\widetilde{S},S,S,1,1)(\psi,4)(2,\psi,2)(1,\psi,3)(S,S,1,1,\widetilde{S},\sigma)\\
&\quad(\psi,3)(\Delta,\Delta,1)\psi_{12}(\Delta,1)\\
&= \tau_2(m,m)(1,\psi,1)(\psi,2)(m,m,m,\sigma)(\widetilde{S},\widetilde{S},S,S,1,1)(\psi,4)(2,\psi,2)(1,\psi,3)(S,S,1,1,\widetilde{S},\sigma)\\
&\quad(\psi,3)\psi_{14}(1,\Delta,\Delta)(\Delta,1)\\
&= \tau_2(m,m)(1,\psi,1)(\psi,2)(m,m,m,\sigma)(\widetilde{S},\widetilde{S},S,S,1,1)(1,S,1,S,\widetilde{S},\sigma)\\
&\quad(\psi,3)(2,\psi,1)(1,\psi,2)(\psi,3)\psi_{14}((1,\Delta)\Delta,\Delta)\\
&= \tau_2(1,m)(m,2)(\psi_{12},1)(m,m,m,\sigma)(\widetilde{S},\widetilde{S}S,S,S^2,\widetilde{S},\sigma)(\psi,3)(2,\psi,1)(1,\psi,2)(\psi,3)\\
&\quad\psi_{14}((\Delta,1)\Delta,\Delta)\\
&= \tau_2(1,m)(\psi,1)(1,m,1)(m,m,m,\sigma)(\widetilde{S},\widetilde{S}S,S,S^2,\widetilde{S},\sigma)(\psi,3)(2,\psi,1)(1,\psi,2)(\psi,3)\\
&\quad\psi_{14}(\Delta,3)(\Delta,2)(1,\Delta)\\
&= \tau_2(1,m)(\psi,1)(m,m(m,m),\sigma)(\widetilde{S},\widetilde{S}S,S,S^2,\widetilde{S},\sigma)(\psi,3)(2,\psi,1)(1,\psi,2)(\psi,3)\\
&\quad(3,\psi)(2,\psi,1)(1,\psi,2)(\psi,3)(\Delta,3)(\Delta,2)(1,\Delta)\\
&= \tau_2(1,m)(\psi,1)(m,(m(m,1)(1,m,1)),\sigma)(\widetilde{S},\widetilde{S}S,S,S^2,\widetilde{S},\sigma)(\psi,3)(2,\psi,1)\\
&\quad(3,\psi)(1,\psi,2)(\psi,3)(2,\psi,1)(1,\psi,2)(\psi,3)(\Delta,3)(\Delta,2)(1,\Delta)\\
&= \tau_2(1,m)(\psi,1)(m,1,\sigma)(2,m(m,1))(3,m,1)(\widetilde{S},\widetilde{S}S,S,S^2,\widetilde{S},\sigma)(\psi,3)(2,\psi_{21})(\psi_{22},1)\\
&\quad(\psi\Delta,3)(\Delta,2)(1,\Delta)\\
&= \tau_2(1,m)(\psi,1)(m,1,\sigma)(2,m(m,1))(3,m,1)(\widetilde{S},\widetilde{S}S,S,S^2,\widetilde{S},\sigma)(3,\psi\Delta)\\
&\quad(\psi,2)(2,\psi)(\psi_{12},1)(\Delta,2)(1,\Delta)\\
&= \tau_2(1,m)(\psi,1)(m,1,\sigma)(2,m(m,1))(\widetilde{S},\widetilde{S}S,S,m(S^2,\widetilde{S})\psi\Delta,\sigma)(\psi,2)(2,\psi)(\psi_{12},1)(\Delta,2)(1,\Delta)\\
&\overset{(1)}{=} \tau_2(1,m)(\psi,1)(m,1,\sigma)(2,m(m,1))(\widetilde{S},\widetilde{S}S,S,\eta\delta,\sigma)(\psi,2)(2,\psi)(\psi_{12},1)(\Delta,2)(1,\Delta)\\
&= \tau_2(1,m)(\psi,1)(m,1,\sigma)(2,m(m,1))(3,\eta,1)(\widetilde{S},\widetilde{S}S,S,\sigma)(3,\delta)(\psi,2)(2,\psi)(\psi_{12},1)(\Delta,2)(1,\Delta)\\
&= \tau_2(1,m)(\psi,1)(m,1,\sigma)(2,m(m(1,\eta),1))(\widetilde{S},\widetilde{S}S,S,\sigma)(\psi,1)(\delta,3)(\Delta,2)(1,\Delta)\\
&= \tau_2(1,m)(\psi,1)(m,1,\sigma)(2,m)(\psi,2)(\widetilde{S}S,\widetilde{S},S,\sigma)(\delta,3)(\Delta,2)(1,\Delta)\\
&= \tau_2(1,m)(\psi,1)(m,m,\sigma)(\psi,2)(\widetilde{S}(\delta,S)\Delta,\widetilde{S},S,\sigma)(1,\Delta)\\
&= \tau_2(1,m)(m,m,\sigma)(\psi_{22})(\psi,2)(\widetilde{S}^2,\widetilde{S},S,\sigma)(1,\Delta)\\
&= \tau_2(1,m)(m,m,1)(2,\psi,1)(\psi_{22},1)(\widetilde{S}^2,\widetilde{S},S,\sigma,\sigma)(1,\Delta)\\
&= \tau_2(m,m(m,1))(2,\psi,1)(1,\psi,2)(\psi,3)(2,\psi,1)(1,\psi,2)(\widetilde{S}^2,\widetilde{S},S,\sigma,\sigma)(1,\Delta)
\end{aligned}
$$

