Luigi Accardi
Quantum Markovianity: a survey
Talk given at:
Workshop on Noncommutative Dynamics
Fields Institute, Toronto
July 16–20, 2007
Email: accardi@volterra.mat.uniroma2.it
WEB page: http://volterra.mat.uniroma2.it

**Expected Markovianity**

**Theorem 1** *Let:*

– $\mathcal{A}$ *be a* $*$–*algebra,*

– $\mathcal{A}_I, \mathcal{A}_{\partial I}, \mathcal{A}_{I'}$ *sub–*$*$*–algebras of* $\mathcal{A}$*, such that*

$$\mathcal{A}_{\partial I} \subseteq \mathcal{A}_I \cap \mathcal{A}_{I'} \tag{1}$$

*Let*

$$E_X : \mathcal{A} \to \mathcal{A}_X \qquad ; \qquad X = I, I', \partial I \tag{2}$$

*be Umegaki conditional expectations onto the respective ranges, satisfying the projectivity condition*

$$E_{\partial I} E_I = E_{\partial I} E_{I'} = E_{\partial I} \tag{3}$$

*If* $E_{\partial I}$ *is faithful, then the following four identities are equivalent (*$E_X \mid_{\mathcal{A}_Y}$ *denotes restriction of* $E_X$ *on* $\mathcal{A}_Y$*):*

$$E_I \mid_{\mathcal{A}_{I'}} = E_{\partial I} \mid_{\mathcal{A}_{I'}} \tag{4}$$

$$E_I(\mathcal{A}_{I'}) \subseteq \mathcal{A}_{\partial I} \tag{5}$$

$$E_{\partial I}(a_I a_{I'}) = E_{\partial I}(a_I) \cdot E_{\partial I}(a_{I'}) \ ; \ \forall a_I \in \mathcal{A}_I \ \ a_{I'} \in \mathcal{A}_{I'} \tag{6}$$

$$E_{I'} \mid_{\mathcal{A}_I} = E_{\partial I} \mid_{\mathcal{A}_I} \tag{7}$$

$$E_{I'}(A_I) \subseteq A_{\partial I} \tag{8}$$

**Intuitive interpretation** *:*

– $\mathcal{A}_I$ *the past, or interior algebra*

*the present or boundary algebra*

– $\mathcal{A}_{I'}$ *the future or exterior algebra*

*If $E_{\partial I}$ is not faithful, then*

*(4)*$\longleftrightarrow$ *(5)* $\longrightarrow$ *(6)* $\longleftarrow$ *(8)* $\longleftrightarrow$ *(7).*

*and, if (6) holds, then for each $a_I \in \mathcal{A}_I$ and $a' \in \mathcal{A}_{I'}$*

$$E_{\partial I}\Big(|E_{\partial I}(a_{I'}) - E_I(a_{I'})|^2\Big) = 0 \qquad (9)$$

$$E_{\partial I}\Big(|E_{\partial I}(a_I) - E_{I'}(a_I)|^2\Big) = 0 \qquad (10)$$

**Factorizability implies Markovianity**

**Definition** Let be given

– a $*$–algebra $\mathcal{A}$

– a measurable space $(X, \mathcal{O})$

– for any $I \in \mathcal{O}$ a sub–$*$–algebra $\mathcal{A}_I$ of $\mathcal{A}$, such that

$$I \subseteq J \Rightarrow \mathcal{A}_I \subseteq \mathcal{A}_J$$

– for any $I \in \mathcal{O}$ a surjective Umegaki conditional expectation

$$E_I : \mathcal{A} \to \mathcal{A}_I \tag{11}$$

The family $(E_I)$ $(I \in \mathcal{O})$ is called factorizable if for any $I, J \in \mathcal{O}$

$$E_I E_J = E_{I \cap J} \tag{12}$$

if (12) holds only when $I \subseteq J$, i.e. if one only requires that

$$I \subseteq J \Rightarrow E_I E_J = E_I$$

then the family $(E_I)$ is called projective.

Notice that in both cases $E_I$ and $E_J$ commute.

Factorizability implies projectivity (particular case).

