Noncommutative Geometry Workshop;

Fields Institute, Toronto May 27-31, 2008

Monopoles and Laplacians on quantum Hopf bundles

Giovanni Landi

Trieste

Math motivation:

(M, g) a cmpt Riemannian manifold
$P \to M$ a principal bundle with cmpt structure group G
a connection on P with covariant derivative ∇

(ρ, V) a representation of G ; the identification of sections of the associated vector bundle $E = P \times_G V$ on M with equivariant maps from P to V:
$\Gamma(M, E) \simeq C^\infty(P, V)_G \subset C^\infty(P) \otimes V$

The Laplacians,

$\Delta^P = -(\mathrm{dd}^* + \mathrm{d}^*\mathrm{d})$ on P; it acts on $C^\infty(P)$

$\Delta^E = -(\nabla\nabla^* + \nabla^*\nabla)$ on E; it acts on $\Gamma(M, E)$

are related by

$$\Delta^E = \left(\Delta^P \otimes 1 + 1 \otimes C_G\right)\Big|_{C^\infty(P,V)_G}$$

$C_G = \sum_a \rho(e_a)^2 \in End(E)$; the quadratic Casimir op. of G

Δ^E is the gauged Laplacian:

$\Delta^M \mapsto \Delta^E$ as $\mathrm{d} \mapsto \nabla$

For $P = H$ a cmpt group

$\Delta^P = C_H$

and diagonalization of Δ^E is easy

Phys motivation:

The Laughlin wave functions for the fractional quantum Hall effect (on the plane) is not translationally invariant.

This problem was overcome by Haldane with a model on a sphere with a magnetic monopole at the origin.

The full Euclidean group of symmetries of the plane is recovered from the rotation group $SO(3)$ of symmetries of the sphere.

One is considering the Hopf fibration of the sphere S^3 over the sphere S^2 with $U(1)$ as gauge (or structure) group

and needs to diagonalize the
Laplacian of S^2 gauged with the monopole connection

Two classes of examples:

q-spaces: manly the monopoles over $A(\mathsf{S}^2_q)$

θ-spaces: manly instantons over $A(S^4_\theta)$

but also the nctorus

GL, C Reina, A Zampini
Gauged Laplacians on quantum Hopf bundles, arXiv:0801.3376

GL,
Spin-Hall effect with quantum group symmetries Lett. Math. Phys., 75 (2006) 187–200

and work in progress

The geometry of quantum $\mathrm{SU}_q(2)$

The algebra:

With $0 < q < 1$, let $\mathcal{A} = A(\mathrm{SU}_q(2))$ be the $*$-algebra generated by a and c, with relations:

$$ac = qca, \qquad ac^* = qc^*a, \qquad cc^* = c^*c,$$

$$a^*a + c^*c = aa^* + q^2cc^* = 1$$

these state that the defining matrix is unitary

$$U = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix}$$

$A(\mathsf{SU}_q(2))$ is a Hopf $*$-algebra (a quantum group) with

- coproduct:

$$\Delta \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} := \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} \dot{\otimes} \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix}$$

- counit:

$$\varepsilon \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- antipode:

$$S \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} = \begin{pmatrix} a^* & c^* \\ -qc & a \end{pmatrix}$$

The quantum universal envelopping algebra $\mathcal{U} = \mathcal{U}_q(\mathrm{su}(2))$ is the $*$-algebra generated by E, F, K, with K invertible, and relations

$$KE = qEK, \quad KF = q^{-1}FK, \quad K^2 - K^{-2} = (q - q^{-1})(EF - FE),$$

The $*$-structure: $K^* = K, \quad F^* = E, \quad E^* = F.$

The Hopf $*$-algebra structure

- coproduct:

$$\Delta K = K \otimes K, \quad \Delta F = F \otimes K + K^{-1} \otimes F, \quad \Delta E = E \otimes K + K^{-1} \otimes E$$

- counit:

$$\epsilon(K) = 1, \quad \epsilon(F) = 0, \quad \epsilon(E) = 0$$

- antipode:

$$SK = K^{-1}, \quad SE = -qE, \quad SF = -q^{-1}F$$

The Casimir operator

$$C_q = (q - q^{-1})^{-2}(q^{\frac{1}{2}}K - q^{-\frac{1}{2}}K^{-1})^2 + FE - \tfrac{1}{4}$$

