On endomorphisms of von Neumann algebras from braid group representations

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(joint with Rolf Gohm)

Fields Institute Workshop on
Noncommutative Dynamics and Applications

Toronto, July 20, 2007

References

To be posted on arXiv:

- C. Köstler, *A noncommutative dual version of the extended De Finetti theorem*
- R. Gohm, C. Köstler, *Spreadable noncommutative random sequences from braid group representations*

Outline

1. Motivation and Terminology
2. Noncommutative De Finetti Theorem
3. Braid Group Representations
4. Applications and Examples

Motivation and Terminology

Motivation

Though many probabilistic symmetries are conceivable [...], four of them - **stationarity**, **contractability**, **exchangeablity** *and* **rotatability** *- stand out as especially interesting and important in several ways: Their study leads to some deep structural theorems of great beauty and significance [...].*

Olav Kallenberg (2005)

Question:

Can one transfer the related concepts to **noncommutative probability theory** and do they turn out to be fruitful in the study of the structure of **operator algebras**?

Hierarchy of distributional symmetries

invariant objects	transformations
stationary	shifts
contractable	sub-sequences
exchangeable	permutations
rotatable	isometries

Topic of this talk:

- invariant objects are generated by an **infinite sequence** of random variables
- only the **first three symmetries** are considered
- **contractable = spreadable**

Motivating Example for De Finetti theorem

"Any exchangeable process is an average of i.i.d. processes."

(De Finetti 1931)

$X_1, X_2, \ldots$ infinite sequence of $\{0,1\}$-valued random variables s.t.

$$P(X_1 = e_1, \ldots, X_n = e_n) = P(X_{\pi(1)} = e_1, \ldots, X_{\pi(n)} = e_n)$$

holds for all $n \in \mathbb{N}$ and permutations $\pi\colon [n] \to [n]$ and for every $e_1, \ldots, e_n \in \{0,1\}$.
Then there exists a unique probability measure μ on $[0,1]$ such that

$$P(X_1 = e_1, \ldots, X_n = e_n) = \int p^s (1-p)^{n-s} d\mu(p),$$

where $s = e_1 + e_2 + \ldots + e_n$.

Terminology of noncommutative probability

- **(noncommutative) probability space:**
  $(\mathcal{A}, \varphi)$ sep. von Neumann algebra $\mathcal{A}$ with f.n. state φ
  where $\mathcal{A}$ is represented on GNS Hilbert space
- **(noncommutative) random variable:**

$$\iota \colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{A}, \varphi)$$

  injective *-homomorphism from $\mathcal{A}_0$ to $\mathcal{A}$ such that

$$\begin{aligned} \iota(\mathbb{1}_{\mathcal{A}_0}) &= \mathbb{1}_{\mathcal{A}} && \text{(unitality)} \\ \varphi \circ \iota &= \varphi_0 && \text{(state-preserving)} \\ \sigma_t^{\varphi}\, \iota &= \iota\, \sigma_t^{\varphi_0} && \text{(intertwining)} \end{aligned}$$

- **Automorphisms of a probability space:**
  $\mathrm{Aut}(\mathcal{A}, \varphi)$ φ-preserving *-automorphisms of $\mathcal{A}$

Noncommutative independence and commuting squares

Definition

Given the probability space $(\mathcal{A}, \varphi)$, let $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ be three von Neumann subalgebras of $\mathcal{A}$ such that the φ-preserving conditional expectations $E_i \colon \mathcal{A} \to \mathcal{A}_i$ exist $(i = 1, 2, 3)$. Then $\mathcal{A}_1$ and $\mathcal{A}_2$ are said to be $\mathcal{A}_0$-**independent** or **conditionally independent** if

$$E_1 \circ E_2 = E_0$$

Equivalent formulation

$\mathcal{A}_1$ and $\mathcal{A}_2$ are $\mathcal{A}_0$-independent if and only if the diagram

$$\begin{array}{ccc} \mathcal{A}_1 & \subset & \mathcal{A}, \\ \cup & & \cup \\ \mathcal{A}_0 & \subset & \mathcal{A}_2, \end{array}$$

is a **commuting square**.