$$
\begin{aligned}
&= \tau_2(m, m(m,1))(\psi_{13},1)(2,\psi,1)(1,\psi,2)(\widetilde{S}^2, \widetilde{S}, S, \sigma, \sigma)(1,\Delta) \\
&= (m,m)(1,\psi,1)(\psi,2)(\widetilde{S}, S, 1, \sigma)(\Delta,1)(m, m(m,1))(\psi_{13},1)(2,\psi,1)(1,\psi,2)(\widetilde{S}^2, \widetilde{S}, S, \sigma, \sigma)(1,\Delta) \\
&= (m,m)(1,\psi,1)(\psi,2)(\widetilde{S}, S, 1, \sigma)(\Delta m, m(m,1))(\psi_{13},1)(2,\psi,1)(1,\psi,2)(\widetilde{S}^2, \widetilde{S}, S, \sigma, \sigma)(1,\Delta) \\
&= (m,m)(1,\psi,1)(\psi,2)(\widetilde{S}, S, 1, \sigma)((m,m)(1,\psi,1)(\Delta,\Delta), m(m,1)) \\
&\quad (\psi_{13},1)(2,\psi,1)(1,\psi,2)(\widetilde{S}^2, \widetilde{S}, S, \sigma, \sigma)(1,\Delta) \\
&= (m,m)(1,\psi,1)(\psi,2)(\widetilde{S}, S, 1, \sigma)(m, m, m(m,1))(1,\psi,4)(\Delta,\Delta,3) \\
&\quad (\psi_{13},1)(2,\psi,1)(1,\psi,2)(\widetilde{S}^2, \widetilde{S}, S, \sigma, \sigma)(1,\Delta) \\
&= (m,m)(1,\psi,1)(\psi,2)(\widetilde{S}m, Sm, m(m,1), \sigma)(1,\psi,4)(\psi_{15},1)(1,\Delta,\Delta,2) \\
&\quad (2,\psi,1)(1,\psi,2)(\widetilde{S}^2, \widetilde{S}, S, \sigma, \sigma)(1,\Delta) \\
&= (m,m)(1,\psi,1)(\psi,2)(m\psi(\widetilde{S},\widetilde{S}), m\psi(S,S), m(m,1), \sigma)(1,\psi,4)(\psi_{15},1) \\
&\quad (3,\psi_{12},1)(1,\Delta,1,\Delta,1)(1,\psi,2)(\widetilde{S}^2, \widetilde{S}, S, \sigma, \sigma)(1,\Delta) \\
&= (m,m)(1,\psi,1)(\psi,2)(m\psi, m\psi, m(m,1), 1)(\widetilde{S}, \widetilde{S}, S, S, 3, \sigma)(1,\psi,4)(\psi_{15},1) \\
&\quad (3,\psi_{12},1)(1,\psi_{12},3)(1,1,\Delta,\Delta,1)(\widetilde{S}^2, \widetilde{S}, S, \sigma, \sigma)(1,\Delta) \\
&= (m,m)(1,\psi,1)(\psi,2)(m\psi, m\psi, m(m,1), 1)(\widetilde{S}, \widetilde{S}, S, S, 3, \sigma)(1,\psi,4)(\psi_{15},1) \\
&\quad (3,\psi_{12},1)(1,\psi_{12},3)(\widetilde{S}^2, \widetilde{S}, \Delta S, \Delta\sigma, \sigma)(1,\Delta) \\
&= (m,m)(1,\psi,1)(\psi,2)(m\psi, m\psi, m(m,1), 1)(\widetilde{S}, \widetilde{S}, S, S, 3, \sigma)(1,\psi,4)(\psi_{15},1) \\
