INCLUSIONS OF C*-ALGEBRAS

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1. Problem

Let $1 \in A \subset B$ be a pair of C*-algebras with common unit and $E\colon B \to A$ be a faithful conditional expectation. We consider the following problem:

Problem 1.1. Let $\mathcal{A}$ be a set of unital C*-algebras with some property "$\mathcal{P}$". Suppose that $A \in \mathcal{A}$. Then when $B \in \mathcal{A}$?

The following problem is my motivation to consider the above problem.

Problem 1.2. (1988, Blackadar [4]) Let A be an AF algebra, G a finite group, and α an action of G on A. Is it true that $\mathrm{tsr}(A \times_\alpha G) = 1$?

This is a still open question.

In this talk we will consider a generalized intersting problem and show some affirmative data.

2. Properties for C*-algebras

We consider the following four properties which are important for C*-algebras Theory.

(1) (1986 Rieffel [24]) For a unital C*-algebra A, the *topological stable rank* $\mathrm{tsr}(A)$ of A is defined to be the least integer n such that the set $\mathrm{Lg}_n(A)$ of all n-tuples $(a_1, , a_2 \dots, a_n) \in A^n$ which generate A as a left ideal is dense in A^n. The topological stable rank of a nonunital C*-algebra is defined to be that of its smallest unitization. Note that $\mathrm{tsr}(A) = 1$ is equivalent to density of the set of invertible elements in A.

(2) (1990 Brown and Pedersen [7]) For a unital C*-algebra A, the *real rank* $\mathrm{RR}(A)$ of A is to be the least integer n such that for any $\varepsilon > 0$ and any $n+1$ elements $a_0, a_1, , a_2 \dots, a_n \in A_{sa}$ there exist $n+1$ elements $b_0, b_1, , b_2 \dots, b_n \in A_{sa}$ such that $\|a_i - b_i\| < \varepsilon$ for $0 \leq i \leq n$ and $\sum_{i=0}^n b_i^2$ is invertible. The real rank of a nonunital C*-algebra is defined to be that of its smallest unitization. Note that $\mathrm{RR}(A) = 0$ is equivalent to density of the set of invertible self-adjoint elements in A.

(3) A C*-algebra has *Property (SP)* if there exists a non-zero projection in any non-zero hereditary subalgebra of A.

(4) A C*-algebra A is said to have *cancellation of projections* if whenever $p, q, r \in A$ are projections with $p \perp r$, $q \perp r$, and $p+r \sim q+r$, then $p \sim q$, where $p \sim q$ means Murray-von Neumann equivalent. If the matrix algebra $M_n(A)$ over A has cancellation of projections for each $n \in \mathbb{N}$, we simply say that A has *cancellation*. Every C*-algebra with cancellation is stably finite.

Note that for a unital C*-algebra A,

- $\mathrm{tsr}(A) = 1$ implies that A has cancellation
- $\mathrm{RR}(A) = 0$ implies that A has property (SP).
- When A has $\mathrm{RR}(A) = 0$, $\mathrm{tsr}(A) = 1$ is equivalent to that A has cancellation (1982 Blackadar and Handelman [6]).

3. Property (SP)

Let $1 \in A \subset B$ be a pair of C*-algebras with common unit and $E\colon B \to A$ be a faithful conditional expectation.

When A is simple and α be an outer action on A. Kishimoto [15] proved the following statement (as a special case): For every element x of A and every nonzero hereditary C*-subalgebra C of A,

$$\inf\{\|cx\alpha(c)\|\colon c \in C_+, \|c\| = 1\} = 0.$$

One of applications of this result is that it implies that the reduced crossed product of a simple C*-algebra by an outer action of a discrete group is again simple.

A counterpart of Kishimoto's theorem for a conditional expectation E is the following statement: for every element $x \in A$ and nonzero hereditary C*-subalgebra C of A,

$$\inf\{\|c(x - E(x))c\|\colon c \in C_+, \|c\| = 1\} = 0.$$

Then E is called *outer*.