Order independence and conditionally i.i.d.

Let $I, J \subset \mathbb{N}_0$ (**ordered set!**). A family of random variables

$$\iota = (\iota_i)_{i \in \mathbb{N}_0} \colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{A}, \varphi)$$

is said to be

- **order $\mathcal{B}$-independent** if $\bigvee\{\iota_i(\mathcal{A}_0) \,|\, i \in I\}$ and $\bigvee\{\iota_j(\mathcal{A}_0) \,|\, j \in J\}$ are $\mathcal{B}$-independent whenever $I < J$
- **conditionally i.i.d. over $\mathcal{B}$** if $\bigvee\{\iota_i(\mathcal{A}_0) \,|\, i \in I\}$ and $\bigvee\{\iota_j(\mathcal{A}_0) \,|\, j \in J\}$ are $\mathcal{B}$-independent whenever $I \cap J = \emptyset$ **and** $\varphi(\iota_1(x)^k) = \varphi(\iota_i(x)^k)$ for all $k \in \mathbb{N}$, $i \in \mathcal{I}$ and $x \in \mathcal{A}_0$

Remark:

What about Boolean algebra as index set? $\longrightarrow$ 'factorizations'

Distributional Symmetries I

Definition

Two n-tuples $\mathbf{i}, \mathbf{j} \colon [n] \to \mathbb{N}_0$ are

1. **translation equivalent** ($\mathbf{i} \sim_\theta \mathbf{j}$), if there exists $k \in \mathbb{N}_0$ such that
$$\mathbf{i} = \theta^k \circ \mathbf{j} \qquad \text{or} \qquad \theta^k \circ \mathbf{i} = \mathbf{j}.$$
2. **order equivalent** ($\mathbf{i} \sim_o \mathbf{j}$), if there exists $\pi \in \mathbb{S}_\infty$ with
$$\mathbf{i} = \pi \circ \mathbf{j} \qquad \text{and} \qquad \pi|_{\mathbf{j}([n])} \text{ is order preserving.}$$
3. **symmetric equivalent** ($\mathbf{i} \sim_\pi \mathbf{j}$), if there exists $\pi \in \mathbb{S}_\infty$ such that
$$\mathbf{i} = \pi \circ \mathbf{j}$$

Note: $(\mathbf{i} \sim_\theta \mathbf{j}) \Rightarrow (\mathbf{i} \sim_o \mathbf{j}) \Rightarrow (\mathbf{i} \sim_\pi \mathbf{j})$

Distributional symmetries II

Speicher's notation of multilinear maps

Let $\iota \equiv (\iota_i)_{i\in\mathbb{N}_0} \colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{A}, \varphi)$ be given. We put, for $\mathbf{i} \colon [n] \to \mathbb{N}_0$, $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{A}_0^n$ and $n \in \mathbb{N}$,

$$\begin{aligned} \mathbf{a} \mapsto \iota[\mathbf{i}; \mathbf{a}] &:= \iota_{\mathbf{i}(1)}(a_1)\iota_{\mathbf{i}(2)}(a_2)\cdots\iota_{\mathbf{i}(n)}(a_n) \\ \mathbf{a} \mapsto \varphi_\iota[\mathbf{i}; \mathbf{a}] &:= \varphi\big(\iota[\mathbf{i}; \mathbf{a}]\big) \end{aligned}$$

Definition (Distributional Symmetries)

A random sequence $\iota \equiv (\iota_i)_{i\in\mathbb{N}_0} \colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{A}, \varphi)$ is

(i) **exchangeable** if, $\forall n \in \mathbb{N}$, $\varphi_\iota[\mathbf{i}; \cdot\,] = \varphi_\iota[\mathbf{j}; \cdot\,]$ whenever $\mathbf{i} \sim_\pi \mathbf{j}$

(ii) **spreadable** if, $\forall n \in \mathbb{N}$, $\varphi_\iota[\mathbf{i}; \cdot\,] = \varphi_\iota[\mathbf{j}; \cdot\,]$ whenever $\mathbf{i} \sim_o \mathbf{j}$