**Factorizability implies Markovianity**

Example, if:

– $X$ is a topological space

$\mathcal{O}$ is its Borel $\sigma$–algebra

$\mathcal{I} := \{I \subseteq X : I = I^{-}(closure)$ and $(I^{o})^{-} = (I^{-})^{o}(interior)$

$$I' := (I^{o})^{c}$$

Then

$$I \cap I' = I \cap (I^{o})^{c} = I^{-} \setminus I^{o} = \partial I$$

The Markov property follows because:

$$E_I(a_{I'}) = E_I E_{I'}(a_{I'}) = E_{I \cap I'}(a_{I'}) = E_{\partial I}(a_{I'})$$

Many other examples are possible depending on how, given $I$, one defines $I'$.

**Markov systems as local perturbations of product systems**

Factorizable families of surjective Umegaki conditional expectations arise naturally in the theory of product systems.

In the discrete case they are the conditional expectations naturally associated to product measures.

Basic idea:

Start from such a family $(E^0_I)$ and perturb it by a localized multiplicative family (LMF).

Simplest example:

$$I \mapsto M_I \in \mathcal{A}_{\overline{I}}$$

$$M_I M_J = M_{I \cup J}$$

$$E^0_I(|M_I|^2) = E^0_I(M_I^* M_I) = 1$$

(e.g. $M_I$ is an isometry). Then $\forall \varphi \in \mathcal{S}(\mathcal{A})$ $(E_I^0)$–compatible (i.e. $\varphi \circ E_I^0 = \varphi$ for any $I_0$)

$$\lim_{I \uparrow S} \varphi_{I_0} \left( M_I^* a_{I_0} M_I ) \right) =: \psi_{I_0} \left( a_{I_0} \right)$$

exists and defines a projective family of states.

**Problems with expected Markovianity**

– commutative case: none

– extreme non–commutative case (all local algebras are factors): triviality

**Definition 1** *A* **quasi–conditional expectation** *with respect to the triple*

$$\mathcal{A}_{I^0} \subseteq \mathcal{A}_I \subseteq \mathcal{A}_J$$

*is a completely positive identity preserving map*

$$E_{J,I} : \mathcal{A}_J \to \mathcal{A}_I$$

*such that*

$$E_{J,I}(ba) = bE_{J,I}(a) \ ; \ \forall b \in \mathcal{A}_{I^0} \ , \ \forall a \in \mathcal{A}_J \tag{13}$$

**Lemma 1** *Let* $\mathcal{B}_0 \subseteq \mathcal{B} \subseteq \mathcal{A}$ *be* $C^*$*–algebras and let* $E : \mathcal{A} \to \mathcal{B}$ *a quasi–conditional expectation with respect to the triple*

$$\mathcal{B}_0 \subseteq \mathcal{B} \subseteq \mathcal{A}$$

*then*

$$E(\mathcal{B}_0' \cap \mathcal{A}) \subseteq \mathcal{B}_0' \cap \mathcal{B}$$

1**–dimensional case: notations**

$H_n$ separable Hilbert space $n \in \mathbf{N}$
$\mathcal{B}_n := \mathcal{B}(H_n)$
For $n \in \mathbf{N}$ let
$j_n : \mathcal{B}_n \to j_n(\mathcal{B}_n) =: \mathcal{A}_n \subset \mathcal{A} = \otimes_{\mathbf{N}}\mathcal{B}$
be the natural embedding onto the $n$-th factor:

$$j_n : b \in \mathcal{B}_n \to \underbrace{1 \otimes 1 \otimes \ldots \otimes 1}_{m \neq n} \otimes \underbrace{b}_{n} \in \mathcal{A} \tag{14}$$

$$j_o(a_o)j_1(a_1)\cdot\ldots\cdot j_n(a_n) = a_o \otimes a_1 \otimes \ldots \otimes a_n \otimes 1 \otimes \ldots \otimes a_n \otimes 1 \otimes \ldots \tag{15}$$

For $I \subseteq \mathbf{N}$ denote:

$\mathcal{A}_I$ the $C^*$-subalgebra of $\mathcal{A}$ generated by $j_n(M) : n \in I$

we shall write simply

$$\mathcal{A}_n := \mathcal{A}_{\{n\}} \ ; \ \mathcal{A}_{n]} = \mathcal{A}_{(-\infty,n]} \ ; \quad \mathcal{A}_{[n} = \mathcal{A}_{[n,+\infty)}$$