Theorem 3.1. (1998 Osaka [18]) Let $1 \in A \subset B$ be a pair of C*-algebras with common unit and $E\colon B \to A$ be a faithful conditional expectation. Suppose that A has Property (SP). If E is outer, then B has Property (SP).

Moreover, every nonzero hereditary C*-subalgebra of B has a projection which is equivalent to some projection in A.

In this case any nonzero hereditary C*-subalgebra of B has nonzero projection which is Murray-von Neumann equivalent to some projection in A.

Let A be a simple C*-algebra and α be an action of a finite group G on A such that α_g is outer for $g \neq 1$. Then E is outer. Hence the crossed product algebra $B = A \times_\alpha G$ has Property (SP) when so does A. This observation can be extended using C*-index theory in the sense of Watatani [28] In fact Izumi proved the following result:

Theorem 3.2. (2002 Izumi [12]) Let $1 \in A \subset B$ be of irreducible and finite depth inclusion of simple C*-algebras, and $E\colon B \to A$ is of index finite type. Then E is outer.

Here we introduce C*-index Theory briefly.

Definition 3.3. Let $1 \in A \subset B$ be an inclusion of unital C*-algebras with a faithful conditional expectation $E\colon B \to A$.

A *quasi-basis* for E is a finite family $\big((u_1, u_1^*),\ (u_2, u_2^*),\ \ldots,\ (u_n, u_n^*)\big)$ of elements of $B \times B$ such that

$$b = \sum_{j=1}^{n} u_j E(u_j^* b) = \sum_{j=1}^{n} E(bu_j)u_j^*$$

for all $b \in B$. The expectation E has *index-finite type* if E has a quasi-basis, and the index of E is then defined by $\mathrm{Index}(E) = \sum_{j=1}^{n} u_j u_j^*$. The index is a positive invertible central element of B that does not depend on the choice of the quasi-basis. In particular, if $1 \in A \subset B$ is a pair of simple unital C*-algebras, then $\mathrm{Index}(E)$ is a positive scalar.

We will say that $1 \in A \subset B$ has index-finite type if there is a faithful conditional expectation $E \colon B \to A$ with index-finite type.

Set $B_0 = A$, $B_1 = B$, and $E_1 = E$. Recall the C*-algebra version of the basic construction. (Definition 2.2.10 of [28], where it is called the C* basic construction). We inductively define $e_k = e_{B_{k-1}}$ and $B_{k+1} = C^*(B_k, e_k)$, the Jones projection and C*-algebra for the basic construction applied to $E_k \colon B_k \to B_{k-1}$, and take $E_{k+1} \colon B_{k+1} \to B_k$ to be the dual conditional expectation E_{B_k} of Definition 2.3.3 of [28]. This gives the *tower of iterated basic constructions*

$$B_0 \subset B_1 \subset B_2 \subset \cdots \subset B_k \subset \cdots ,$$

with $B_0 = A$ and $B_1 = B$. It follows from Proposition 2.10.11 of [28] that this tower does not depend on the choice of E.

We then say that the inclusion $A \subset B$ has *finite depth* if there is $n \in \mathbb{N}$ such that $(A' \cap B_n)e_n(A' \cap B_n) = A' \cap B_{n+1}$. We call the least such n the *depth* of the inclusion.

Example 3.4. Let A be a unital C*-algebra, let G be a finite group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of G on A. For $g \in G$, let $u_g \in A \times_\alpha G$ be the standard unitary in the crossed product, implementing α_g. Then the function $E \colon A \times_\alpha G \to$

A, given by $E\big(\sum_{g\in G} a_g u_g\big) = a_1$, is a conditional expectation with index-finite type, $\big((u_g, u_g^*)\big)_{g\in G}$ is a quasi-basis for E, and $\mathrm{Index}(E) = \mathrm{card}(G)\cdot 1_{A\times_\alpha G}$.

The following result is a final version of this type.

Theorem 3.5. (1998 Osaka [18]) Let $1 \in A \subset B$ be a pair of C*-algebras with common unit and $E\colon B \to A$ be a faithful conditional expectation of index finite type. Suppose that A is simple and has Property (SP). Then B has Property (SP).