(iii) **stationary** if, $\forall n \in \mathbb{N}$, $\varphi_\iota[\mathbf{i}; \cdot\,] = \varphi_\iota[\mathbf{j}; \cdot\,]$ whenever $\mathbf{i} \sim_\theta \mathbf{j}$

Note: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Noncommutative De Finetti Theorem

Classical dual version of extended De Finetti theorem

Let $\iota \equiv (\iota_i)_{i\in\mathbb{N}_0} \colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{A}, \varphi)$ be a random sequence with tail algebra

$$\mathcal{A}^{\text{tail}} := \bigcap_{n\geq 0} \bigvee_{k\geq n} \iota_k(\mathcal{A}_0)$$

and consider the following conditions:

(a) ι is exchangeable

(b) ι is spreadable

(c) ι is stationary and order $\mathcal{A}^{\text{tail}}$-independent

(d) ι is conditionally i.i.d. over $\mathcal{A}^{\text{tail}}$

Theorem (De Finetti (1931), Ryll-Nardzewski (1957))

$\mathcal{A} \simeq L^\infty(\Omega, \Sigma, \mu) \quad \Longrightarrow \quad$ (a) to (d) are equivalent.

Noncommutative dual version of extended De Finetti theorem

Theorem (K.)

Let $\iota \equiv (\iota_i)_{i\in\mathbb{N}_0} \colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{A}, \varphi)$ be a random sequence with tail algebra

$$\mathcal{A}^{\text{tail}} := \bigcap_{n\geq 0} \bigvee_{k\geq n} \iota_k(\mathcal{A}_0)$$

and consider the following conditions:

(a) ι is exchangeable

(b) ι is spreadable

(c) ι is stationary and order $\mathcal{A}^{\text{tail}}$-independent

(d) ι is conditionally i.i.d. over $\mathcal{A}^{\text{tail}}$

Then we have

$$\text{(a)} \Rightarrow \text{(b)} \Rightarrow \text{(c)} \Rightarrow \text{(d)}.$$

Discussion of the noncommutative De Finetti theorem

Exchangeability (a), spreadability (b), stationarity and order independence (c) and conditionally i.i.d. (d) are **no longer equivalent in presented noncommutative setting !**

Theorem (K. 2007)

Speicher's **University rules** on noncommutative independence imply **tensor product independence** or **free independence**. As a consequence, **conditions (a) to (d) are equivalent under the presence of universality rules**.

Noncommutative versions of De Finetti theorem in literature

- **involving tensor product constructions or other commutativity conditions:** E. Stoermer (1969), R.L. Hudson (1981), R.L. Hudson & G.R. Moody(1976), D. Petz (1990), L. Accardi & Y.G. Lu (1993), among others
- **Free version with cumulants:** Lehner (2004)
- **Extended versions with spreadability:** ????

Are this **'good news'** or **'bad news'**?!

Indeed there is no hope to obtain a general De Finetti's theorem without imposing additional conditions.

Franz Lehner, 2004

Braid Group Representations

Artin's Braid groups

Algebraic Definition (Artin 1925)

The braid group $\mathbb{B}_n$ is presented by $n-1$ generators $\sigma_1, \ldots, \sigma_{n-1}$ satisfying

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \qquad \text{if } \mid i-j \mid = 1 \tag{B1}$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \text{if } \mid i-j \mid > 1 \tag{B2}$$

Notation

- $\mathbb{B}_1 = \langle \sigma_0 \rangle$, *where* σ_0 *denotes identity.*
- $\mathbb{B}_1 \subset \mathbb{B}_2 \subset \mathbb{B}_3 \subset \ldots \subset \mathbb{B}_\infty$ *(inductive limit)*

Fixed point algebras of braid group representation

Suppose the representation $\rho\colon \mathbb{B}_\infty \to \mathrm{Aut}(\mathcal{A}, \varphi)$ is given.

$$\mathcal{A}_{n-2} := \mathcal{A}^{\rho(\mathbb{B}_{n,\infty})} := \bigcap_{k \geq n} \mathcal{A}^{\rho(\sigma_k)}$$