The **shift** on $\mathcal{A}$ is defined, if $\mathcal{A}_{\{n\}} \equiv \mathcal{B}\ \forall n$, to be the unique endomorphism $u^o$ of $\mathcal{A}$ into itself satisfying

$$u^o \circ j_n = j_{n+1} \qquad \forall n \in \mathbf{N} \tag{16}$$

or equivalently

$$u^o(j_o(a_o)j_1(a_1)\cdot\ldots\cdot j_n(a_n)) = j_1(a_o)j_2(a_1)\cdot\ldots\cdot j_{n+1}(a_n)$$

It follows that, for each natural integer $n$ and each subset I of $\mathbf{N}$ :

$$u^o(\mathcal{A}_I) = \mathcal{A}_{I+n} \tag{17}$$

i.e., the family of local algebras $(\mathcal{A}_I)$ is covariant with respect to the shift. A state $\varphi$ which is invariant for the shift i.e.

$$\varphi \circ u^o = \varphi \tag{18}$$

will be called **translation invariant** or **stationary**
.

**Definition 2** *A* **quasi–conditional expectation** *with respect to the triple*

$$\mathcal{A}_{[0,n-1]} \subseteq \mathcal{A}_{[0,n]} \subseteq \mathcal{A}_{[0,n+1]}$$

*is called Markovian if*

$$E_{n+1,n}(\mathcal{A}_{[n}) \subseteq \mathcal{A}_n \tag{19}$$

**Definition 3** *A state $\varphi$ on $\mathcal{A}$ is called a* **Markov state** *with respect to the localization $\mathcal{A}_{[0,n]}$ if for each $n \in \mathbf{N}$ there exists a quasi-conditional expectation $E_{n+1,n}$ with respect to the triple*

$\mathcal{A}_{[0,n-1]} \subseteq \mathcal{A}_{[0,n]} \subseteq \mathcal{A}_{[0,n+1]}$ *such that*

$$\varphi \circ E_{n+1,n} = \varphi \tag{20}$$

*In this case we shall say that the quasi-conditional expectation $E_{n+1,n}$ is compatible with the state $\varphi$.*

In the following we shall simply say that $\varphi$ is a Markov state on $\mathcal{A}$ without explicitly mentioning the localization $\{\mathcal{A}_{[0,n]}\}$.

**The structure of QMS**

Let $\varphi$ be a Markov state on $\mathcal{A}$ and let $(E_{n+1,n})$ be an associated family of quasi-conditional expectations. Define, for each $n \in \mathbf{N}$,

$$\mathcal{E}_n = j_n^* \circ E_{n+1,n} \circ (j_n \otimes j_{n+1}) : \mathcal{B}_n \otimes \mathcal{B}_{n+1} \to \mathcal{B}_n$$

Because of the Markov property, $\mathcal{E}_n$ is a transition expectation on $M$.

The invariance condition

$$\varphi \circ E_{n+1,n} = \varphi$$

implies that

$$\varphi(a_{[o,n-1]} j_n(a_n) j_{n+1}(a_{n+1})) = \tag{21}$$

$$= \varphi(a_{[o,n-1]} E_{n+1,n}(j_n(a_n) j_{n+1}(a_{n+1}))) =$$

$$= \varphi(a_{[o,n-1]} j_n \circ \mathcal{E}_n(a_n \otimes a_{n+1}))) =$$

$$= \varphi(a_{[o,n-1]} j_n \circ \mathcal{E}_n((a_n \otimes \mathcal{E}_{n+1}(a_{n+1} \otimes 1))))$$

Iterating one finds

$$\varphi(j_o(a_o) \ldots j_{n-1}(a_{n-1}) j_n(a_n)) = \tag{22}$$

$$= \varphi_o\Big(\mathcal{E}_o(a_o \otimes \ldots \otimes \mathcal{E}_{n-1}(a_{n-1} \otimes \mathcal{E}_n(a_n \otimes 1)))))$$

Thus $\varphi$ is completely determined by the pair

$\{\varphi_o; (\mathcal{E}_n)\}$

Conversely:

**Proposition 1** *For any pair $\{\varphi_o(\mathcal{E}_n)\}$ such that $\varphi_o$ is a state on $M$ and each $\mathcal{E}_n$ is a transition expectation $\mathcal{B}_n \otimes \mathcal{B}_{n+1} \to \mathcal{B}_n$, the right hand side of (21) determines a unique state $\varphi$ on $\mathcal{A}$. Such state is called a backward, one sided Markov chain.*

## Obstructions for a QMC to be a QMS

They come from the invariance condition

$$\varphi \circ E_{n+1,n} = \varphi$$

Recall that

$$\varphi(a_{[o,n-1]} j_n(a_n) j_{n+1}(a_{n+1})) =$$

$$= \varphi(a_{[o,n-1]} j_n \circ \mathcal{E}_n((a_n \otimes \mathcal{E}_{n+1}(a_{n+1} \otimes 1))))$$

But the left hand side of (2) is also equal to

$$\varphi(a_{[o,n-1]} E_{n+1,n}((j_n(a_n) E_{n+2,n+1}(j_{n+1}(a_{n+1}))))) \tag{23}$$

We write simply that for each $a, b \in \mathcal{B}$

$$\mathcal{E}_n(a \otimes b) = \mathcal{E}_n((a \otimes \mathcal{E}_{n+1}(b \otimes 1))) \qquad , \quad mod\{\varphi_o, (\mathcal{E}_k)\} \tag{24}$$

## Structure of locally faithful QMS

**Theorem 2** *Suppose that there exists a Markovian CP1 map*

$$E_{n]} : \mathcal{A} \to \mathcal{A}_{n]} \tag{25}$$

$$E_{n]}(\mathcal{A}_{[n}) \subseteq \mathcal{A}_n$$

*such that*

$$\varphi(a_{n-1]}a_{[n}) = \varphi(a_{n-1]}E_{n]}(a_{[n})) \tag{26}$$

*Then there exist:*

**(i)** *an algebra*

$$\mathcal{A}_n^p \subseteq \mathcal{A}_n$$

**(ii)** *a Umegaki conditional expectation*

$$E_{n]}^p : \mathcal{A} \to \mathcal{A}_{n]}^p := \mathcal{A}_{n-1]} \vee \mathcal{A}_n^p \subseteq \mathcal{A}_{n]} \tag{27}$$

*with the following properties:*

$$\varphi(a_{n-1]}a_{[n}) = \varphi(a_{n-1]}E_{n]}^p(a_{[n})) \tag{28}$$

$$E^p_{n]}(\mathcal{A}_{[n}) \subseteq \mathcal{A}^p_n \subseteq \mathcal{A}_n \tag{29}$$

**Proof**. Then because of (25), for each $k \in \mathbf{N}$

$$\varphi(a_{n-1]}a_{[n}) = \varphi(a_{n-1]}E^k_{n]}(a_{[n}))$$

and therefore also:

$$\varphi(a_{n-1]}a_{[n}) = \varphi\left(a_{n-1]}\frac{1}{k}\sum_{h=1}^{k} E^h_{n]}(a_{[n})\right) \tag{30}$$

Going to the GNS representation, where the ergodic theorem holds we deduce that the weak limit

$$\lim_{k\to\infty} \frac{1}{k}\sum_{h=1}^{k} E^h_{n]} =: E^p_{n]}$$

exists and is a Umegaki conditional expectation onto the fixed points of $E_{n]}$:

$$E^p_{n]} : \mathcal{A} \to \mathcal{A}^p_{n]} \subseteq \mathrm{Fix}\ E_{n]}$$

and, from (30) we deduce that (28) holds.

(28) also implies that

$$\mathcal{A}_{n-1]} \subseteq \mathcal{A}^p_{n]} \subseteq \mathcal{A}_{n]}$$

therefore, defining the algebra $\mathcal{A}^p_n$ by:

$$\mathcal{A}^p_{n]} = \mathcal{A}_{n-1]} \vee \mathcal{A}^p_n$$

and from this (27) follows.