4. Topological stable rank

Theorem 4.1. (2005 O-Teruya [20])
Let B be a unital C*-algebra, let $A \subset B$ be a unital subalgebra, let $E\colon B \to A$ be a faithful conditional expectation with index-finite type, and let $\big((v_k, v_k^*)\big)_{1\le k\le n}$ be a quasi-basis for E. Then $\mathrm{tsr}(B) \le n \times \mathrm{tsr}(A)$.

Using this estimate we can get the following result.

Theorem 4.2. (2006 O-Teruya [21]) Let $1 \in A \subset B$ be an inclusion of unital C*-algebras of index-finite type and depth 2. Suppose that A is infinite dimensional simple with $\mathrm{tsr}(A) = 1$ and the property SP. Then $\mathrm{tsr}(B) \le 2$.

The point is the following observation.

Proposition 4.3. (2005 O-Teruya [20]) Let $1 \in A \subset B$ be an inclusion of unital C*-algebras of index-finite type and depth 2. Suppose that $\mathrm{tsr}(A) = 1$. Then we have

$$\sup_{p\in P(A)} \mathrm{tsr}(pBp) < \infty,$$

where $P(A)$ denotes the set of all prjections in A.

Indeed, Since A is simple with the property SP, there is a sequence of mutually orthogonal equivalent projections $\{p_i\}_{i=1}^N$ in A such that $N > K$.

Set $p = \sum_{i=1}^N p_i$. Then pBp has a matrix unite such that

$$pBp \cong M_N(p_1Bp_1).$$

Then using Rieffel's formula [24]

$$\begin{aligned}\mathrm{tsr}(pBp) &= \mathrm{tsr}(M_N(p_1Bp_1)) \\ &= \{\frac{\mathrm{tsr}(p_1Bp_1) - 1}{N}\} + 1 \\ &\leq \{\frac{K}{N}\} + 1 = 2,\end{aligned}$$

where $\{a\}$ denotes least integer greater than a. Since A is simple, p is a full projection in A, and moreover, in B. Hence from Blackadar's formula [5] for corner algebras, we have

$$\mathrm{tsr}(B) \leq \mathrm{tsr}(pBp) \leq 2.$$

∎

Theorem 4.2 implies that if A is AF C*-algebra, G a finite group, and α an action of G on A. Then $\mathrm{tsr}(A \times_\alpha G) \leq 2$. We hope that $\mathrm{tsr}(A \times_\alpha G) = 1$.

The following estimate will give a support to this problem.

Theorem 4.4. (2007 Osaka [19]) Let A be a simple unital C*-algebra with $\mathrm{tsr}(A) = 1$ and Property (SP), $\{G_k\}_{k=1}^n$ finite groups, α_k actions from G_k to $\mathrm{Aut}((\cdots((A \times_{\alpha_1} G_1) \times_{\alpha_2} G_2)\cdots) \times_{\alpha_{k-1}} G_{k-1})$. ($G_0 = \{1\}$) Then

$$\mathrm{tsr}((\cdots((A \times_{\alpha_1} G_1) \times_{\alpha_2} G_2)\cdots) \times_{\alpha_n} G_n) \leq 2.$$

5. Cancellation property

In this section we prove a cancellation theorem for inclusions of simple C*-algebras with index-finite type. We need the following modification of Blackadar's cancellation theorem, which is itself a modification of an argument of Rieffel's one. It is proved on an argument of Goodearl [9].

Theorem 5.1 (1983 Blackadar [3])**.** Let A be a simple C*-algebra. Let $P \subset M_\infty(A)$ be a set of nonzero projections with the following two properties:

(1) For every nonzero projection $q \in M_\infty(A)$, there exists $p \in P$ such that $2[p] \leq [q]$ in $K_0(A)$.
(2) $\sup_{p \in P} \mathrm{tsr}(pM_\infty(A)p) < \infty$.

Then the projections in $M_\infty(A)$ satisfy cancellation.