Tower of von Neumann algebras:

$$\mathcal{A}^{\rho(\mathbb{B}_\infty)} = \mathcal{A}_{-1} \subset \mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_{n-2} \subset \cdots \subset \mathcal{A}_\infty.$$

Main results

Theorem (Gohm & K. 2007)

Let the (not necessarily faithful) representation

$$\rho\colon \mathbb{B}_\infty \to \mathrm{Aut}(\mathcal{A}, \varphi)$$

be given. Furthermore, suppose $\mathcal{C}_0$ is a von Neumann subalgebra of $\mathcal{A}$ such that the φ-preserving conditional expectation from $\mathcal{A}$ onto $\mathcal{C}_0$ exists.
If $\mathcal{C}_0$ satisfies the **localization property** $\mathcal{C}_0 \subset \mathcal{A}^{\rho(\mathbb{B}_{2,\infty})}$, then

$$\begin{aligned} \iota_n\colon \mathcal{C}_0 &\to \mathcal{A}; \\ x &\mapsto \rho(\sigma_n\sigma_{n-1}\cdots\sigma_2\sigma_1\sigma_0)(x) \end{aligned}$$

defines a **spreadable random sequence** $(\iota_n)_{n\in\mathbb{N}_0}$.

Braided counterpart of Hewitt-Savage 0-1 law

Theorem (Gohm &K. 2007)

Under the assumptions of the previous theorem, it holds

$$\mathcal{A}^{\mathsf{tail}} = \mathcal{A}^{\rho(\mathbb{B}_\infty)}$$

In particular, these two algebras are trivial if the random sequence is conditionally i.i.d. over $\mathbb{C}$.

Endomorphisms from braid group representations

Proposition

Suppose $(\mathcal{A}, \varphi)$ admits a (not necessarily faithful) representation

$$\rho\colon \mathbb{B}_\infty \to \mathrm{Aut}(\mathcal{A}, \varphi)$$

and let $\mathcal{A}_{n-1} := \mathcal{A}^{\rho(\mathbb{B}_{n+1,\infty})}$. Then

$$\alpha(x) = \text{SOT-}\lim_{n\to\infty} \rho(\sigma_1\sigma_2\cdots\sigma_n)(x) \tag{BPR-0}$$

defines an endomorphism α on $\mathcal{A}_\infty := (\bigcup_n \mathcal{A}_n)''$ and

$$\rho(\sigma_k)(\mathcal{A}_n) = \mathcal{A}_n \tag{BPR-1}$$

$$\rho(\sigma_{n+1})|_{\mathcal{A}_{n-1}} = \mathrm{id}\,|_{\mathcal{A}_{n-1}} \tag{BPR-2}$$

for all $0 \leq k \leq n < \infty$.

Tower of commuting squares

Theorem (Gohm & K. 2007)

Under the assumptions of the previous proposition, one obtains a triangular tower of inclusions such that each cell forms a commuting square:

$$
\begin{array}{ccccccccccccc}
\mathcal{A}_{-1} & \subset & \mathcal{A}_0 & \subset & \mathcal{A}_1 & \subset & \mathcal{A}_2 & \subset & \mathcal{A}_3 & \subset & \cdots & \subset & \mathcal{A}_\infty \\
 & & \cup & & \cup & & \cup & & \cup & & & & \cup \\
 & & \mathcal{A}_{-1} & \subset & \alpha(\mathcal{A}_0) & \subset & \alpha(\mathcal{A}_1) & \subset & \alpha(\mathcal{A}_2) & \subset & \cdots & \subset & \alpha(\mathcal{A}_\infty) \\
 & & & & \cup & & \cup & & \cup & & & & \cup \\
 & & & & \mathcal{A}_{-1} & \subset & \alpha^2(\mathcal{A}_0) & \subset & \alpha^2(\mathcal{A}_1) & \subset & \cdots & \subset & \alpha^2(\mathcal{A}_\infty) \\
 & & & & & & \cup & & \cup & & & & \cup \\
 & & & & & & \vdots & & \vdots & & & & \vdots
\end{array}
$$

Here $\mathcal{A}_{-1}$ equals the fixed point algebra of α.