From our assumptions it follows that

$$\mathcal{A}_{[n} \subseteq (\mathcal{A}_{n]})$$

**Remark**. In tensor product algebras one has

$$\mathcal{A}^p_n := \mathcal{A}'_{n-1]} \cap \mathcal{A}^p_{n]} \subseteq \mathcal{A}'_{n-1]} \cap \mathcal{A}_{n]} = \mathcal{A}_n$$

**Cecchini's theorem (for matrix algebras)**

**Theorem 3** *In the notations of Theorem (2), suppose that for each $n$, $\mathcal{A}_n^p$ is a factor and denote*

$$\mathcal{A}_n^f := (\mathcal{A}_n^p)' \cap \mathcal{A}_n \tag{31}$$

$$\varphi_n := \varphi | \mathcal{A}_n^p \vee \mathcal{A}_{n+1}^f$$

*Then if the $\mathcal{A}_n$ are matrix algebras, $\varphi$ is a 2–block factor for the localization*

$$\bigvee_n (\mathcal{A}_k^f \vee \mathcal{A}_{k+1}^p) \equiv \bigvee_n (\mathcal{A}_k^p \vee \mathcal{A}_k^f) \equiv \bigvee_n \mathcal{A}_n$$

*Equivalently: for any*

$$\varphi(a_k^f a_{k+1}^p \cdot a_{k+1}^f a_{k+2}^p \cdot \ldots \cdot a_n^f a_{n+1}^p) = \tag{32}$$

$$= \varphi_k(a_k^f a_{k+1}^p) \varphi_{k+1}(a_{k+1}^f a_{k+2}^p) \ldots \varphi_n(a_n^f a_{n+1}^p)$$

**Remark**. If the $\mathcal{A}_n$ are matrix algebras then

$$\mathcal{A}_n^p \vee \mathcal{A}_n^f = \mathcal{A}_n$$

In the general case this property is satisfied under additional conditions (of split type). There exist examples of subalgebras with trivial relative commutant (singular subalgebras).

**Proof**. From the Definition (31) it follows that $\mathcal{A}_n^f \vee \mathcal{A}_{[n+1} \subseteq \mathcal{A}_{[n}$ is in the commutant of $\mathcal{A}_n^p$ and therefore, if $\mathcal{A}_n^p$ is a factor, then by Lemma (1) it follows that

$$E_{n]}^p(\mathcal{A}_n^f \vee \mathcal{A}_{[n+1}) \subseteq (\mathcal{A}_n^p)' \cap \mathcal{A}_n^p = \mathbf{C}1$$

Hence there exists a state

$$\varphi_{n]}^p \in \mathcal{S}(\mathcal{A}_n^f \vee \mathcal{A}_{[n+1})$$

such that for any, one has

$$E_{n]}^p(a_n^f a_{[n+1}) = \varphi_{n]}^p(a_n^f a_{[n+1})$$

But from (28), choosing $a_{n-1]} = 1$, $a_{[n} = a_n^p a_n^f a_{[n+1}$, it follows that

$$\varphi(a_n^p a_n^f a_{[n+1}) = \varphi(a_n^p)\varphi_{n]}^p(a_n^f a_{[n+1})$$

Therefore

$$\varphi_{n]}^p = \varphi|\mathcal{A}_n^f \vee \mathcal{A}_{[n+1}$$

Iterating this formula we obtain (32).

**Markov fields on graphs**

$$\mathcal{B}_x = \mathcal{H}(x) \ (\equiv \mathcal{B})$$

$$\mathcal{B}_\Lambda := \mathcal{B}(\mathcal{H}_\Lambda) \quad ; \qquad \forall \, \Lambda \subseteq_{\text{fin}} L$$

$\mathcal{A} := C^*\text{−ind−lim } \mathcal{B}_\Lambda \ (C^*\text{−inductive limit }) \equiv \otimes_{x \in L} \mathcal{B}_x$

$$j_x : b \in \mathcal{B}(x) \to j_x(b) = b \otimes 1_{\{x\}^c} \in \mathcal{A} \qquad (33)$$

natural embedding of $\mathcal{B}_x$ into $\mathcal{A}$

$$j_\Lambda := \otimes_{x \in \Lambda} j_x \qquad ; \qquad \Lambda \subseteq L$$

identification

$$\mathcal{B}_\Lambda \equiv \otimes_{x \in \Lambda} \mathcal{B}_x \equiv \mathcal{B}_\Lambda \otimes 1_{\Lambda^c} = j_\Lambda(\mathcal{B}_\Lambda)$$

the elements of the $*$–sub–algebra of $\mathcal{A} := \mathcal{B}_L$ defined by

$\mathcal{B}_{\text{loc}} = \mathcal{B}_{L,\text{loc}} := \bigcup_{\Lambda \subset_{\text{fin}} L} \mathcal{B}_\Lambda$ (set theoretical union)

will be called local operators (observables if self–adjoint).