&\quad (3,\psi_{12},1)(1,\psi_{12},3)(\widetilde{S}^2, \widetilde{S}, (S,S)\psi\Delta, \sigma, \sigma, \sigma)(1,\Delta) \\
&= (m,m)(1,\psi,1)(\psi,2)(m\psi, m\psi, m(m,1), 1)(\widetilde{S}, \widetilde{S}, S, S, 3, \sigma)(1,\psi,4)(\psi_{15},1) \\
&\quad (3,\psi_{12},1)(1,\psi_{12},3)(\widetilde{S}^2, \widetilde{S}, S, S, \sigma, \sigma, \sigma)(2,\psi)(2,\Delta)(1,\Delta) \\
&= (m,m)(1,\psi,1)(\psi,2)(m\psi, m\psi, m(m,1), 1)(\widetilde{S}, \widetilde{S}, S, S, 3, \sigma)(1,\psi,4)(\psi_{15},1) \\
&\quad (3,\psi_{12},1)(1,\psi_{12},3)(\widetilde{S}^2, \widetilde{S}, S, S, \sigma, \sigma, \sigma)(2,\psi)(2,\Delta)(1,\Delta) \\
&= (m,m)(1,\psi,1)(\psi,2)(m\psi, m\psi, m(m,1), 1)(\widetilde{S}, \widetilde{S}, S, S, 3, \sigma)(1,\psi,4)(\psi_{15},1) \\
&\quad (3,\psi_{12},1)(\widetilde{S}^2, S, S, \widetilde{S}, \sigma, \sigma, \sigma)(1,\psi_{12})(2,\psi)(2,\Delta)(1,\Delta) \\
&= (m,m)(1,\psi,1)(\psi,2)(m\psi, m\psi, m(m,1), 1)(\widetilde{S}, \widetilde{S}, S, S, 3, \sigma)(1,\psi,4)(\psi_{15},1)(\widetilde{S}^2, S, S, \sigma, \sigma, \widetilde{S}, \sigma) \\
&\quad (1,\psi_{12})(2,\psi)(2,\Delta)(1,\Delta) \\
&= (m,m)(1,\psi,1)(\psi,2)(m\psi, m\psi, m(m,1), 1)(\widetilde{S}, \widetilde{S}, S, S, 3, \sigma)(1,\psi,4)(S, S, \sigma, \sigma, \widetilde{S}, \widetilde{S}^2, \sigma) \\
&\quad (\psi_{13},1)(1,\psi_{12})(2,\psi)(2,\Delta)(1,\Delta) \\
&= (m,m)(1,\psi,1)(\psi,2)(m\psi, m\psi, m(m,1), 1)(\widetilde{S}, \widetilde{S}, S, S, 3, \sigma)(S, \sigma, S, \sigma, \widetilde{S}, \widetilde{S}^2, \sigma) \\
&\quad (\psi_{13},1)(1,\psi_{12})(2,\psi)(2,\Delta)(1,\Delta) \\
&= (m,m)(1,\psi,1)(\psi,2)(m, m, m(m,1), 1)(\psi,\psi,4)(\widetilde{S}S, \widetilde{S}\sigma, S^2, S\sigma, \widetilde{S}, \widetilde{S}^2, \sigma, \sigma) \\
&\quad (\psi_{13},1)(1,\psi_{12})(2,\psi)(2,\Delta)(1,\Delta) \\
&= (m,m)(m, m(m,1), m, 1)(2,\psi_{23},1)(\psi_{22},4)(\widetilde{S}\sigma, \widetilde{S}S, S\sigma, S^2, \widetilde{S}, \widetilde{S}^2, \sigma, \sigma) \\