Remark

$(\mathcal{A}_\infty, \varphi_\infty, \alpha, \mathcal{A}_0; \mathcal{A}_{-1})$ *is a 'discrete-time' version of continuous Bernoulli shifts as defined in Hellmich -K.- Kümmerer (2004).*

Applications and Examples

Unitary braid group representations

Given $(\mathcal{A}, \varphi)$, suppose the sequence $(u_n)_{n\in\mathbb{N}}$ of unitary operators $u_n \in \mathcal{A}^\varphi$ satisfies the braid relations:

$$\begin{aligned} u_i u_j &= u_j u_i && \text{for } |i-j| > 1; \\ u_i u_j u_i &= u_j u_i u_j && \text{for } |i-j| = 1. \end{aligned}$$

Let $u_0 = 1$ and

$$\mathcal{J}_n \quad := \quad \bigvee_{0\le k\le n} \{u_k\} \qquad (0 \le n \le \infty)$$

Then $\rho(\sigma_n)(x) := \mathrm{Ad}\, u_n(x) = u_n x u_n^*$, with $x \in \mathcal{J}_\infty$, defines the representation

$$\rho\colon \mathbb{B}_\infty \to \mathrm{Aut}(\mathcal{J}_\infty, \mathrm{tr}_{\mathcal{J}_\infty}).$$

Unitary braid group representations II

An application of the main result gives

$$\begin{array}{ccccccccc}
\mathcal{Z}(\mathcal{J}_\infty) & \subset & \mathcal{J}_\infty^{\rho(\mathbb{B}_{2,\infty})} & \subset & \mathcal{J}_\infty^{\rho(\mathbb{B}_{3,\infty})} & \subset & \cdots & \subset & \mathcal{J}_\infty \\
 & & \cup & & \cup & & & & \cup \\
 & & \mathcal{Z}(\mathcal{J}_\infty) & \subset & \alpha(\mathcal{J}_\infty^{\rho(\mathbb{B}_{2,\infty})}) & \subset & \cdots & \subset & \alpha(\mathcal{J}_\infty) \\
 & & & & \cup & & & & \cup \\
 & & & & \vdots & & & & \vdots
\end{array}$$

Some elementary properties

1. $\alpha(u_n) = u_{n+1}$
2. $u_n \in \mathcal{J}_\infty^{\rho(\mathbb{B}_{n+2,\infty})} \quad \Rightarrow \quad \mathcal{J}_n \subset \mathcal{J}_\infty^{\rho(\mathbb{B}_{n+2,\infty})}$
3. $\mathcal{J}_\infty \cap \left(\alpha(\mathcal{J}_\infty)\right)' \subset \mathcal{Z}(\mathcal{J}_\infty) \quad \Longleftrightarrow \quad \mathcal{J}_\infty^{\rho(\mathbb{B}_{2,\infty})} \subset \mathcal{Z}(\mathcal{J}_\infty)$

Group von Neumann algebra $L(\mathbb{B}_\infty)$

Theorem (Gohm & K. 2007)

$L(\mathbb{B}_\infty)$ *is a* **nonhyperfinite factor of type** II_1 *and the inclusion* $L(\mathbb{B}_{2,\infty}) \subset L(\mathbb{B}_\infty)$ *is* **irreducible**,*i.e.*

$$L(\mathbb{B}_\infty) \cap \left(L(\mathbb{B}_{2,\infty})\right)' \simeq \mathbb{C}.$$

Putting $\mathcal{J}_\infty = L(\mathbb{B}_\infty)$ *we have the following* **commuting squares**:

$$\begin{array}{ccccccccc}
\mathcal{Z}(\mathcal{J}_\infty) & \subset & \mathcal{J}_\infty^{\rho(\mathbb{B}_{2,\infty})} & \subset & \mathcal{J}_\infty^{\rho(\mathbb{B}_{3,\infty})} & \subset & \mathcal{J}_\infty^{\rho(\mathbb{B}_{4,\infty})} & \subset \cdots \subset & \mathcal{J}_\infty \\
\| & & \| & & \cup & & \cup & & \| \\
\mathbb{C} & \subset & \mathbb{C} & \subset & \mathcal{J}_1 & \subset & \mathcal{J}_2 & \subset \cdots \subset & \mathcal{J}_\infty \\
 & & \cup & & \cup & & \cup & & \cup \\
 & & \mathbb{C} & \subset & \mathbb{C} & \subset & \alpha(\mathcal{J}_1) & \subset \cdots \subset & \alpha(\mathcal{J}_\infty) \\
 & & & & \cup & & \cup & & \\
 & & & & \vdots & & \vdots & &
\end{array}$$