## (Forward) Markov fields

A local family of $C^*$–algebras is a quadruple

$$\{\mathcal{A}\ ,\ L\ ,\ \mathcal{I}\ ,\ \{\mathcal{A}_F\}_{F\in\mathcal{I}}\}$$

such that
– $\mathcal{A}$ and $\mathcal{A}_F$ are $C^*$–algebras
– $L$ is a set
– $\mathcal{I}$ is a directed family of subsets of $L$ closed under difference,
– the map $F \in \mathcal{I} \mapsto \mathcal{A}_F \subseteq \mathcal{A}$ is order preserving ($\subseteq$)
– if $\{F_a\}$ is any family in $\mathcal{I}$ such that $\bigcup_a F_a = F \in \mathcal{I}$, then $\bigcup_a \mathcal{A}_{F_a}$ is dense in $\mathcal{A}_F$.
– $\bigcup\{\mathcal{A}_F \,|\, F \in \mathcal{I}\}$ is dense in $\mathcal{A}$ (usually $L \notin \mathcal{I}$)

**Definition**
$d : \mathcal{I} \to \mathcal{I}$ a map such that

(i) $\bigcup\{dF \,|\, F \in \mathcal{I}\} = L$,

(ii) $dF \subseteq F\,, \qquad F \in \mathcal{A}$,

(iii) $F \subseteq G \Longrightarrow dF \subseteq dG$,

(iv) if $\{F_a\}$ is any family in $\mathcal{I}$ such that $F_a \uparrow L$ (i.e. $\bigcup_a F_a = L$) then $dF_a \uparrow L$.

For $F, G \in \mathcal{I}$ with $F \subseteq G$, a linear map $E_{G,F} : \mathcal{A}_G \to \mathcal{A}_F$ is said to have the $d$–Markov property if

$$E_{G,F}(\mathcal{A}_{G \setminus F}) \subseteq \mathcal{A}_{F \setminus dF} \,.$$

(unification of Nelson's topological Markov property with Dobrushin's discrete $d$–Markov property)

**Definition**

A state $\varphi \in \mathcal{S}(\mathcal{A})$ is called a $d$–Markov chain if there exist

(i) an increasing sequence $\{F_n\}_{n \in \mathbf{N}}$ in $\mathcal{I}$ such that

$$F_{n-1} \subseteq dF_n \subseteq F_n \uparrow L \,,$$

(ii) a sequence of $d$–Markovian quasi–conditional expectations

$$E_{F_{n+1},F_n} : \mathcal{A}_{F_{n+1}} \mapsto \mathcal{A}_{F_n}$$

w.r.t. the triplet $\mathcal{A}_{dF_n} \subseteq \mathcal{A}_{F_n} \subseteq \mathcal{A}_{F_{n+1}}$, i.e. CP1 maps satisfying

$$E_{F_{n+1},F_n}\left(a_{dF_n}a_{F_{n+1}}\right) = a_{dF_n}E_{F_{n+1},F_n}\left(a_{F_{n+1}}\right)$$

$$a_{dF_n} \in \mathcal{A}_{dF_n} \quad ; \quad a_{F_{n+1}} \in \mathcal{A}_{F_{n+1}}$$

(iii) a state $\varphi_{F_0} \in \mathcal{S}(\mathcal{A}_{F_0})$.
such that

$$\varphi = \lim_n \varphi_{F_0} \circ E_{F_1,F_0} \circ \ldots \circ E_{F_n,F_{n-1}} \tag{34}$$

in the $*$–weak topology, for some sequence $\{E_{F_{n+1},F_n}\}_{n\in\mathbf{N}}$ of quasi–conditional expectation as above.

