E_0-dilation of strongly commuting pairs of CP_0-semigroups

Orr Shalit, Technion
(under supervision of Baruch Solel)

Workshop on Noncommutative Dynamics and Applications
Fields Institute, July 17, 2007

The Problem

- A separable Hilbert space H.

- A von Neumann algebra $\mathcal{M} \subseteq B(H)$.

- Two commuting CP-semigroups $\Phi = \{\Phi_t\}_{t\geq 0}$, $\Psi = \{\Psi_t\}_{t\geq 0}$ on $\mathcal{M}$, i.e., for all $s, t \geq 0$,

$$\Phi_s \circ \Psi_t = \Psi_t \circ \Phi_s.$$

- **Question:** *Is it possible to dilate Φ and Ψ to a pair of commuting E-semigroups on some larger von Neumann algebra?*

The expected answer is *yes*. We have shown that the answer is *yes* under some additional assumptions.

Definitions

- **CP-semigroup** - A semigroup $\Theta = \{\Theta_t\}_{t \geq 0}$ ($\Theta_s \circ \Theta_t = \Theta_{s+t}$) of normal, contractive, completely positive maps on $\mathcal{M} \subseteq B(H)$, continuous in the following sense: for all $h, g \in H$, and all $a \in \mathcal{M}$,

$$\lim_{t \to t_0} \langle \Theta_t(a)h, g \rangle = \left\langle \Theta_{t_0}(a)h, g \right\rangle$$

- **E-semigroup** a CP-semigroup with every element a $*$-endomorphism.

- **CP$_0$-semigroup** - a CP-semigroup with unital elements.

- **E$_0$-semigroup** - an E-semigroup with unital elements.

E-dilation of a CP-semigroup

Given a semigroup $\Theta = \{\Theta_s\}_{s\in\mathcal{S}}$ of CP maps acting on $\mathcal{M} \subseteq B(H)$, an E-dilation is a triple $(\alpha, \mathcal{R}, K)$, where

- $K \supseteq H$ is a Hilbert space,
- $\mathcal{R} \subseteq B(K)$ is a vN algebra such that $\mathcal{M} = P_H \mathcal{R} P_H$,
- $\alpha = \{\alpha_s\}_{s\in\mathcal{S}}$ is an E-semigroup such that for all $T \in \mathcal{R}, s \in \mathcal{S}$

$$\Theta_s(P_H T P_H) = P_H \alpha_s(T) P_H.$$

- α is to have the same kind of continuity as Θ.

E-dilation of CP-semigroups (Bhat-Skeide, SeLegue, Muhly-Solel, Arveson)

Theorem 1. *Let* $\Theta = \{\Theta_t\}_{t\geq 0}$ *be a CP-semigroup acting on* $\mathcal{M} \subseteq B(H)$. *Then* Θ *has an E-dilation.*

That is, There exists a Hilbert space $K \supseteq H$, *a vN algebra* $\mathcal{R} \subseteq B(K)$ *such that* $\mathcal{M} = P_H \mathcal{R} P_H$, *and an E-semigroup* $\{\alpha_t\}_{t\geq 0}$ *on* $\mathcal{R}$ *such that for all* $T \in \mathcal{R}, t \geq 0$

$$\Theta_t(P_H T P_H) = P_H \alpha_t(T) P_H.$$

Remarks:

- Unitality.

- Minimality + uniqueness.

E-dilation of a commuting pair of CP maps (Bhat, Solel)

Theorem 2. *Let* Φ *and* Ψ *be two commuting CP maps acting on* $\mathcal{M} \subseteq B(H)$. *Then the semigroup* Θ *(over* $\mathbb{N}^2$*) of CP-maps given by*

$$\Theta_{(m,n)} = \Phi^m \circ \Psi^n$$

has an E-dilation.

That is, there exists a Hilbert space $K \supseteq H$, *a vN algebra* $\mathcal{R} \subseteq B(K)$ *such that* $\mathcal{M} = P_H \mathcal{R} P_H$, *and there exists two commuting normal* $*$*-endomorph* α *and* β *on* $\mathcal{R}$ *such that for all* $T \in \mathcal{R}, m, n \in \mathbb{N}$

$$\Phi^m \circ \Psi^n(P_H T P_H) = P_H \alpha^m \circ \beta^n(T) P_H.$$

Remarks:

- Unitality?

- Minimality + uniqueness?

@

Our main result

Theorem 3. *Let* $\Phi = \{\Phi_t\}_{t\geq 0}$, $\Psi = \{\Psi_t\}_{t\geq 0}$ *be two* **strongly commuting** *CP_0-semigroups on* $\mathcal{M} \subseteq B(H)$. *Then the semigroup* Θ *(over* $\mathbb{R}^2_+$*) of CP-maps given by*

$$\Theta_{(s,t)} = \Phi_s \circ \Psi_t$$

has an E_0-dilation.