The action of $\mathcal{U}$ on $\mathcal{A}$

A natural bilinear pairing between $\mathcal{U}$ and $\mathcal{A}$,

$$\langle K, a\rangle = q^{-\frac{1}{2}}, \quad \langle K, a^*\rangle = q^{\frac{1}{2}}, \qquad \langle E, c\rangle = 1, \qquad \langle F, c\rangle = -q^{-1}$$

gives commuting left and right $\mathcal{U}$-module algebra structures on $\mathcal{A}$:

$$h \triangleright x := x_{(1)} \left\langle h, x_{(2)} \right\rangle, \qquad x \triangleleft h := \left\langle h, x_{(1)} \right\rangle x_{(2)}$$

with notation $\Delta(x) = x_{(1)} \otimes x_{(2)}$

A left-covariant calculus on $\mathsf{SU}_q(2)$; it is three dimensional

The quantum tangent space $\mathcal{X}_{\mathsf{SU}_q(2)}$ is generated by:

$$X_z = \frac{1-K^4}{1-q^{-2}}, \qquad X_- = q^{-1/2}FK, \qquad X_+ = q^{1/2}EK$$

The dual basis for the one-forms: $\Omega^1(\mathsf{SU}_q(2))$

$$\omega_z = a^*\mathrm{d}a + c^*\mathrm{d}c, \qquad \omega_- = c^*\mathrm{d}a^* - qa^*\mathrm{d}c^*, \qquad \omega_+ = a\mathrm{d}c - qc\mathrm{d}a$$

comes with left-invariance: $\Phi_L(\omega_s) = 1 \otimes \omega_s$;

with $\Phi_L(x\mathrm{d}x') = \Delta(x)(id \otimes \mathrm{d})\Delta(x')$

The exterior derivative is expressed as

$$\mathrm{d}x = (X_s \triangleright x)\omega_s$$

Higher dimensional forms:

$$\mathrm{d}\omega_z = -\omega_- \wedge \omega_+$$

$$\mathrm{d}\omega_+ = q^2(1+q^2)\omega_z \wedge \omega_+, \qquad \mathrm{d}\omega_- = -(1+q^{-2})\omega_z \wedge \omega_-$$

commutation relations among forms

A $U(1)$ principal bundle.

On $A(\mathsf{SU}_q(2))$ a right coaction of $U(1) = \mathbb{C} < z, z^{-1} >$:

$$\Delta_R : A(\mathsf{SU}_q(2)) \to A(\mathsf{SU}_q(2)) \otimes U(1)$$

$$\Delta_R \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} \dot{\otimes} \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$$

the subalgebra of coinvariants

$$A(\mathsf{S}^2_q) := \{ p \in A(\mathsf{SU}_q(2)) \ , \ \Delta_R(p) = p \otimes 1 \}$$

is Podleś standard sphere. Possible generators:

$$b_- := -q(1+q^2)^{-\frac{1}{2}} \ ac^*, \qquad b_+ := q^2(1+q^2)^{-\frac{1}{2}} \ ca^*$$
$$b_0 := aa^* - (1+q^2)^{-1}$$

A left coaction of $A(\mathsf{SU}_q(2))$ on $A(\mathsf{S}^2_q)$:

$$\Delta : A(\mathsf{S}^2_q) \to A(\mathsf{SU}_q(2)) \otimes A(\mathsf{S}^2_q),$$

$$\Delta(b_-) = a^2 \otimes b_- - (1+q^{-2}) b_- \otimes b_0 + c^{*2} \otimes b_+$$

$$\Delta(b_0) = (1+q^2)^{\frac{1}{2}} ac \otimes b_- + (1+q^{-2}) b_0 \otimes b_0 - (1+q^{-2})^{\frac{1}{2}} c^* a^* \otimes b_+$$

$$\Delta(b_+) = q^2 c^2 \otimes b_- + (1+q^{-2}) b_+ \otimes b_0 + a^{*2} \otimes b_+$$

The left action of the group-like element K on $A(\mathsf{SU}_q(2))$ defines (modules of sections) of line bundles over S^2_q:

$$\mathcal{L}_n := \{x \in A(\mathsf{SU}_q(2)) \ : \ K \triangleright x = q^{n/2} x\}$$

this has winding number n, with $n \in \mathbb{Z}$.