When $1 \in A \subset B$ is an inclusion of simple C*-algebras of index-finite type and depth 2, we could conclude that B has cancellation under the assumption that A has Cancellation with $\mathrm{tsr}(A) = 1$ and Property (SP) by Theorems 3.1, 3.2, 5.1 and Proposition 4.3.

Making counterpart of subfactors theory for inclusions of C*-algebras (which may be well known) we can get the following result.

Theorem 5.2 (2007 Jeong-O-Phillips-Teruya [14])**.** Let $1 \in A \subset B$ be an inclusion of unital C*-algebras of index-finite type and with finite depth. Suppose that A is simple, $\mathrm{tsr}(A) = 1$, and A has Property (SP). Then B has cancellation.

Let $\mathcal{P}$ be the set of all nonzero projections in A. Since A is simple, using Izumi's observation there exist projections $z_1, z_2, \ldots, z_k$ in the center of B such that Bz_j is simple and

$$B = Bz_1 \oplus Bz_2 \oplus \cdots \oplus Bz_k.$$

By several observations $z_k \in z_j A z_j \subset Bz_j$ has index-finite type and finite depth. Hence we may assume that B is simple.

Using finite depth property, tere exists $k \in \mathbb{N}$ such that

$$(A' \cap B_{k+1})e_{k+1}(A' \cap B_{k+1}) = A' \cap B_{k+2},$$

and B_k is stably isomorphic to B and B_{k+1} is stably isomorphic to A. Then there are $n \in \mathbb{N}$ and $u_1, u_2, \ldots, u_n \in A' \cap B_{k+1}$ such that for every $p \in \mathcal{P}$, the family $\{(pu_j, u_j^* p)\}_{1 \le j \le n}$ is a quasi-basis for the conditional expectation $E_{k+1}|_{pB_{k+1}p} \colon pB_{k+1}p \to pB_k p$. Then we have

$$\mathrm{tsr}(pB_k p) \le n \times \mathrm{tsr}(pB_{k+1}p) + n^2 - 2n + 1 = n^2 - n + 1$$

by a similar argument in the proof of Theorem 4.1.

Showing that B_k has Property (SP), we can conclude that B_k has cancellation.

Since B_k is stably isomorphic to B, B has cancellation. ∎

The following result gives an affirmative data to Problem 1.2.

Corollary 5.3. Let A be an infinite dimensional simple unital C*-algebra, let G be a finite group, and let α be an action of G on A. Suppose that $\mathrm{tsr}(A) = 1$ and A has Property (SP). Then $A \times_\alpha G$ has cancellation. Moreover, if $A \times_\alpha G$ has real rank zero, then $\mathrm{tsr}(A \times_\alpha G) = 1$.

Remark 5.4. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a unital C*-algebra A. Taking the crossed product $A \times_\alpha G$ can increase the topological stable rank if G is finite and A is not simple (see Example 8.2.1 of [4]) or if G is infinite and A is simple (see Example 8.2.2 of [4]). Blackadar asked, in Question 8.2.3 of [4], whether the crossed product of an AF algebra by a finite group has topological stable rank one. This question remains open, even for simple AF algebras and $\mathbb{Z}/2\mathbb{Z}$. We have seen in Corollary 5.3 that if A is a simple unital C*-algebra with $\mathrm{TR}(A) = 0$ and G is finite, then $A \times_\alpha G$ has cancellation. It often happens that cancellation for a simple unital C*-algebra B implies $\mathrm{tsr}(B) = 1$, for example if B has real rank zero. However, a crossed product of a simple AF algebra by a finite group may have nonzero real rank (Example 9 of [8]), and cancellation for a simple unital C*-algebra A does not imply $\mathrm{tsr}(A) = 1$ ([26]).

References

[1] D. Bisch, *On the structure of finite depth subfactors*, pages 175–194 in: *Algebraic Methods in Operator Theory*, Birkhäuser Boston, Boston MA, 1994.

[2] B. Blackadar, *A stable cancellation theorem for simple C*-algebras*, Appendix to: *The cancellation theorem for projective modules over irrational rotation C*-algebras* [M. A. Rieffel, Proc. London Math. Soc. (3) **47**(1983), 285–302], Proc. London Math. Soc. (3) **47**(1983), 303–305.