&\quad (\psi_{13},1)(1,\psi_{12})(2,\psi)(2,\Delta)(1,\Delta) \\
&= (m,m)(m, m(m,1), m, 1)(2,\psi_{23},1)(S\sigma, S^2, \widetilde{S}\sigma, \widetilde{S}S, \widetilde{S}, \widetilde{S}^2, \sigma, \sigma) \\
&\quad (\psi,2)(\psi_{13},1)(1,\psi_{12})(2,\psi)(2,\Delta)(1,\Delta) \\
&= (m,m)(m, m(m,1), m, 1)(S\sigma, S^2, \widetilde{S}, \widetilde{S}^2, \sigma, \widetilde{S}\sigma, \widetilde{S}S, \sigma) \\
&\quad (1,\psi_{12})(\psi,2)(\psi_{13},1)(1,\psi_{12})(2,\psi)(2,\Delta)(1,\Delta) \\
&= (m,m)(m, m(m,1), m, 1)(S\sigma, S^2, \widetilde{S}, \widetilde{S}^2, \sigma, \widetilde{S}\sigma, \widetilde{S}S, \sigma) \\
&\quad (2,\psi)(1,\psi,1)(\psi,2)(2,\psi)(1,\psi,1)(\psi,2)(2,\psi)(1,\psi,1)(2,\psi)(2,\Delta)(1,\Delta) \\
&= (m,m)(m, m(m,1), m, 1)(S\sigma, S^2, \widetilde{S}, \widetilde{S}^2, \sigma, \widetilde{S}\sigma, \widetilde{S}S, \sigma) \\
&\quad (2,\psi)(1,\psi,1)(\psi,2)(2,\psi)(1,\psi,1)(2,\psi)(\psi,2)(1,\psi_{21})(2,\Delta)(1,\Delta) \\
&= (m,m)(m, m(m,1), m, 1)(S\sigma, S^2, \widetilde{S}, \widetilde{S}^2, \sigma, \widetilde{S}\sigma, \widetilde{S}S, \sigma) \\
&\quad (2,\psi)(1,\psi,1)(\psi,2)(1,\psi,1)(2,\psi)(1,\psi,1)(\psi,2)(1,\psi_{21})(2,\Delta)(1,\Delta) \\
&= (m,m)(m, m(m,1), m, 1)(S\sigma, S^2, \widetilde{S}, \widetilde{S}^2, \sigma, \widetilde{S}\sigma, \widetilde{S}S, \sigma) \\
&\quad (2,\psi)(\psi,2)(\psi_{12},1)(1,\psi_{12})(\psi,2)(1,\psi_{21})(1,\Delta,1)(1,\Delta) \\
&= (m,m)(m, m(m,1), m, 1)(S\sigma, S^2, \widetilde{S}, \widetilde{S}^2, \sigma, \widetilde{S}\sigma, \widetilde{S}S, \sigma) \\
&\quad (2,\psi)(\psi,2)(\psi_{12},1)(1,\psi_{12})(\psi,2)(2,\Delta)(1,\psi)(1,\Delta) \\
&= (m,m)(m, m(m,1), m, 1)(S\sigma, S^2, \widetilde{S}, \widetilde{S}^2, \sigma, \widetilde{S}\sigma, \widetilde{S}S, \sigma) \\
&\quad (2,\psi)(\psi,2)(\psi_{12},1)(1,\Delta,1)(1,\psi)(\psi,1)(1,\psi)(1,\Delta)
\end{aligned}
$$