In particular $A(\mathsf{S}^2_q) = \mathcal{L}_0$. Also: $A(\mathsf{SU}_q(2)) = \oplus_n \mathcal{L}_n$

$$\mathcal{L}_n^* \subset \mathcal{L}_{-n}, \qquad \mathcal{L}_n \mathcal{L}_m \subset \mathcal{L}_{n+m}$$

$$E \triangleright \mathcal{L}_n \subset \mathcal{L}_{n+2}, \qquad F \triangleright \mathcal{L}_n \subset \mathcal{L}_{n-2}$$

$$\mathcal{L}_n \triangleleft h \subset \mathcal{L}_n, \qquad h \in \mathcal{U}_q(\mathsf{su}(2))$$

The corresponding projections:

For $n > 0$: $$\mathfrak{p}^{(n)} = \left|\Psi^{(n)}\right\rangle \left\langle\Psi^{(n)}\right|$$

$$\left|\Psi^{(n)}\right\rangle_{\mu} \sim c^{*\mu} a^{*n-\mu}$$

For $n < 0$: $$\breve{\mathfrak{p}}^{(n)} = \left|\breve{\Psi}^{(n)}\right\rangle \left\langle\breve{\Psi}^{(n)}\right|$$

$$\left|\breve{\Psi}^{(n)}\right\rangle_{\mu} \sim c^{|n|-\mu} a^{\mu}$$

Let $\mathcal{E}_n := (A(\mathsf{S}^2_q))^{n+1}\mathfrak{p}^{(n)}$ a left $A(\mathsf{S}^2_q)$-modules isomorphism:

$$\mathcal{L}_n \xrightarrow{\simeq} \mathcal{E}_n, \quad \phi_f \to \sigma_f := \phi_f \left\langle\Psi^{(n)}\right| = \langle f|\,\mathfrak{p}^{(n)},$$

with inverse

$$\mathcal{E}_n \xrightarrow{\simeq} \mathcal{L}_n, \quad \sigma_f = \langle f|\,\mathfrak{p}^{(n)} \to \phi_f := \left\langle f, \Psi^{(n)}\right\rangle,$$

and similar maps for the case $n \leq 0$.

The differential calculus on S^2_q:

$$\Omega(A(\mathsf{S}^2_q)) \simeq A(\mathsf{S}^2_q) \oplus \left(\mathcal{L}_{-2} \oplus \mathcal{L}_2\right) \oplus A(\mathsf{S}^2_q)$$

In particular

$$\Omega^1(A(\mathsf{S}^2_q)) = \Omega^+ \oplus \Omega^- \simeq \mathcal{L}_{-2} \oplus \mathcal{L}_2$$

$$\partial b_- = \frac{q^3}{(1+q^2)^{\frac{1}{2}}}\, c^{*2}\omega_+, \quad \partial b_0 = -q^2\, c^* a^* \omega_+, \quad \partial b_+ = \frac{q^3}{(1+q^2)^{\frac{1}{2}}}\, a^{*2}\omega_+$$

$$\bar\partial b_- = \frac{1}{(1+q^2)^{\frac{1}{2}}}\, a^2\omega_-, \quad \bar\partial b_0 = ac\omega_-, \quad \bar\partial b_+ = \frac{q^2}{(1+q^2)^{\frac{1}{2}}}\, c^2\omega_-$$

Ω^+ is generated by $\{c^{*2}, c^*a^*, a^{*2}\}\omega_+ = \{\partial b_-, \partial b_0, \partial b_+\}$:

$$\partial b_0 = (q^{-1} + q^{-3}) b_- \partial b_+ - (q + q^3) b_+ \partial b_-$$

Ω^- is generated by $\{c^2, ca, a^2\}\omega_- = \{\bar\partial b_-, \bar\partial b_0, \bar\partial b_+\}$:

$$\bar\partial b_0 = (q^{-1} + q)b_+\bar\partial b_- - (q^{-5} + q^{-3})b_-\bar\partial b_+$$

$$\mathrm{d} = \partial + \bar\partial, \qquad \mathrm{d}x = (X_+ \triangleright x)\omega_+ + (X_- \triangleright x)\omega_-$$

$$\partial x = (X_+ \triangleright x)\omega_+, \quad \bar\partial x = (X_- \triangleright x)\omega_-$$

Also,

$$\Omega^2(A(\mathsf{S}^2_q)) = A(\mathsf{S}^2_q)(\omega_+ \wedge \omega_-) = (\omega_+ \wedge \omega_-)A(\mathsf{S}^2_q)$$

The calculus on S_q^2 can be realized via the Dirac operator

$$d = [D, \ \cdot \ ]$$

$D = D_{irac}$

Dabrowski-Sitarz , Schmüdgen-Wagner

The Laplacian operator on S^2_q

First, a Hodge $\star$-operator on the forms

$$\star\, 1 = \omega_+ \wedge \omega_-, \qquad \star(\omega_+ \wedge \omega_-) = 1$$

$$\star\, \partial f = \partial f, \qquad \star\bar\partial f = -\bar\partial f$$