Another presentation by square roots of free generators

Set of generators

$$\gamma_i := (\sigma_0\sigma_1\sigma_2\cdots\sigma_{i-1})\sigma_i(\sigma_{i-1}^{-1}\cdots\sigma_2^{-1}\sigma_1^{-1}\sigma_0^{-1})$$

Proposition (Gohm & K. 2007)

$\mathbb{B}_n$ is presented by $\{\gamma_k | 1 \leq k \leq n-1\}$ subject to

$$\gamma_k\left(\gamma_{k-1}\gamma_{k-2}\cdots\gamma_{j+1}\gamma_j\right)\gamma_k = \left(\gamma_{k-1}\gamma_{k-2}\cdots\gamma_{j+1}\gamma_j\right)\gamma_k\gamma_{k-1}$$

for $1 \leq j < k \leq n-1$.

Corollary

The free group $\mathbb{F}_n$ admits the presentation $\langle\gamma_1^2, \ldots, \gamma_n^2\rangle$.

Is there a braided extension of free probability?

Let

$$\beta := \lim_{n\to\infty} \mathrm{Ad}(u_2^* u_3^* u_4^* \cdots u_n^*)$$

Then, in the left regular representation $\sigma \to \ell(\sigma)$, with $v_n := \ell(\gamma_n)$

$$\beta(v_n) = \beta(v_{n+1})$$

Theorem (Gohm & K. 2007)

The **square root of free generator presentation** *gives the triangular tower of commuting squares:*

$$\begin{array}{ccccccccc}
\mathbb{C} & \subset & \langle v_1 \rangle & \subset & \langle v_1, v_2 \rangle & \subset & \langle v_1, v_2, v_3 \rangle & \subset \cdots \subset & \mathcal{J}_\infty \\
 & & \cup & & \cup & & \cup & & \| \\
 & & \mathbb{C} & \subset & \langle v_2 \rangle & \subset & \langle v_2, v_3 \rangle & \subset \cdots \subset & \beta(\mathcal{J}_\infty) \\
 & & & & \cup & & \cup & & \cup \\
 & & & & \mathbb{C} & \subset & \langle v_3 \rangle & \subset \cdots \subset & \beta^2(\mathcal{J}_\infty) \\
 & & & & \cup & & \cup & & \\
 & & & & \vdots & & \vdots & &
\end{array}$$

Subfactor theory

Subfactor theory

Definition (Jones Index)

Let $\mathcal{M}_0 \subset \mathcal{M}_1$ be an inclusion of separable type II_1 factors on $\mathcal{H}$.

$$[\mathcal{M}_1 : \mathcal{M}_0] := \frac{\dim_{\mathcal{M}_0} \mathcal{H}}{\dim_{\mathcal{M}_1} \mathcal{H}}$$

Theorem (Jones)

Let $\mathcal{M}_0 \subset \mathcal{M}_1$ be a type II_1 subfactor. Then

$$[\mathcal{M}_1 \colon \mathcal{M}_0] \in \{4\cos(2\pi/n) \mid n \in \mathbb{N}, n \geq 3\} \cup [4, \infty]$$

Each value in this index set is realized by subfactors.

