Noncommutative Geometry and Lower Dimensional Volumes in Riemannian Geometry

Raphaël Ponge
(University of Toronto)

⋆⋆ Einstein-Hilbert Action and Area ⋆⋆

Let M be a compact Riemannian spin manifold with Dirac operator $D\!\!\!/$.

Observation (Connes '96)**:**

We have

$$⨍ D\!\!\!/^{-n+2} = -c_n \int_M \kappa(x)\sqrt{g(x)}d^n x,$$

where $⨍$ is the Dixmier trace and κ is the scalar curvature.

Observation (Connes '96):

In QFT $ds := D\!\!\!/^{-1}$ is the free propagator for Fermions. Therefore:

- ds has the physical dimension of a length;

- $⨍ ds^2$ can be interpreted as the area of M.

Consequence:

When $\dim M = 4$ the Einstein-Hilbert action,

$$\int_M \kappa(x)\sqrt{g(x)}d^4x,$$

yields a differential geometric expression of the area of M.

⋆⋆ Lower Dimensional Volumes ⋆⋆

Let (M^n, g) be a compact Riemannian manifold.

Question:

For any $k = 1, 2, \ldots, n-1$ can we also give sense to the k'th dimensional volume of M?

Answer (RP '07):

The answer uses 2 main tools:

- Noncommutative residue trace of Wodzicki and Guillemin;
- Quantized calculus of Connes.

⋆⋆ Noncommutative Residue ⋆⋆
(Wodzicki, Guillemin)

• Trace on the algebra of ΨDOs on a compact manifold M^n independently found by Wodzicki and Guillemin.

• Numerous applications and generalizations, e.g., this is an essential tool in the framework for the local index formula in NCG of Connes-Moscovici.

• Elementary construction in terms of the logarithmic singularity of the kernel near the diagonal (Connes-Moscovici).

Let $U \subset \mathbb{R}^n$ be a local chart.

Symbols of order $\mathbf{m}$:

• Smooth functions $p(x, \xi)$ on $U \times \mathbb{R}^n$ with an asymptotic expansion,

$$p(x,\xi) \sim \sum_{j \geq 0} p_{m-j}(x,\xi),$$

$$p_{m-j}(x, t\xi) = t^{m-j} p_{m-j}(x,\xi) \quad \forall t > 0.$$

ΨDOs of order m:

• To a symbol $p(x,\xi)$ we associate the operator $P = p(x,D)$ from $C_c^\infty(U)$ to $C^\infty(U)$ such that

$$Pu(x) = (2\pi)^{-n} \int e^{i\langle x,\xi\rangle} p(x,\xi)\hat{u}(\xi) d\xi.$$

• A ΨDO of order m on the manifold M is a continuous operator $P : C^\infty(M) \to C^\infty(M)$ which is locally of the form:

$$P = p(x,D) + R,$$

with $p(x,\xi)$ symbol of order m and R smoothing operator.

Logarithmic singularity:

• The kernel $k_P(x,y)$ of P has a behavior near the diagonal $y=x$ of the form:

$$k_P(x,y) = \sum_{-(m+n)\leq l\leq 0} a_l(x,x-y) - c_P(x)\log|x-y| + \mathrm{O}(1),$$

where

$$a_l(x,ty) = t^l a_l(x,y) \quad \forall t>0,$$
$$c_P(x) = (2\pi)^{-n}\int_{|\xi|=1} p_{-n}(x,\xi)d\xi.$$

Lemma. The coefficient $c_P(x)$ makes sense globally on M as a density which is functorial with respect to diffeomorphisms.

Noncommutative residue:

- The noncommutative residue of P is

$$\operatorname{Res} P := \int_M c_P(x).$$

Proposition (Guillemin, Wodzicki)**.** The following hold:

1. Locality: $\operatorname{Res} P = 0$ if $\operatorname{ord} P < -n$.

2. Invariance: we have $\operatorname{Res}_{M'} \phi^* P = \operatorname{Res}_M P$ for any diffeomorphism $\phi : M' \to M$.

3. Trace: $\operatorname{Res} P_1 P_2 = \operatorname{Res} P_2 P_1$.

⋆⋆ Quantized Calculus ⋆⋆
(Connes)

- $\mathcal{H} =$ separable Hilbert space.

Classical	Quantum
Complex variable	Operator on $\mathcal{H}$
Real variable	Selfadjoint operator
Infinitesimal variable	Compact operator
Infinitesimal of order $\alpha > 0$	Compact operator T s.t. $\mu_k(T) = \mathrm{O}(k^{-\alpha})$
Integral	Dixmier trace $⨍$

- Here $\mu_k(T) := (k+1)$'th eigenvalue of $|T|$.

- If $T \in \mathcal{K}_+$, then:

$$\frac{1}{\log N} \sum_{k<N} \mu_k(T) \to L \Longrightarrow ⨍ T = L.$$

- For $\mathcal{H} = L^2(M)$ we have:

Theorem (Connes '88)**.** Let P be a ΨDO of order m, $m < 0$.