Then the Laplacian operator on S^2_q is defined as:

$$\Delta^{\mathsf{S}^2_q} f \;:= -\tfrac{1}{2} \star \mathsf{d} \star \mathsf{d} f$$

One finds also

$$\Delta^{\mathsf{S}^2_q} f \;= -\,\bar\partial \partial f \;=\; \partial \bar\partial f$$

$$\Delta^{\mathsf{S}^2_q} f \;=\; \tfrac{1}{2} \Big[X_+ X_- + q^{-2} X_- X_+ \Big] \triangleright f = q^{-1} F E \triangleright f$$

Easy to diagonalise; it is the Casimir operator restrict to $A(\mathsf{S}^2_q)$:

$$\Delta^{\mathsf{S}^2_q} = C_q\Big|_{A(\mathsf{S}^2_q)} + q^{-1}(\tfrac{1}{4} - [\tfrac{1}{2}]^2)$$

Note also that: $\qquad \Delta^{\mathsf{S}^2_q} \sim (D_{irac})^2$

Decompose $A(\mathsf{S}^2_q)$ for the right action of $\mathcal{U}_q(\mathrm{su}(2))$:
this yields the eigenspaces of the Laplacian since

$$\Delta^{\mathsf{S}^2_q}(f \triangleleft h) = (\Delta^{\mathsf{S}^2_q} f) \triangleleft h, \quad h \in \mathcal{U}_q(\mathrm{su}(2))$$

One has,

$$A(\mathsf{S}_q^2) = \oplus_{J\in 2\mathbb{N}} V_J$$

the lowest weight vector of V_J is $a^J c^{*J}$ and a basis of V_J is given by $(a^J c^{*J}) \triangleleft E^m$ with $m = 0, 1, \dots, 2J$.

Also the eigenvalues are

$$\lambda_J = q^{-1}[J][J+1]$$

a direct computation gives

$$\Delta^{\mathsf{S}_q^2}(a^J c^{\star J}) = q^{-1}([J][J+1]) a^J c^{*J}$$

Enter the connection

a calculus on $U(1)$:

$$\mathsf{d}z = z\omega_z, \qquad \mathsf{d}z^{-1} = -q^2 z^{-1}\omega_z, \qquad \omega_z = z^{-1}\mathsf{d}z$$
$$\omega_z z = q^{-2} z\omega_z, \qquad \omega_z z^{-1} = q^2 z^{-1}\omega_z, \qquad z\mathsf{d}z = q^2 \mathsf{d}z z$$

A principal connection; a right invariant splitting:

$$\Omega_{\mathsf{SU}_q(2)} = \Omega^{ver}_{\mathsf{SU}_q(2)} \oplus \Omega^{hor}_{\mathsf{SU}_q(2)}, \qquad \Pi : \Omega_{\mathsf{SU}_q(2)} \mapsto \Omega^{ver}_{\mathsf{SU}_q(2)}$$

with $\Delta_R^{(1)}\,\Pi = (\Pi \otimes id)\,\Delta_R^{(1)}$

Now, $\Delta_R^{(1)}(\omega_z) = \omega_z \otimes 1$

a natural choice of a connection is to define ω_z to be vertical:

$$\Pi_z(\omega_z) = \omega_z \ , \qquad \Pi_z(\omega_\pm) = 0$$

The corresponding covariant derivative on co-equivariant maps:

$$\nabla\phi := (1 - \Pi_z)\,\mathrm{d}\phi$$

explicitly

$$\nabla\phi = \Big(X_+ \triangleright \phi\Big)\,\omega_+ + (X_- \triangleright \phi)\,\omega_-$$

On the sections: $\mathcal{E} \simeq \mathcal{L}_n \simeq \mathfrak{p}^{(n)}\left[A(\mathsf{S}^2_q)\right]^n$

$$\nabla\sigma_\phi := (\mathfrak{p}^{(n)}\mathrm{d})\sigma_\phi = \sigma_{\nabla\phi}$$

The gauged Laplacian

$$\Delta^{\mathcal{E}} : \mathcal{E} \to \mathcal{E}, \qquad \Delta^{\mathcal{E}} := -\tfrac{1}{2} \star \nabla \star \nabla$$

