Generalized CCR flows

Let K be a complex Hilbert space and let $\{S_t\}$ be the shift semigroup on $L^2((0,\infty),K)$ defined by

$$\begin{aligned}(S_t f)(s) &= 0, \quad s<t,\\ &= f(s-t), \quad s\geq t,\end{aligned}$$

for $f\in L^2((0,\infty),K)$. The CCR flow of index $\dim K$ is the E_0-semigroup α acting on $\mathbb{B}(\Gamma(L^2((0,\infty),K)))$ defined by

$$\alpha_t(W(f)) = W(S_t f),$$

where $\Gamma(L^2((0,\infty),K)))$ the symmetric Fock space of $L^2((0,\infty),K)$ and $W(x)\in\mathbb{B}(\Gamma(K))$ be the Weyl operator corresponding to x.

To generalize the CCR flows, we ask the following question. Let G be a real Hilbert space and $H=\Gamma(G^{\mathbb{C}})$, where $G^{\mathbb{C}}$ is the complexification of $\mathbb{C}$. Suppose we have two semigroups of linear operators, $S_t,\ T_t : G\mapsto G$ for $t\geq 0$, Consider the association

$$\alpha_t(W(x))\mapsto W(S_t x),\ \alpha_t(W(iy))\mapsto W(iT_t y),\ x,y\in G.$$

When can we extend this map to an E_0-semigroup on $\mathbb{B}(H)$? The continuity and the semigroup property of $\{\alpha_t\}$ will immediately imply that both $\{S_t\}$ and $\{T_t\}$ have to be C_0-semigroups. Also, α_t being an endomorphism satisfies

$$\alpha_t(W(u)W(v)) = \alpha_t(W(u))\alpha_t(W(v)), u,v\in G^{\mathbb{C}}.$$

Comparing both sides, using the canonical commutation relation, we get

$$\langle S_t x, T_t y\rangle = \langle x,y\rangle\ \forall\ x,y\in G$$

which is same as saying $T_t^* S_t = 1$.

Assume that these conditions are satisfied, α_t extends to an endomorphism to the whole of $\mathbb{B}(H)$ if and only if $T_t - S_t$ is a Hilbert-Schmidt operator.

Definition 0.1. Let $\{S_t\}$ and $\{T_t\}$ be C_0-semigroups acting on a real Hilbert space G. We say that $\{T_t\}$ is a perturbation of $\{S_t\}$, if they satisfy,

(i) ${T_t}^* S_t = 1$.
(ii) $S_t - T_t$ is a Hilbert Schmidt operator.

Given a perturbation $\{T_t\}$ of $\{S_t\}$, we say that the E_0-semigroup $\{\alpha_t\}$ acting on $\mathbb{B}(\Gamma(G^{\mathbb{C}}))$ given by

$$\alpha_t(W(x+iy)) = W(S_t x + iT_t y), \quad x,y\in G$$

is a generalized CCR flow associated with the pair $\{S_t\}$ and $\{T_t\}$.

Boris Tsirelson constructed an uncountable family of non-isomorphic type III product systems. The construction involves associating measure type spaces to Gaussian spaces. A construction from an operator algebraic view point was given by Bhat and Sr. They were called as 'product systems arising from a sum system'. This construction is described as follows.

For two Hilbert spaces G_1, G_2, define

$$\mathcal{S}(G_1, G_2) = \{A \in \mathbb{B}(G_1, G_2); A \text{ invertible and } I - (A^*A)^{\frac{1}{2}} \text{ Hilbert-Schmidt}\}.$$

For two real Hilbert spaces G_1, G_2 and $A \in \mathcal{S}(G_1, G_2)$, define a real liner operator $S_A : G_1^{\mathbb{C}} \to G_2^{\mathbb{C}}$ by $S_A(u + iv) = Au + i(A^{-1})^*v$ for $u, v \in G_1$. In general S_A is not complex linear, unless A is unitary.