The state $\varphi$ is called a $d$–Markov state if

$$\varphi \circ E_{F_{n+1},F_n} = \varphi\,, \qquad n \in \mathbf{N}$$

for some sequence sequence $\{E_{F_{n+1},F_n}\}_{n\in\mathbf{N}}$ of quasi–conditional expectation as above.

A $d$–Markov state is also a $d$–Markov chain.

The converse is not true in general.

For $d$–Markov states each $E_{G,F}$ can be chosen to be the $\varphi$–conditional expectation from $\mathcal{A}_G$ into $\mathcal{A}_F$

$L = \mathbf{N}$: two interpretations of the points of $L$

In one (time) he order structure is important.
In the other one it is not.

**I: Time** In this case there may be a privileged point 0 and an orientation which allows to distinguish between past and fugure (Fugure 1). This interpretation is frequent in probability

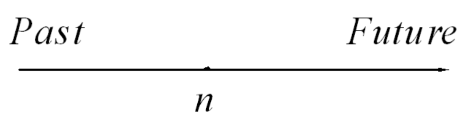


*Figure 1*




*Figure 4a*

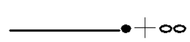


*Figure 4b*

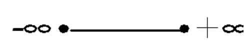


*Figure 4c*

**II: Position** In this case there is no privileged instant inside $L$ but, if one considers the localization given by the finite subsets of $L$ (inside) and their complements (outside, Figure 2), one can consider the 1–point compactification of $L$ and $\infty$ is the privileged point (Figure 3a).
This interpretation has the advantage of being possible also in multidimensional lattices (Dobrushin's theory).


*Figure 2*

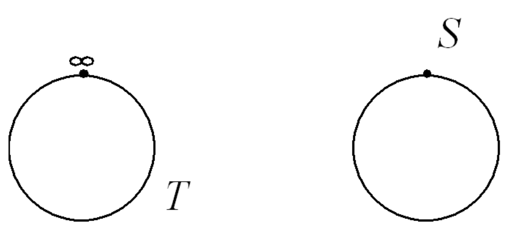


*Figure 3a* *Figure 3b*

In all figures the curved arrows indicate the direction of conditioning.
Alternative interpretations are: a gas with one– or two–sided boundary conditions at $\infty$ (Figure 2).
This interpretation is frequent in statistical mechanics.

**(Backward) QMF**

Luigi Accardi, Hiromichi Ohno:
Quantum bio information and Markov fields on graphs,

to appear in: QP – PQ Series,
Proceedings of QBIC-I, Tokyo university of Science, 14–19 March, 2007

notations as above, but:

For $\Lambda \subseteq_{fin} L$ we denote

$$d\Lambda =: \overline{\Lambda}$$

$\Lambda \subset\subset \Lambda_1$ means $\overline{\Lambda} \subseteq \Lambda_1$

**Definition**
A state $\varphi$ on $\mathcal{A} = \mathcal{B}_L$ is called
a generalized quantum Markov state on $\mathcal{B}_L$
if there exist:
– an increasing sequence $\Lambda_n \uparrow L$
(i.e.: eventually absorbing any finite subset)

– for each $\Lambda_n$, a quasi-conditional expectation $E_{\Lambda_n^c}$ with respect to the triplet

$$\mathcal{B}_{\overline{\Lambda_n}^c} \subseteq \mathcal{B}_{\Lambda_n^c} \subseteq \mathcal{B}_L \tag{35}$$

– for each $\Lambda_n$, a state

$$\widehat{\varphi}_{\Lambda_n^c} \in \mathcal{S}(\mathcal{B}_{\Lambda_n^c})$$

such that for any $\Lambda_0 \subset\subset \Lambda_n$ one has

$$\varphi|\mathcal{B}_{\Lambda_0} = \widehat{\varphi}_{\Lambda_n^c} \circ E_{\Lambda_n^c}|\mathcal{B}_{\Lambda_0} \tag{36}$$

If, in condition (36), one can choose

$$\widehat{\varphi}_{\Lambda_n^c} = \varphi|\mathcal{B}_{\Lambda_n^c} \tag{37}$$

then $\varphi$ is called a quantum Markov state.