One finds on equivariant maps $\mathcal{L}_n$:

$$\Delta^{\mathcal{E}}\phi = \tfrac{1}{2} K^{-4} \left[X_+ X_- + q^{-2} X_- X_+ \right] \triangleright \phi$$

$\Rightarrow$

$$qK^2 \Box_\nabla = C_q + \tfrac{1}{4} - \tfrac{1}{2} \left(\frac{qK^2 - 2 + q^{-1}K^{-2}}{(q-q^{-1})^2} + \frac{q^{-1}K^2 - 2 + qK^{-2}}{(q-q^{-1})^2} \right)$$

To diagonalize $\Delta^{\mathcal{E}}$, one decomposes $\mathcal{L}_n$ for the right action of $\mathcal{U}_q(\mathrm{su}(2))$: this yields the eigenspaces of the Laplacian since

$$\Delta^{\mathcal{E}}(\phi_f \triangleleft h) = (\Delta^{\mathcal{E}}\phi_f) \triangleleft h, \quad h \in \mathcal{U}_q(\mathrm{su}(2))$$

One has, $\mathcal{L}_n = \bigoplus V_J^{(n)}$ with $J = \frac{|n|}{2} + s$, $s \in \mathbb{N}$

in $V_J^{(n)}$ the highest weight elements is $\phi_{n,J} = c^{J-n/2} a^{*J+n/2}$, and the $2J$ basis vectors are obtained via the right action of $\triangleleft E$

On the vectors $\phi_{n,J,l} = (c^{J-n/2} a^{*J+n/2}) \triangleleft E^l$ on finds:

$$\Box_\nabla \phi_{n,J,l} = \lambda_{n,J} \phi_{n,J,l},$$

with the $(2J+1)$-degenerate energies:

$$\lambda_{n,J} = q^{-n-1} \left\{ [J + \tfrac{1}{2}]^2 - \tfrac{1}{2} \left([\tfrac{n+1}{2}]^2 + [\tfrac{n-1}{2}]^2 \right) \right\}$$

A remarkable fact is that, contrary to what happens in the classical limit, the energies are not symmetric under the exchange $n \leftrightarrow -n$ ('quantization removes degeneracy')

Writing $J = \frac{|n|}{2} + s$, with $s \in \mathbb{N}$, the energies become:

$$\lambda_{n,s} = q^{-n-1}\left([s][n+s+1] + \tfrac{1}{2}[n]\right), \qquad \text{for} \quad n \geq 0,$$

with $(n+2s+1)$ eigenfunctions $\phi_{n,s,l} = (c^s a^{*n+s}) \triangleleft E^l$,

$$\lambda_{n,s} = q^{-n-1}\left([s-n][s+1] + \tfrac{1}{2}[n]\right), \qquad \text{for} \quad n \leq 0,$$

with $(\mid n \mid + 2s + 1)$ eigenfunctions $\phi_{n,s,l} = (c^{s+|n|} a^{*s}) \triangleleft E^l$.

A physics parallel with the quantum Hall effect: the integer s labels Landau levels and the $\phi_{n,s,l}$ are the (‘one excitation’) Laughlin wave functions with energies $\lambda_{n,s}$. The lowest Landau, $s = 0$, is $\mid n \mid$-degenerate with energy

$$\lambda_{n,0} = \frac{1}{2} q^{-n-1} [\mid n \mid]$$

The classical limit. At the value $q = 1$, the energies of the gauged Laplacian become

$$\lambda_{n,s}(q \to 1) = J(J+1) - \frac{1}{4} n^2 = \mid n \mid (s + \frac{1}{2}) + s(s+1)$$

and coincide with the energies of the classical gauged Laplacian. They are symmetric under the exchange $n \leftrightarrow -n$ which corresponds to inverting the direction of the magnetic field.

The winding numbers:

The Chern character has a non trivial component in degree zero $\mathsf{ch}_0(\mathfrak{P}^{(n)}) \in \mathsf{HC}_0(\mathsf{S}^2_q)$ given by a (partial) matrix trace:

$$\mathsf{ch}_0(\mathfrak{P}^{(n)}) = \begin{cases} \sum_{\mu=0}^{n} \beta_{n,\mu} (c^*c)^{\mu} \prod_{j=0}^{n-\mu-1} (1 - q^{-2j} c^*c), & n \geq 0 \\ \\ \sum_{\mu=0}^{|n|} \alpha_{n,\mu} (c^*c)^{|n|-\mu} \prod_{j=0}^{\mu-1} (1 - q^{2j} c^*c), & n \leq 0 \end{cases},$$