Theorem 0.2. (i) *Let G_1, G_2 be real Hilbert spaces and $A \in \mathcal{S}(G_1, G_2)$, then there exists a unique unitary operator $\Gamma(A) : \Gamma(G_1^{\mathbb{C}}) \to \Gamma(G_2^{\mathbb{C}})$ such that*

$$\Gamma(A)W(u)\Gamma(A)^* = W(S_A u) \;\forall\; u \in G_1^{\mathbb{C}} \tag{0.1}$$

$$\langle \Gamma(A)\Phi_1, \Phi_2\rangle \in \mathbb{R}^+ \tag{0.2}$$

where Φ_1 and Φ_2 are the vacuum vectors in $\Gamma(G_1^{\mathbb{C}})$ and $\Gamma(G_2^{\mathbb{C}})$ respectively.

(ii) *Suppose G_1, G_2, G_3 be three real Hilbert spaces, and $A \in \mathcal{S}(G_1, G_2)$, $B \in \mathcal{S}(G_2, G_3)$, then*

$$\Gamma(A^{-1}) = \Gamma(A)^* \tag{0.3}$$

$$\Gamma(BA) = \Gamma(B)\Gamma(A) \tag{0.4}$$

Definition 0.3. A sum system is a two parameter family of real Hilbert spaces $\{G_{s,t}\}$ for $0 < s < t \leq \infty$, satisfying $G_{s,t} \subset G_{s',t'}$ if the interval (s, t) is contained in the interval (s', t'), together with a one parameter semigroup $\{S_t\}$, of bounded linear operators on $G_{(0,\infty)}$ for $t \in (0, \infty)$ such that

(i) $S_s|_{G_{0,t}} \in \mathcal{S}(G_{0,t}, G_{s,s+t}) \;\forall\; t \in (0, \infty],\; s \in [0, \infty)$.

(ii) If $A_{s,t} : G_{0,s} \oplus G_{s,s+t} \mapsto G_{0,s+t}$, is the map $A_{s,t}(x \oplus y) = x + y$, for $x \in G_{0,s}, y \in G_{s,s+t}$, then $A_{s,t} \in \mathcal{S}(G_{0,s} \oplus G_{s,s+t}, G_{0,s+t})$, $\forall\; s, t \in (0, \infty)$.

(iii) The semigroup $\{S_t\}$ is strongly continuous.

Given a sum system $(\{G_{s,t}\}, \{S_t\})$, we define Hilbert spaces $H_t = \Gamma(G_{0,t}^{\mathbb{C}})$, and unitary operators $U_{s,t} : H_s \otimes H_t \mapsto H_{s+t}$, by $U_{s,t} = \Gamma(A_{s,t})(1_{H_s} \otimes \Gamma(S_s|_{G_{0,t}}))$.

$(\{H_t\}, \{U_{s,t}\})$ forms a product system.

Fix a sum system $(\{G_{a,b}\},\{S_t\})$ and $(\{H_t\},\{U_{s,t}\})$ be the product system constructed out of it. Denote $G = G_{0,\infty}$, $A_t = A_{t,\infty}$.

We may consider S_t as a bounded linear invertible map between $G \mapsto G_{t,\infty}$. Hence $(S_t^*)^{-1}$ is a well-defined bounded operator between $G \mapsto G_{t,\infty}$. When there is no confusion, by misusing the notation, we consider $(S_t^*)^{-1}$ as an element of $\mathbb{B}(G)$ itself. Define $T_t \in \mathbb{B}(G)$, by

$$T_t = (A_t^*)^{-1} A_t^{-1} (S_t^*)^{-1} \ \forall\ t \in [0,\infty).$$

Lemma 0.4. *$\{T_t\}$ forms a C_0-semigroup on G and $\{T_t\}$ is a perturbation of $\{S_t\}$.*

The E_0-semigroup associated with the product system $(H_t, U_{s,t})$ can be described in terms of these two semigroups, S_t, T_t as follows. Let $H = \Gamma(G^{\mathbb{C}})$.

Proposition 0.5. *Let the notation be as above. Then there is a unique E_0-semigroup α_t on $\mathbb{B}(H)$ satisfying*

$$\alpha_t(W(x)) = W(S_t x),\ \ \alpha_t(W(iy)) = W(iT_t y),\ \ x, y \in G.$$

Moreover the product system associated with this E_0-semigroup is the one constructed out of the sum system.