That is, there exists a Hilbert space $K \supseteq H$, *a vN algebra* $\mathcal{R} \subseteq B(K)$ *such that* $\mathcal{M} = P_H \mathcal{R} P_H$, *and there exists two commuting E_0-semigroups* α *and* β *on* $\mathcal{R}$ *such that for all* $T \in \mathcal{R}, s, t \in \mathbb{R}^2_+$

$$\Phi_s \circ \Psi_t(P_H T P_H) = P_H \alpha_s \circ \beta_t(T) P_H.$$

Proof. This theorem is proved using Muhly and Solel's approach to dilation. We will give the idea for the case $\mathcal{M} = B(H)$. □

Toy example

Assume that $\{T_t\}_{t\geq 0}, \{S_t\}_{t\geq 0}$ are two commuting contractive semigroups, and assume that Φ and Ψ are given by

$$\Phi_t(a) = T_t a T_t^*, \ \Psi_t(a) = S_t a S_t^*,$$

for $a \in B(H)$.

Φ and Ψ are E-semigroups $\Leftrightarrow$ $\{T_t\}_{t\geq 0}, \{S_t\}_{t\geq 0}$ are isometry semigroups.

Let $\{V_t\}_{t\geq 0}, \{U_t\}_{t\geq 0}$ be the (Słociński) isometric dilation on $K \supseteq H$. Define for all $a \in B(K)$

$$\alpha_t(a) = V_t a V_t^*, \ \beta_t(a) = U_t a U_t^*.$$

Then α and β are an E-dilation of Φ and Ψ.

In general, CP-semigroups are not given by such simple formulas. However, this example captures the essence of the Muhly-Solel approach.

The Muhly-Solel strategy for dilation

Given: a semigroup $\mathcal{S}$, and CP-semigroup $\Theta = \{\Theta_s\}_{s\in\mathcal{S}}$ on $B(H)$.

1. Construct product system (of Hilbert spaces), $X = \{X(s)\}_{s\in\mathcal{S}}$. Construct product system representation T of X on H such that

$$\Theta_s(a) = \tilde{T}_s \left(I_{X(s)} \otimes a \right) \tilde{T}_s^*.$$

2. Dilate T to an isometric representation V on $K \supseteq H$.

3. Check that

$$\alpha_s(a) := \tilde{V}_s \left(I_{X(s)} \otimes a \right) \tilde{V}_s^*$$

defined on the algebra $B(K)$ is the sought after dilation of Θ.

Step 1: Product system representations

E - a Hilbert space.

C.C. Representation: (of E on a Hilbert space H) A linear, completely contractive map $T : E \to B(H)$

$E \otimes H$ is the usual tensor product of the spaces.

$\tilde{T} : E \otimes H \to H$ is defined: $\tilde{T}(\xi \otimes h) = T(\xi)h$.

Product system: $X = \{X(s)\}_{s \in \mathcal{S}}$, $X(0) = \mathbb{C}$,

$$X(s) \otimes X(t) \cong X(s+t).$$

Product system representation: A family $T = \{T_s\}_{s \in \mathcal{S}}$, T_s is a c.c. representation of $X(s)$ on H,

$$T_{s+t}(x_s \otimes x_t) = T_s(x_s) T_t(x_t).$$

Step 1 (cont.): Representing the CP-semigroup

- M-S constructed $E = \{E(t)\}_{t \geq 0}$, $F = \{F(t)\}_{t \geq 0}$ with representation $T^E : E \to B(H), T^F : F \to B(H)$ such that

$$\Phi_t(a) = \tilde{T}_t^E \left(I_{E(t)} \otimes a \right) (\tilde{T}_t^E)^*,$$

$$\Psi_t(a) = \tilde{T}_t^F \left(I_{F(t)} \otimes a \right) (\tilde{T}_t^F)^*.$$

- Define $X = \{X(s,t)\}_{(s,t) \in \mathbb{R}_+^2}$ by

$$X(s,t) = E(s) \otimes F(t).$$

This is rather technical. Here the **strong commutativity** assumption plays.

- Define $T = \{T_{(s,t)}\}_{(s,t) \in \mathbb{R}_+^2}$ by

$$T_{(s,t)}(e_s \otimes f_t) = T_s^E(e_s) T_t^F(f_t).$$

Step 2: Dilating the product system representation

The product system X and representation T constructed in step 1 satisfy

$$\Phi_s(\Psi_t(a)) = \tilde{T}_{(s,t)}(I_{X(s)} \otimes a)\tilde{T}^*_{(s,t)}.$$

$\tilde{T}$ is isometric $\Rightarrow$ the CP-semigroup is actually an E-semigroup.