Dually, one needs a cyclic 0-cocycle on $A(\mathsf{S}^2_q)$; Masuda et al. :

$$\mu\left((c^*c)^k\right) = (1 - q^{2k})^{-1}, \qquad k > 0.$$

The pairing results in (Hajac)

$$\left\langle [\mu], [\mathfrak{P}^{(n)}] \right\rangle := \mu \left(\mathrm{ch}_0(\mathfrak{P}^{(n)}) \right) = -n$$

This integer is a topological quantity that depends only on the bundle, both on the quantum sphere than on its classical limit

In the limit is also computed by integrating the curvature 2-form of a connection (indeed any connection) on the classical sphere

To integrate the gauge curvature on the quantum sphere S^2_q one needs a 'twisted integral'; furthermore the result is not an integer any longer but rather a q-integer

Integrating the curvature

h the Haar state on $A(\mathsf{S}^2_q)$; ϑ the modular automorphism:

$$\vartheta(x) := x \triangleleft K^2, \qquad x \in A(\mathsf{S}^2_q)$$

then the linear functional

$$\int : \Omega^2(A(\mathsf{S}^2_q)) \to \mathbb{C}, \qquad \int x \; \omega_+ \wedge \omega_- := h(x)$$

defines a non-trivial ϑ-twisted cyclic 2-cocycle on $A(\mathsf{S}^2_q)$

$$\tau(a_0, a_1, a_2) = \int a_0 \wedge \mathrm{d}a_1 \wedge \mathrm{d}a_2$$

$$b_\vartheta \tau = 0, \qquad \lambda_\vartheta \tau = \tau$$

Schmüdgen-Wagner

b_ϑ the ϑ-twisted coboundary operator:

$$(b_\vartheta\tau)(f_0,f_1,f_2,f_3) := \tau(f_0f_1,f_2,f_3) - \tau(f_0,f_1f_2,f_3) \\ + \tau(f_0,f_1,f_2f_3) - \tau(\vartheta(f_3)f_0,f_1,f_2),$$

λ_ϑ is the ϑ-twisted cyclicity operator:

$$(\lambda_\vartheta\tau)(f_0,f_1,f_2) := \lambda_\tau(\vartheta(f_2),f_0,f_1)$$

$[\tau] \in \mathsf{HC}^2_\vartheta(\mathsf{S}^2_q)$
the degree 2 twisted cyclic cohomology of the sphere S^2_q

Couple τ with the bundles over S^2_q, via a twisted Chern character

It is enough to consider the lowest term, given by a twisted or ‘quantum trace’

If $M \in \mathsf{Mat}_{m+1}(A(\mathsf{S}^2_q))$, its (partial) quantum trace

$$\mathsf{tr_q}(M) := \mathsf{tr}\left(M\sigma_{m/2}(K^2)\right) := \sum_{jl} M_{jl}\left(\sigma_{m/2}(K^2)\right)_{lj} \quad \in \quad A(\mathsf{S}^2_q)$$

$\sigma_{m/2}(K^2)$ is the spin $J = m/2$ representation of $\mathcal{U}_q(\mathsf{su}(2))$

The q-trace is 'twisted' by the automorphism ϑ

$$\mathsf{tr_q}(M_1 M_2) = \mathsf{tr_q}\left((M_2 \triangleleft K^2)M_1\right) = \mathsf{tr_q}\left(\vartheta(M_2)M_1\right)$$

One finds ($n > 0$ say):

$$F_\nabla = \mathfrak{p}^{(n)}\, \mathsf{d}\mathfrak{p}^{(n)} \wedge\, \mathsf{d}\mathfrak{p}^{(n)} \;=\; -q^{-n-1}[n]\; \mathfrak{p}^{(n)}\, \omega_+ \wedge \omega_-$$

$$\mathsf{tr}_\mathsf{q}(\mathfrak{p}^{(n)}) := \mathsf{tr}\left(\pi_{n/2}(K^2)\; \mathfrak{p}^{(n)}\right) = q^n$$

$\pi_{n/2}$ is the spin $n/2$ representation of $\mathcal{U}_q(\mathsf{su}(2))$

$$\Rightarrow \qquad (q\tau) \circ \mathsf{tr}_\mathsf{q}\,(\mathfrak{p}^{(n)}, \mathfrak{p}^{(n)}, \mathfrak{p}^{(n)}) = q \int \mathsf{tr}_\mathsf{q}\left(\mathfrak{p}^{(n)}\, \mathsf{d}\mathfrak{p}^{(n)} \wedge\, \mathsf{d}\mathfrak{p}^{(n)}\right)$$
$$= q \int \mathsf{tr}_\mathsf{q}\, F_\nabla = -[n]$$

it is the q-index of the Dirac operator on S^2_q
Wagner, Neshveyev-Tuset

A Hopf-Galois extension with $SU_q(2)$ as 'structure quantum group'