1. P is an infinitesimal of order $\frac{|m|}{n}$.

2. If $\mathrm{ord} P = -n$, then

$$⨍ P = \frac{1}{n} \mathrm{Res}\, P.$$

Consequence:

We can integrate any ΨDO, even if it is not an infinitesimal of order ≤ 1, by setting

$$⨍ P := \frac{1}{n} \mathrm{Res}\, P.$$

⋆⋆ Lower dimensional volumes ⋆⋆
(n even, M spin)

• Assume M has even dimension and is spin with Dirac operator $D\!\!\!/$.

• As $\sigma_{-n}(D\!\!\!/^{-n}) = \sigma_{-n}[(D\!\!\!/^2)^{-\frac{n}{2}}] = |\xi|^{-n}$ we get

$$c_{D\!\!\!/^{-n}}(x) = n.\frac{(2\pi)^{-\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}\sqrt{g(x)}d^nx.$$

• From this we deduce that, for any $f \in C^\infty(M)$,

$$⨍ fD\!\!\!/^{-n} = \frac{(2\pi)^{-\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}\int_M f(x)\sqrt{g(x)}d^nx.$$

Thus $D\!\!\!/^{-n}$ allows us to recover the Riemannian volume form.

Noncommutative length element:

- It is defined to be

$$ds := c_n \not{D}^{-1}, \quad c_n = \sqrt{2\pi}\,\Gamma\left(\frac{n}{2}+1\right)^{\frac{1}{n}}.$$

- We have

$$⨍ ds^n = \mathrm{Vol}_g\, M.$$

Lower dimensional volumes:

- For $k = 1, \ldots, n$ the kth dimensional volume is

$$\mathrm{Vol}_g^{(k)} M := ⨍ ds^k.$$

- The area of M is

$$\mathrm{Area}_g M := \mathrm{Vol}_g^{(2)} M = ⨍ ds^2.$$

Proposition (RP '07).

1. $\mathrm{Vol}_g^{(k)} M$ vanishes when k is odd.

2. When k is even, we have

$$\mathrm{Vol}_g^{(k)} M = \nu_{n,k} \int_M \gamma_{n-k}(x)\sqrt{g(x)}d^n x,$$

$$\nu_{n,k} = \frac{k}{n}(2\pi)^{\frac{k-n}{2}} \frac{\Gamma(\frac{n}{2}+1)^{\frac{k}{n}}}{\Gamma(\frac{k}{2}+1)},$$

where $\gamma_{n-k}(x)$ is a universal polynomial in complete contractions of the covariant derivatives of the curvature tensor depending only on $n-k$.

- For $n-k=0,2,4$ we have

$$\gamma_0(x)=1, \quad \gamma_2(x)=\frac{-\kappa(x)}{12},$$
$$\gamma_4(x)=\frac{1}{1440}(5\kappa(x)^2-8|\rho(x)|^2-7|R(x)|^2).$$

where R denotes the curvature tensor, ρ is the Ricci tensor and κ is the scalar curvature.

⋆⋆ Lower dimensional volumes ⋆⋆
(n even, general case)

• In general the kth dimensional volume is

$$\mathrm{Vol}_g^{(k)}\, M := \begin{cases} \nu_{n,k} \int_M \gamma_{n-k}(x)\sqrt{g(x)}d^n x & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

• This definition is purely differential geometric and does not make reference to noncommutative geometry anymore.

Examples:

- If $\dim M = 4$, then

$$\text{Area}_g\, M = \frac{-1}{96\pi\sqrt{2}} \int_M \kappa(x)\sqrt{g(x)}d^4x.$$

- If $\dim M = 6$, then

$$\text{Area}_g\, M = \frac{\sqrt[3]{6}}{69120\pi^2} \int_M (5\kappa(x)^2 - 8|\rho(x)|^2 - 7|R(x)|^2)\sqrt{g(x)}d^6x.$$

⋆⋆ **Lower dimensional volumes (n odd)** ⋆⋆

- For $k = 1, \dots, n$ the kth dimensional volume is

$$\mathrm{Vol}_g^{(k)} M = \begin{cases} \nu'_{n,k} \int_M \gamma_{n-k}(x) \sqrt{g(x)} d^n x & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even,} \end{cases}$$

where

$$\nu'_{n,k} = \frac{k}{n} 2^{\frac{k-n}{2n}} (2\pi)^{\frac{k-n}{2}} \frac{\Gamma(\frac{n}{2}+1)^{\frac{k}{n}}}{\Gamma(\frac{k}{2}+1)}.$$

- The length of M is

$$\mathrm{Length}_g M := \mathrm{Vol}_g^{(1)} M = \nu'_{n1} \int_M \gamma_{n-1}(x) \sqrt{g(x)} d^n x.$$

Examples:

- If $\dim M = 3$, then

$$\mathrm{Length}_g M = \frac{-1}{72\pi^{\frac{5}{6}}} \int_M \kappa(x)\sqrt{g(x)}d^3x.$$

- If $\dim M = 5$, then

$$\mathrm{Length}_g M = \frac{1}{1800\pi^2}\sqrt[5]{\frac{15\pi^2}{2}}.$$
$$\int_M (5\kappa(x)^2 - 8|\rho(x)|^2 - 7|R(x)|^2)\sqrt{g(x)}d^5x.$$