$$
\begin{aligned}
&= (m,m)(m,m(m,1),m,1)(S\sigma,S^2,\widetilde{S},\widetilde{S}^2,\sigma,\widetilde{S}\sigma,\widetilde{S}S,\sigma)\\
&\quad (2,\psi)(\psi,2)(\Delta,2)(\psi,1)(1,\psi)(\psi,1)(1,\psi)(1,\Delta)\\
&= (m,m)(m,m(m,1),m,1)(S\sigma,S^2,\widetilde{S},\widetilde{S}^2,\sigma,\widetilde{S}\sigma,\widetilde{S}S,\sigma)\\
&\quad (\psi\Delta,2)(1,\psi)(\psi,1)(1,\psi)(\psi,1)(1,\psi)(1,\Delta)\\
&\overset{(2)}{=} (m(m,1),m(m,1))(2,m,3)(1,m,5)(S\sigma,S^2,\widetilde{S},\widetilde{S}^2,\sigma,\widetilde{S}\sigma,\widetilde{S}S,\sigma)\\
&\quad (\psi\Delta,2)(1,\psi)(\psi,1)(1,\psi)(\psi,1)(1,\psi)(1,\Delta)\\
&= (m(m,1),m(m,1))(2,m,3)(S\sigma,m(S^2,\widetilde{S})\psi\Delta,\widetilde{S}^2,\sigma,\widetilde{S}\sigma,\widetilde{S}S,\sigma)\\
&\quad (1,\psi)(\psi,1)(1,\psi)(\psi,1)(1,\psi)(1,\Delta)\\
&\overset{(1)}{=} (m(m,1),m(m,1))(2,m,3)(S\sigma,\eta\delta,\widetilde{S}^2,\sigma,\widetilde{S}\sigma,\widetilde{S}S,\sigma)\\
&\quad (1,\psi)(\psi,1)(1,\psi)(\psi,1)(1,\psi)(1,\Delta)\\
&= (m(m,1),m(m,1))(1,\eta,4)(1,m,3)(S\sigma,\widetilde{S}^2,\sigma,\widetilde{S}\sigma,\widetilde{S}S,\sigma)\\
&\quad (\psi)(\delta,2)(\psi,1)(1,\psi)(\psi,1)(1,\psi)(1,\Delta)\\
&= (m(m(1,\eta),1),m(m,1))(1,m,3)(S\sigma,\widetilde{S}^2,\sigma,\widetilde{S}\sigma,\widetilde{S}S,\sigma)(\psi)(2,\delta)(\psi,1)(1,\psi)(1,\Delta)\\
&= (m,m(m,1))(1,m,3)(S\sigma,\widetilde{S}^2,\sigma,\widetilde{S}\sigma,\widetilde{S}S,\sigma)(\psi)(\psi)(1,\delta,1)(1,\Delta)\\
&= (m,m(m,1))(1,m,3)(S\sigma,\widetilde{S}^2,\sigma,\widetilde{S}\sigma,\widetilde{S},\sigma)(1,S)\psi^2(1,\delta,1)(1,\Delta)\\
&= (m,m(m,1))(1,m,3)(S\sigma,\widetilde{S}^2,\sigma,\widetilde{S}\sigma,\widetilde{S},\sigma)\psi^2(1,S)(1,\delta,1)(1,\Delta)\\
&= (m,m(m,1))(1,m,3)(S\sigma,\widetilde{S}^2,\sigma,\widetilde{S}\sigma,\widetilde{S},\sigma)\psi^2(1,(\delta,S)\Delta)\\
&= (m,m(m,1))(1,m,3)(S\sigma,\widetilde{S}^2,\sigma,\widetilde{S}\sigma,\widetilde{S},\sigma)\psi^2(1,\widetilde{S})\\
&= (m(1,m),m(m,1))(S\sigma,\widetilde{S}^2,\sigma,\widetilde{S}\sigma,\widetilde{S},\sigma)(1,\widetilde{S})\psi^2\\
&= (m(1,m)(S\sigma,\widetilde{S}^2,\sigma),m(m,1)(\widetilde{S}\sigma,\widetilde{S}^2,\sigma))\psi^2\\
&\overset{(3)}{=} (1,1)\psi^2\\
&= \psi^2
\end{aligned}
$$

$\square$

Remark 4.1.

In general we have

$$\tau_n^{n+1} = \psi^n_{H^{(n-1)},H},$$

which is equal to id **if** $\psi^2 = id$**.**

Remark 4.2.

The above procedure of eliminating the symmetry condition can be extended to the more general case of braided triples in Theorem (1.1)**.**

WITH SO MUCH THANKS TO MY SUPERVISOR AND TO ORGANIZERS FOR HAVING ME HERE.

AND THANK YOU FOR LISTENING.