$SU_q(2)$ co-acts on a quantum sphere S_q^7

coming from the symplectic groups $Sp_q(2)$

the co-fixed-point subalgebra is a quantum sphere S_q^4

GL, C. Pagani, C. Reina, CMP 263 (2006) 65-88

A noncommutative Hopf fibration on S^4_θ

θ a real parameter, the coordinate algebra $A(S^4_\theta)$ of the sphere S^4_θ is generated by elements $z_0 = z_0^*, z_j, z_j^*$, $j = 1, 2$, with

$$z_\mu z_\nu = \lambda_{\mu\nu} z_\nu z_\mu, \quad z_\mu z_\nu^* = \lambda_{\nu\mu} z_\nu z_\mu^*, \quad z_\mu^* z_\nu^* = \lambda_{\mu\nu} z_\nu^* z_\mu^*, \quad \mu, \nu = 0, 1, 2,$$

and deformation parameters

$$\lambda_{12} = \bar{\lambda}_{21} =: \lambda = e^{2\pi \mathrm{i} \theta}, \quad \lambda_{j0} = \lambda_{0j} = 1, \quad j = 1, 2,$$

also: $\sum_\mu z_\mu^* z_\mu = 1$

The sphere S^4_θ comes with a noncommutative vector bundles endowed with an anti-self-dual gauge connection

SU(2) noncommutative principal fibration $S^7_{\theta'} \to S^4_\theta$

With $\lambda'_{ab} = e^{2\pi\,\mathrm{i}\,\theta'_{ab}}$; $(\theta'_{ab} = -\theta'_{ba})$, the coordinate algebra $A(S^7_{\theta'})$ of the sphere $S^7_{\theta'}$: generators ψ_a, ψ_a^*, $a = 1, \ldots, 4$, relations

$$\psi_a \psi_b = \lambda_{ab} \psi_b \psi_a, \quad \psi_a \psi_b^* = \lambda_{ba} \psi_b^* \psi_a, \quad \psi_a^* \psi_a^* = \lambda_{ab} \psi_b^* \psi_a^*,$$

$\sum_a \psi_a^* \psi_a = 1$.

The choice

$$\lambda'_{ab} = \begin{pmatrix} 1 & 1 & \bar{\mu} & \mu \\ 1 & 1 & \mu & \bar{\mu} \\ \mu & \bar{\mu} & 1 & 1 \\ \bar{\mu} & \mu & 1 & 1 \end{pmatrix}, \quad \mu = \sqrt{\lambda},$$

is the only one that allows the algebra $A(S^7_{\theta'})$ to carry an action of SU(2) by automorphisms s. t.

$$A(S^7_{\theta'})^{\mathrm{SU}(2)} = A(S^4_\theta)$$

A matrix-valued function on $A(S^7_{\theta'})$

$$\Psi = \begin{pmatrix} \psi_1 & \psi_2 & \psi_3 & \psi_4 \\ -\psi_2^* & \psi_1^* & -\psi_4^* & \psi_3^* \end{pmatrix}^t, \qquad \Psi^\dagger \Psi = \mathbb{I}_2$$

$p = \Psi\Psi^\dagger$ is a projection, $p^2 = p = p^\dagger$

its entries are (the generating) elements of $A(S^4_\theta)$

$$p = \frac{1}{2} \begin{pmatrix} 1 + z_0 & 0 & z_1 & -\bar{\mu} z_2^* \\ 0 & 1 + z_0 & z_2 & \mu z_1^* \\ z_1^* & z_2^* & 1 - z_0 & 0 \\ -\mu z_2 & \bar{\mu} z_1 & 0 & 1 - z_0 \end{pmatrix},$$

The z_μ's are quadratic in the ψ_a's.