Let G be a real Hilbert space and $H = \Gamma(G^{\mathbb{C}})$. We assume that a C_0-semigroup $\{T_t\}$ is a perturbation of another C_0-semigroup $\{S_t\}$ acting on G.

Define

$$G_{0,t} = \mathrm{Ker}\,(T_t^*),\ G_{(0,\infty)} = \overline{\bigcup_{t>0} G_{0,t}},\ G_{a,b} = S_a(G_{0,b-a}).$$

Let $P : G \to G_{0,\infty}$ be the orthogonal projection. We define S_t^0 and T_t^0 by

$$S_t^0 = PS_tP,\ T_t^0 = PT_tP.$$

Then $\{S_t^0\}$ and $\{T_t^0\}$ are C_0-semigroups and one is a perturbation of the other.

Proposition 0.6. *Let G be a real Hilbert space and let $\{S_t\}$ and $\{T_t\}$ be C_0-semigroups acting on G such that $\{T_t\}$ is a perturbation of $\{S_t\}$. Let $\{G_{s,t}\}$, $\{S_t^0\}$, and $\{T_t^0\}$ be as above. Then*

(a) *The system $(\{G_{a,b}\}, \{S_t^0\})$ forms a sum system.*

(b) *The pair of C_0-semigroups $(\{S_t^0\}, \{T_t^0\})$ is associated with $(\{G_{a,b}\}, \{S_t^0\})$. In consequence, the product system for the generalized CCR flow arising from $(\{S_t^0\}, \{T_t^0\})$ is isomorphic to the one arising from $(\{G_{a,b}\}, \{S_t^0\})$.*

(c) *The product system for the generalized CCR flow arising from $(\{S_t\}, \{T_t\})$ is isomorphic to the product system arising from $(\{G_{a,b}\}, \{S_t^0\})$. In consequence, the generalized CCR flow arising from the pair $(\{S_t\}, \{T_t\})$ is cocycle conjugate to that arising from $(\{S_t^0\}, \{T_t^0\})$.*

Definition 0.7. Let $(\{G_{a,b}\},\{S_t\})$ be a sum system. A real addit for the sum system $(\{G_{(a,b)}\},\{S_t\})$ is a family $\{x_t\}_{t\in(0,\infty)}$ such that $x_t \in G_{0,t},\ \forall\, t \in (0,\infty)$, satisfying the following conditions.

(i) The map $t \mapsto \langle x_t, x\rangle$ is measurable for any $x \in G_{0,\infty}$.

(ii) $x_s + S_s x_t = x_{s+t},\ \forall s,t,\in(0,\infty)$, (i. e.) $A_{s,t}(x_s \oplus S_s x_t) = x_{s+t}$.

An imaginary addit for the sum system $(\{G_{a,b}\},\{S_t\})$ is a family $\{y_t\}_{t\in(0,\infty)}$ such that $y_t \in G_{0,t},\ \forall\, t\in(0,\infty)$, satisfying the following conditions.

(i) The map $t \mapsto \langle y_t, y\rangle$ is measurable for any $y \in G_{0,\infty}$.

(ii) $\{y_t\}$ satisfies $(A^*_{s,t})^{-1}(y_s \oplus (S^*_s)^{-1} y_t) = y_{s+t},\ \forall s,t,\in(0,\infty)$.

We denote by $R\mathcal{AU}$ and $I\mathcal{AU}$ the set of all real and imaginary addits respectively, which are real linear spaces. For a given real addit $\{x_t\}$, define $x_{s,t} = S_s(x_{t-s}) \in G_{s,t}$. Similarly for a given imaginary addit $\{y_t\}$ define $y_{s,t} = (S^*_s)^{-1}(y_{t-s}) \in G_{s,t}$.