Fundamental construction of Jones towers

- $\mathcal{M}_0 \subset \mathcal{M}_1$ subfactor of type II_1 with **finite index** λ
- GNS representation of $(\mathcal{M}_1, \tau_{\mathcal{M}_1})$ gives Hilbert space $\mathcal{H}_1$
- Let e_1 be the orthogonal projection in the $B(\mathcal{H}_1)$ which implements the trace-preserving conditional expectation from $\mathcal{M}_1$ onto $\mathcal{M}_0$
- $\mathcal{M}_2 := \mathrm{vN}\{\mathcal{M}_1, e_1\}$ is a type II_1 factor (on $\mathcal{H}_1$) and $\mathcal{M}_1 \subset \mathcal{M}_2$ is again a subfactor of type II_1 with **finite index** λ
- Iterate this construction to obtain the towers of inclusions

$$\begin{array}{ccccccccc} \mathcal{M}_0 & \subset & \mathcal{M}_1 & \subset & \mathcal{M}_2 & \subset & \cdots & \subset & \mathcal{M}_\infty \\ \cup & & \cup & & \cup & & & & \cup \\ \mathcal{J}_0 & \subset & \mathcal{J}_1 & \subset & \mathcal{J}_2 & \subset & \cdots & \subset & \mathcal{J}_\infty \end{array}$$

with Jones algebras $\mathcal{J}_n := \mathrm{vN}\{1, e_1, e_2, \ldots e_{n-1}\}$

Definition

Let $g_k := te_k - (1 - e_k)$ with $(1-t)(1-t^{-1}) = \frac{1}{[\mathcal{M}_1 : \mathcal{M}_0]} > 0$

Proposition

The g_k's are unitaries and $\mathcal{J}_n = \mathrm{vN}\{1, g_1, g_2, \ldots, g_{n-1}\}$

Theorem (Jones)

The mapping $\sigma_k \mapsto g_k$ defines a representation of $\mathbb{B}_\infty$ in the unitary operators of $\mathcal{M}_\infty$ such that

$$g_i g_j = g_j g_i \quad \text{if } |i-j| \geq 2 \tag{H1}$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \tag{H2}$$

$$g_i^2 = (t-1)g_i + t \tag{H3}$$

$$1 + g_i + g_{i+1} + g_i g_{i+1} + g_{i+1} g_i + g_i g_{i+1} g_i = 0$$

Commuting squares in subfactor theory

Definition

Let $(\mathcal{A}, \tau)$ be a separable von Neumann algebra with faithful normal tracial state τ on $\mathcal{A}$. The four von Neumann algebras $\mathcal{A}_0 \subseteq \mathcal{A}_i \subset \mathcal{A}$ $(i = 0, 1, 2)$ form a commuting square if there exist conditional expectations $E_i \colon \mathcal{A} \to \mathcal{A}_i$ such that $E_1 E_2 = E_0$.

Proposition

$$\begin{array}{ccc} g_k \mathcal{M}_k g_k^* & \subset & \mathcal{M}_{k+1} \\ \cup & & \cup \\ \mathcal{M}_{k-1} & \subset & \mathcal{M}_k \end{array}$$

is a commuting square for all $k \in \mathbb{N}$.

Conclusion for Jones fundamental tower with finite index

The braid group $\mathbb{B}_\infty$ defines via

$$\alpha(x) := \text{SOT-}\lim_n \mathrm{Ad}(g_1 g_2 \cdots g_n)(x)$$

an endomorphism on $\mathcal{M}_\infty$ and one obtains a triangular tower of inclusions such that each cell forms a commuting square:

$$\begin{array}{ccccccccccc}
\mathcal{M}_0 & \subset & \mathcal{M}_1 & \subset & \mathcal{M}_2 & \subset & \mathcal{M}_3 & \subset & \cdots & \subset & \mathcal{M}_\infty \\
 & & \cup & & \cup & & \cup & & & & \cup \\
 & & \alpha(\mathcal{M}_0) & \subset & \alpha(\mathcal{M}_1) & \subset & \alpha(\mathcal{M}_2) & \subset & \cdots & \subset & \alpha(\mathcal{M}_\infty) \\
 & & & & \cup & & \cup & & & & \cup \\
 & & & & \alpha^2(\mathcal{M}_0) & \subset & \alpha^2(\mathcal{M}_1) & \subset & \cdots & \subset & \alpha^2(\mathcal{M}_\infty) \\
 & & & & & & \vdots & & & & \vdots
\end{array}$$