A vector bundle E over S^4_θ: $\qquad \mathcal{E} = \Gamma(S^4_\theta, E) = p[A(S^4_\theta)]^4$

the connection $\nabla = p \circ \mathrm{d}$ has anti-selfdual curvature $F = p(\mathrm{d}p)^2$:

$$*_\theta F = -F$$

The $su(2)$-valued connection 1 form on $S^7_{\theta'}$ is most simply written in terms of the matrix-valued function Ψ :

$$\omega = \Psi^\dagger \mathrm{d}\Psi$$

The spin-Hall system on S^4_θ

The Hamiltonian of a "single particle" moving on the sphere S^4_θ and coupled to the gauge field ω :

$$H_\omega = -(\mathsf{d}+\omega)^*(\mathsf{d}+\omega),$$

The gauge potential ω in an arbitrary representation J of $su(2)$. The spin label $J \in \frac{1}{2}\mathbb{N}$ and the Casimir operator has value

$$C_{su(2)} = J(J+1)$$

Expand the covariant derivative: $D = \mathsf{d}z_\mu D_\mu + \mathsf{d}z^*_\mu D^*_\mu$. The Hamiltonian becomes,

$$H_\omega = \widetilde{H}_1^2 + \widetilde{H}_2^2 + \sum_{r^+}(\tilde{E}_r\tilde{E}_{-r} + \tilde{E}_{-r}\tilde{E}_r)$$

the operators $\widetilde{H}_j$ and $\tilde{E}_r$ are 'gauged twisted derivations'

$H_{\omega=0}$ is the Casimir operator

$$C = H_1^2 + H_2^2 + \sum_{r^+}(E_r E_{-r} + E_{-r}E_r),$$

of the twisted algebra $U_\theta(so(5))$.
In general, one needs also the curvature F. Expand

$$F = \mathrm{d}z_0 \mathrm{d}z_0 F_{00} + \tfrac{1}{2}\mathrm{d}z_{\varepsilon_\mu \mu}\mathrm{d}z_{\varepsilon_\nu \nu}F_{\varepsilon_\mu\mu,\varepsilon_\nu\nu}$$

with ε_μ and ε_ν taking values ± 1 and $\mathrm{d}z_{-\mu} = \mathrm{d}z_\mu^*$.

The operators

$$H_1 = \widetilde{H}_1 - F_{00}, \qquad H_2 = \widetilde{H}_2 - F_{00}, \qquad E_{\varepsilon_\mu\mu,\varepsilon_\nu\nu} = \tilde{E}_{\varepsilon_\mu\mu,\varepsilon_\nu\nu} - F_{\varepsilon_\mu\mu,\varepsilon_\nu\nu}$$

close the commutation relations of the Lie algebra $so(5)$;

The operators $F_{\varepsilon_\mu\mu,\varepsilon_\nu\nu}$ carry a spin representation labelled by J.

With this, one finds that

$$H_\omega = C_{U_\theta(so(5))} - 2C_{su(2)}.$$

Easy to diagonalize from representation theory. Two fundamental weights $W^1 = \frac{1}{2}(1,1)$ and $W^2 = (1,0)$; each representation is labelled by two integers s, n, with highest weight $W = sW^1 + nW^2$ and has dimension

$$d(s,n) = (1+s)(1+n)(1+\frac{s+n}{2})(1+\frac{s+2n}{3}).$$

The integer s measures the "spinorial content" ; a spin label J, $s = 2J$, takes integer and half integer values. The Casimir is :

$$C(s,n) = \frac{1}{2}(s^2 + 2n^2 + 2sn) + 2s + 3n.$$

The eigenvalues of the Hamiltonian H_ω are the energies

$$E(J,n) = C(s=2J,n) - 2J(J+1)$$
$$= n^2 + n(2J+3) + 2J$$

with degeneracy $d(s=2J,n)$.

The integer n labels Landau levels and J, which plays the role of the magnetic flux, label the degeneracy in each Landau level.

The ground state for a given J is obtained when $n=0$; energy

$$E_0(J) = 2J$$

with degeneration

$$d_0(J) = d(s=2J, n=0) = \frac{1}{6}(1+2J)(2+2J)(3+2J).$$

The representations of $U_\theta(so(5))$, also gives wave functions.

For the ground state: the spinor $\psi = (\psi_1, \ldots, \psi_4)$ is an eigenfunction of the Hamiltonian with $J = \frac{1}{2}$, the fundamental spinorial representation having highest weight vector ψ_4:

$$H_1(\psi_4) = \tfrac{1}{2} = H_2(\psi_4).$$

H_1, H_2 are the Cartan elements.

A basis of eigenfunctions for the ground state, – with is the representation with $s = 2J$ and $n = 0$ – is obtained by the corresponding highest weight vector, $\Phi = (\psi_4)^{2J}$, by repeated action of the lowering operators E_r of $U_\theta(so(5))$.