We also define for an imaginary addit $\{y_t\}$,

$$G_{0,s} \ni {}^s y'_{s_1,s_2} = (A^*)^{-1}(0 \oplus y_{s_1,s_2} \oplus 0),\ \text{ for any } (s_1,s_2) \subset (0,s),$$

where $A : G_{0,s_1} \oplus G_{s_1,s_2} \oplus G_{s_2,s} \to G_{0,s}$ is defined by $x \oplus y \oplus z \mapsto x+y+z$. It is easy to check that ${}^s y'_{s_1,s_2} \in (G_{0,s_1} \bigvee G_{s_2,s})^{\perp} \cap G_{0,s}$. We have

$$x_s + x_{s,s+t} = x_{s+t},\ y'_s + y'_{s,s+t} = y'_{s+t}.$$

Proposition 0.8. *For any sum system $(\{G_{a,b}\},\{S_t\})$ addits exist and generate the sum system, (i. e.)*

$$G_{0,s} = \overline{\text{span}_{\mathbb{R}}[x_{s_1,s_2}; (s_1,s_2) \subseteq (0,s), \{x_t\} \in R\mathcal{AU}]}$$

and

$$G_{0,s} = \overline{\text{span}_{\mathbb{R}}[{}^s y'_{s_1,s_2}; (s_1,s_2) \subseteq (0,s), \{y_t\} \in I\mathcal{AU}]}.$$

Theorem 0.9. *Every product system arising from a sum system is either of type I or type III. Consequently every generalized CCR flow is either of type I or type III*

Definition 0.10. For a divisible sumsystem $(\{G_{a,b}\}, \{S_t\})$, the index $\operatorname{ind} G$ is the number $\dim R\mathcal{AU} = \dim I\mathcal{AU} \in \mathbb{N} \cup \{\infty\}$.

Assume that $\operatorname{ind} G = n$ is finite. In that case, both $R\mathcal{AU}$ and $I\mathcal{AU}$ carry unique linear topologies. Denote

$$G^0_{0,t} = \operatorname{span}_{\mathbb{R}}[x_{s_1,s_2}; (s_1, s_2) \subseteq (0,t),\ \{x_t\} \in R\mathcal{AU}] \subseteq G_{0,t},$$
$${G^0_{0,t}}' = \operatorname{span}_{\mathbb{R}}[{}^t y'_{s_1,s_2}; (s_1, s_2) \subseteq (0,t),\ \{y_t\} \in I\mathcal{AU}] \subseteq G_{0,t}.$$

For a given linear map $J : R\mathcal{AU} \to I\mathcal{AU}$, we set $J_{t,0}$ to be the linear map $J_{t,0} : G^0_{0,t} \to {G^0_{0,t}}'$ determined by

$$J_{t,0}(x_{s_1,s_2}) = {}^t J(x)'_{s_1,s_2},$$

for $(s_1, s_2) \subseteq (0,t)$ and $x \in R\mathcal{AU}$. When $J_{t,0}$ has a bounded extension to $G_{0,t}$ we denote it by J_t.

Theorem 0.11. *Let $(\{G_{(a,b)}\}, \{S_t\})$ be a sum system with finite index and let $(\{H_t\}, \{U_{s,t}\})$ be the product system constructed out of the above sum system. Then the following statements are equivalent.*

(i) *The product system $(H_t, U_{s,t})$ is of type I.*

(ii) *There exists a linear isomorphism $J : R\mathcal{AU} \to I\mathcal{AU}$ satisfying the following property: for each $t > 0$, the operator $J_{t,0}$ extends to a bounded positive operator on $G_{0,t}$ such that $J_t \in \mathcal{S}(G_{0,t}, G_{0,t})$.*

(iii) *There exists a linear isomorphism $J : R\mathcal{AU} \to I\mathcal{AU}$ satisfying the following property: the operator $J_{1.0}$ extends to a bounded positive operator on $G_{0,1}$ such that $J_1 \in \mathcal{S}(G_{0,1}, G_{0,1})$.*

Remark 0.12. Since only type I and type III product systems can be constructed from divisible sum systems. So thanks to the above Theorem, violating the condition $J_1 \in \mathcal{S}(G_{0,1}, G_{0,1})$ is necessary and sufficient for the associated product system to be of type III. This criterion is much more powerful than the necessary condition for type I already proved by Bhat and Sr. In fact we can arrive at that condition just by assuming that J_1 is bounded. there are examples of divisible sum systems of finite index with bounded J_1, which give rise to type III. In particular there are many type III examples, which can not be distinguished from type I examples by the invariants introduced by Tsirelson. product systems.