[3] B. Blackadar, *Comparison theory for simple C*-algebras*, pages 21–54 in: *Operator Algebras and Applications*, D. E. Evans and M. Takesaki (eds.) (London Math. Soc. Lecture Notes Series no. 135), Cambridge University Press, Cambridge, New York, 1988.

[4] B. Blackadar, *Symmetries of the CAR algebra*, Ann. Math. (2) **131**(1990), 589–623.

[5] B. Blackadar, *The stable rank of full corners in C*-algebras*, Proc. Amer. Math. Soc. **132**(2004), 2945–2950.

[6] B. Blackadar and D. Handelman, *Dimension functions and traces on C*-algebras*, J. Funct. Anal. **45**(1982), 297–340.

[7] L. G. Brown and G. K. Pedersen, *C*-algebras of real rank zero*, J. Funct. Anal. **99**(1991), 131–149.

[8] G. A. Elliott, *A classification of certain simple C*-algebras*, pages 373–385 in: *Quantum and Non-Commutative Analysis*, H. Araki etc. (eds.), Kluwer, Dordrecht, 1993.

[9] K. Goodearl, private communication.

[10] F. M. Goodman, P. de la Harpe, and V. F. R. Jones, *Coxeter Graphs and Towers of Algebras*, Mathematical Sciences Research Institute Publications **14**, Springer-Verlag, New York, 1989.

[11] R. H. Herman and L. N. Vaserstein, *The stable range of C*-algebras*, Invent. Math. **77**(1984), 553–555.

[12] M. Izumi, *Inclusions of simple C*-algebras*, J. reine angew. Math. **547**(2002), 97–138.

[13] J. A. Jeong and H. Osaka, *Extremally rich C*-crossed products and the cancellation property*, J. Austral. Math. Soc. (Series A) **64**(1998), 285–301.

[14] J. A. Jeong, H. Osaka, N. C. Phillips, and T. Teruya, *Cancellation of C*-crossed products*, 2007, arXiv:0704.3645.

[15] A. Kishimoto, *Automorphisms of $A\mathbb{T}$ algebras with the Rohlin property*, J. Operator Theory **40**(1998), 277–294.

[16] H. Lin, *Tracially AF C*-algebras*, Trans. Amer. Math. Soc. **353**(2001), 693–722.

[17] H. Lin, *An Introduction to the Classification of Amenable C*-algebras*, World Scientific, River Edge NJ, 2001.

[18] H. Osaka, *SP-property for a pair of C*-algebras*, J. Operator Theory **46**(2001), 159–171.

[19] H. Osaka, *Stable rank for inclusions of C*-algebras*, 2007, preprint.

[20] H. Osaka and T. Teruya, *Topological stable rank of inclusions of unital C*-algebras*, International J. Math. **17**(2006), 19–34.

[21] H. Osaka and T. Teruya, *Stable rank of inclusion of C*-algebras of depth 2*, to appear in Math. Rep. Acad. Sci. Royal Soc. Canada.

[22] M. Pimsner and S. Popa, *Iterating the basic construction*, Trans. Amer. Math. Soc. **310**(1988), 127–133.

[23] M. Pimsner and S. Popa, *Entropy and index for subfactors*, Ann. Sci. École Norm. Sup. (4) **19**(1986), 57–106.

[24] M. A. Rieffel, *Dimension and stable rank in the K-theory of C*-algebras*, Proc. London Math. Soc. (3) **46**(1983), 301–333.

[25] M. A. Rieffel, *The cancellation theorem for projective modules over irrational rotation C*-algebras*, Proc. London Math. Soc. (3) **47**(1983), 285–302.

[26] A. S. Toms, *Cancellation does not imply stable rank one*, preprint (arXiv: math.OA/0509107).

[27] J. Villadsen, *On the stable rank of simple C*-algebras*, J. Amer. Math. Soc. **12**(1999), 1091–1102.

[28] Y. Watatani, *Index for C*-subalgebras*, Mem. Amer. Math. Soc. **83**(1990), no. 424.

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