$\tilde{T}$ is coisometric $\Leftrightarrow$ the CP-semigroup is a CP_0-semigroup (unitality).

Isometric representation: a representation T such that $\tilde{T}_s$ is an isometry for all $s \in \mathbb{R}^2_+$.

Fully-coisometric representation: a representation T such that $\tilde{T}_s$ is a coisometry for all $s \in \mathbb{R}^2_+$.

Step 2 (continued)

Theorem 4. *Let $X = \{X(s)\}_{s\in\mathbb{R}_+^k}$ be a product system, and let T be a fully coisometric representation of X on H. Then there exists a Hilbert space $K \supseteq H$ and a* **fully coisometric** *and* **isometric** *representation V of X on K, such that*

1. $P_H V_s(x)\big|_H = T_s(x)$ *for all* $s \in \mathbb{R}_+^k$, $x \in X(s)$.

2. $P_H V_s(x)\big|_{K\ominus H} = 0$ *for all* $s \in \mathcal{S}$, $x \in X(s)$.

This is the part where the **unitality** of the semigroups is used (fully-coisometric).

Step 2: Proof

Let

$$\mathcal{H} = \bigoplus_{s \in \mathbb{R}_+^k} X(s) \otimes H.$$

We define a family $\hat{T} = \{\hat{T}_s\}_{s \in \mathbb{R}_+^k}$ on $\mathcal{H}$ as follows:

- $\hat{T}_0 = I$.
- $t \not\geq s > 0$: $\hat{T}_s(\delta_t \cdot x_t \otimes h) = 0$.
- $t \geq s > 0$:

  $\hat{T}_s\left(\delta_t \cdot (x_{t-s} \otimes x_s \otimes h)\right) = \delta_{t-s} \cdot \Big(x_{t-s} \otimes \tilde{T}_s(x_s \otimes h)\Big)$

$\hat{T}$ is a semigroup of *coisometries* on $\mathcal{H}$.

Step 2: Proof (continued)

By classical dilation theory, $\widehat{T}$ has a minimal isometric dilation $\widehat{V} = \{\widehat{V}_s\}_{s \in \mathbb{R}^k_+}$ on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ satisfying

$$P_{\mathcal{H}} \widehat{V}^*_{s_-} \widehat{V}_{s_+} P_{\mathcal{H}} = \widehat{T}_{s_+} \widehat{T}^*_{s_-} \ , \ s \in \mathbb{R}^k.$$

Define

$$K = \bigvee \{\widehat{V}_s(\delta_s \cdot (x_s \otimes h)) : s \in \mathcal{S}, x_s \in X(s), h \in H\}.$$

We define the representation V of X on K by

$$V_t(x_t) \left(\widehat{V}_s(\delta_s \cdot (x_s \otimes h))\right) = \widehat{V}_{s+t}(\delta_{s+t} \cdot (x_t \otimes x_s \otimes h)).$$

Step 2: Proof (continued)

V is a well defined, covariant, fully-coisometric and isometric product system representation. For example:

$$
\begin{aligned}
V_{s+t}(x_s \otimes x_t)\widehat{V}_u(\delta_u \cdot x_u \otimes h) = \\
&= \widehat{V}_{s+t+u}(\delta_{s+t+u} \cdot x_s \otimes x_t \otimes x_u \otimes h) \\
&= V_s(x_s)\widehat{V}_{t+u}(\delta_{t+u} \cdot x_t \otimes x_u \otimes h) \\
&= V_s(x_s)V_t(x_t)\widehat{V}_u(\delta_u \cdot x_u \otimes h),
\end{aligned}
$$

the "semigroup property".

V is a dilation of T:

$$
\begin{aligned}
P_H V_s(x)\Big|_H h &= P_H V_s(x)\delta_0 \cdot 1 \otimes h \\
&= P_H \widehat{V}_s(\delta_s \cdot x \otimes h) \\
&= P_H P_{\mathcal{H}} \widehat{V}_s\Big|_{\mathcal{H}}(\delta_s \cdot x \otimes h) \\
&= P_H \widehat{T}_s(\delta_s \cdot x \otimes h) \\
&= P_H(\delta_0 \cdot 1 \otimes T_s(x)h) = T_s(x)h.
\end{aligned}
$$

Step 3: Putting the pieces together

Given: two strongly commuting CP_0-semigroups $\Phi = \{\Phi_t\}_{t\geq 0}$, $\Psi = \{\Psi_t\}_{t\geq 0}$ on $B(H)$.

- Step 1: construct a product system X of correspondences, a representation T:

$$\Phi_s(\Psi_t(a)) = \tilde{T}_{(s,t)}(I_{X(s)} \otimes a)\tilde{T}^*_{(s,t)}.$$

- Step 2: construct isometric dilation V of T on $K \supseteq H$.