Remark

Let $\mathcal{M}_0, \mathcal{M}_1$ be two hyperfinite factors. TFAE:

(a) $\mathcal{M}_0 \subset \mathcal{M}_1$

(b) *There exists an endomorphism β of $\mathcal{M}_1$ with $\beta(\mathcal{M}_1) = \mathcal{M}_0$*

Beyond random sequences

Braid groups have a linear order

Theorem (Dehornoy '92, '94, '00)

There exists a linear order $<_L$ on $\mathbb{B}_\infty$ and an order isomorphism

$$J\colon (\mathbb{B}_\infty, <_L) \to (\mathbb{Q}, <).$$

Question (Dehornoy '00)

Does some 'convenient' binary operation on $\mathbb{B}_n$ correspond to multiplication of integers? Does there exist some (necessarily noncommutative) arithmetic of braids?

Remark

$\mathbb{Q}$ *is a quotient of* $\mathbb{Z} \times \mathbb{Z}^*$ —> *'arrays with equivalence relation'*

Is there are a distributional symmetry 'braidability'?!

Definition (K)

Let $\mathbf{i}, \mathbf{j} \colon [n] \to \mathbb{Q}$ be two 'arrays'.

- $\mathbf{i}$ and $\mathbf{j}$ are $>_L$-*invariant*, in symbols: $\mathbf{i} \sim_L \mathbf{j}$, if there exists $\sigma \in \mathbb{B}_\infty$ such that

$$\mathbf{i} = J \circ \sigma \circ J^{-1} \circ \mathbf{j}.$$

- An 'array' of random variables $(\iota_q)_{q \in \mathbb{Q}} \colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{A}, \varphi)$ is *braidable* if, for any $n \in \mathbb{N}$,

$$\varphi_\iota[\mathbf{i}; \cdot] = \varphi_\iota[\mathbf{j}; \cdot] \text{ whenever } \mathbf{i} \sim_L \mathbf{j}.$$

Braidability and braid group representations

Theorem (K)

Assume that the random 'array' $(\iota_q)_{q\in\mathbb{Q}}$ *is braidable and minimal. Then there exists a representation*

$$\rho\colon \mathbb{B}_\infty \to \mathrm{Aut}(\mathcal{A}, \varphi)$$

such that, for any $n \in \mathbb{N}$,

$$\rho(\sigma_i)(\iota[\mathbf{i}; \mathbf{a}]) = \iota[J \circ \sigma_i \circ J^{-1} \circ \mathbf{i}; \mathbf{a}]$$

for all $i \in \mathbb{N}$, $\mathbf{i}\colon [n] \to \mathbb{Q}$ *and* $\mathbf{a} \in \mathcal{A}_0^n$.

Existence of braidable random 'arrays'

Noncommutative case – Answer

There are many examples!

Commutative case – Question

Does there exist a standard probability space $(\Omega, \mathscr{A}, \mu)$ and a family of random variables

$$(X_q)_{q\in\mathbb{Q}} \colon (\Omega, \mathscr{A}, \mu) \to \mathbb{R}$$

such that, for all $\sigma \in \mathbb{B}_\infty$,

$$(X_{q_1}, X_{q_2}, \dots X_{q_m}) \stackrel{d}{=} (X_{J\circ\sigma\circ J^{-1}(q_1)}, X_{J\circ\sigma\circ J^{-1}(q_2)}, \dots X_{J\circ\sigma\circ J^{-1}(q_m)})$$

for any collection $q_1, \dots, q_m$ of distinct elements in $\mathbb{Q}$?

Summary

- Braid group representations lead to spreadable random sequences
- Representations of $\mathbb{B}_\infty$ lead to factorizations of states
- Subfactor theory and free probability can be treated under one ambrella
- Inclusions of subfactors with infinite index or infinite depth or non-hyperfinite von Neumann algebras are captured and probabilistic approach offers alternative invariants

Conjecture 1

Every unitary representation of $\mathbb{B}_\infty$ in a separable II_1-factor leads to a link invariant.

Conjecture 2

There is a braided extension of free probability.