- Step 3: put the pieces together:

$$\alpha_{(s,t)}(b) = \tilde{V}_{(s,t)}(I_{X(s)} \otimes b)\tilde{V}^*_{(s,t)},$$

for all $b \in B(K)$. α is a dilation of Φ, Ψ!

Strong commutativity 1

- Given a CP map P on $\mathcal{M} \subseteq B(H)$, one may form $\mathcal{M} \otimes_P H$, the Hausdorff completion w.r.t. to $\langle a \otimes h, b \otimes g \rangle = \langle h, P(a^* b) g \rangle$.

- Similarly, given $P_1, \ldots, P_n$ CP maps on $\mathcal{M}$, $\mathcal{M} \otimes_{P_1} \ldots \otimes_{P_{n-1}} \mathcal{M} \otimes_{P_n} H$ is the Hausdorff completion w.r.t.

$$\langle a_1 \otimes \cdot \otimes a_n \otimes h, b_1 \otimes \cdots \otimes b_n \otimes g \rangle =$$

$$= \langle h, P_n(a_n^* P_{n-1}(\cdots P_1(a^* b_1) \cdots) b_n *) g \rangle \, .$$

- The product system $E = \{E(t)\}_{t \geq 0}$ that M-S associate with a CP-semigroup Φ is constructed from the spaces $\mathcal{M} \otimes_{\Phi_{t_1}} \mathcal{M} \otimes_{\Phi_{t_2 - t_1}} \cdots \mathcal{M} \otimes_{\Phi_{t_n - t_{n-1}}} H$.

Strong commutativity 2

Two CP maps P and Q on $\mathcal{M}$ are said to **commute strongly** if there is a unitary

$$u : \mathcal{M} \otimes_P \mathcal{M} \otimes_Q H \to \mathcal{M} \otimes_Q \mathcal{M} \otimes_P H$$

1. $u(a \otimes_P I \otimes_Q h) = a \otimes_Q I \otimes_P h$, for all $a \in \mathcal{M}, h \in H$.

2. $u(ca \otimes_P b \otimes_Q h) = (c \otimes I \otimes I)u(a \otimes_P b \otimes_Q h)$ for all $a, b, c \in \mathcal{M}, h \in H$.

3. $u(a \otimes_P b \otimes_Q dh) = (I \otimes I \otimes d)u(a \otimes_P b \otimes_Q h)$ for all $a, b \in \mathcal{M}, h \in H$ and $d \in \mathcal{M}'$.

Two semigroups $\Phi = \{\Phi_t\}_{t \geq 0}$, $\Psi = \{\Psi_t\}_{t \geq 0}$ are said to **commute strongly** if for all s, t the maps Φ_s and Ψ_t commute strongly. @

Strong commutativity in $B(H)$

Theorem 5. *(Solel) Let* $(T_1, \ldots, T_n)$ *and* $(S_1, \ldots, S_m)$ *(*$m, n \in \mathbb{N} \cup \{\infty\}$*) be two* ℓ^2*-independent row contractions. The CP maps*

$$\Theta(a) = \sum_i T_i a T_i^*,$$

and

$$\Phi(a) = \sum_j S_j a S_j^*,$$

commute strongly if and only if there is an $mn \times mn$ *unitary matrix*

$$U = \left(U_{(i,j)}^{(k,l)} \right)_{(i,j),(k,l)}$$

such that for all i, j*,*

$$T_i S_j = \sum_{(k,l)} U_{(i,j)}^{(k,l)} S_l T_k.$$

So this works at least for the toy example!

Strong commutativity - example

$\mathcal{M} = \mathbb{C}^n$ acting as diagonal matrices on $H = \mathbb{C}^n$ (works for ℓ^∞ on ℓ^2 as well).

A unital, CP map is just a stochastic matrix, that is, a matrix P such that $p_{ij} \geq 0$ for all i, j and such that for all i,

$$\sum_j p_{ij} = 1.$$

Two matrices P and Q strongly commute $\Leftrightarrow$ for all i, k,

$$|\{j : q_{kj} p_{ji} \neq 0\}| = |\{j : p_{kj} q_{ji} \neq 0\}|.$$

All commuting positive matrices do so strongly.

Example for non-strong commutation:

$$P = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \; Q = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}.$$

Strong continuity - examples

1. If Θ and Φ are endomorphisms that commute then they commute strongly.

2. If Θ and Φ commute and either one of them is an automorphism then they commute strongly.

3. If α is a normal automorphism that commutes with Θ, and $\Phi = \Theta \circ \alpha$, then Θ and Φ commute strongly.