$\varphi$ is called a weak Markov state if

for all $a \in \mathcal{B}_{L,loc}$ there exists $\Lambda(a) \subseteq_{\text{fin}} L$

such that $\forall\, \Lambda(a) \subseteq \Lambda \subseteq_{\text{fin}} L$ one has:

$$\varphi(a) = \varphi(E_{\Lambda^c}(a))$$

**Remark**. for quantum Markov states $E_{\Lambda_n^c}$ can be replaced by an Umegaki conditional expectation from $\mathcal{B}_L$ onto a sub–algebra of $\mathcal{B}_{\Lambda_n^c}$

( ergodic argument used in [AcFri80] )

**Remark**. In the case of infinite tensor products (the only one considered here) one has, for any subset, $I \subseteq L$:

$$\mathcal{B}_{I^c} = \mathcal{B}'_I \quad \text{the commutant of } \ \mathcal{B}_I \tag{38}$$

Recall that, by definition [x.], a quasi–conditional expectation with respect to the triplet (35) is a CP1 map $E_{\Lambda_n^c} : \mathcal{B}_L \to \mathcal{B}_{\Lambda_n^c}$ satisfying

$$E_{\Lambda_n^c}(a_{\overline{\Lambda_n}^c} a_{\Lambda_n}) = a_{\overline{\Lambda_n}^c} E_{\Lambda_n^c}(a_{\Lambda_n}) \qquad ; \ \overline{\Lambda}_n^c := (\overline{\Lambda}_n)^c \tag{39}$$

Because of (38) this implies that

$$E_{\Lambda_n^c}(\mathcal{B}_{\Lambda_n}) \subseteq (\mathcal{B}_{\overline{\Lambda}_n^c})' = \mathcal{B}_{(\overline{\Lambda}_n^c)^c} = \mathcal{B}_{\overline{\Lambda_n}}$$

Consequently

$$E_{\Lambda_n^c}(\mathcal{B}_{\Lambda_n}) \subseteq \mathcal{B}_{\Lambda_n^c} \cap \mathcal{B}_{\overline{\Lambda_n}} = \mathcal{B}_{\Lambda_n^c \cap \overline{\Lambda_n}} = \mathcal{B}_{\vec{\partial}\Lambda_n}$$

which is the natural quantum generalization of the multidimensional (discrete) Markov property as originally formulated by Dobrushin .

The above argument shows that, whenever (38) holds (e.g. in the case of infinite tensor products) the Markov property

$$E_{\Lambda_n^c}(\mathcal{B}_{\Lambda_n}) \subseteq \mathcal{B}_{\vec{\partial}\Lambda}$$

follows from the basic property (39) of the quasi–conditional expectations.

This is not true in general when (e.g. in the abelian case or in the case of CAR algebras).
Luigi Accardi, Francesco Fidaleo, Farruh Mukhamedov:
Markov states and chains on the CAR algebra,
IDA–QP (2007), to appear

In all these cases the Markov property should be included in the definition of the various notions of Markov states as an additional requirement.

**Fact**

there are natural classes of states which are weak Markov states but not Markov states.

**Graphs**

$\mathcal{G} = (\mathcal{L}, \mathcal{E})$ (non-oriented simple) graph, that is,

$L$ is a non-empty at most countable set and

$$E \subset \{\{x, y\}; x, y \in L, x \neq y\}$$

$L$ *vertices*

$E$ *edges*

Two vertices $x, y \in L$ are called *adjacent*,

$x \sim y$

or *nearest neighbors*, if $\{x, y\} \in E$,

For each $x \in L$ the set of nearest neighbors of $x$ will be denoted

$$N(x) := \{y \in L : y \sim x\}$$

*degree* of $x \in L$

$$\kappa(x) := |N(x)| = |\{y \in L\,;\, y \sim x\}|,$$

where $|\cdot|$ denotes the cardinality.

we always deal with locally finite graphs i.e., $\kappa(x) < \infty$ for all $x \in L$

A graph can be equivalently assigned by giving the pair

$$\{L, \sim\}$$

of its vertices and the binary symmetric relation $\sim$.

*path* or a *trajectory* or a *walk* from $x \in L$ to $y \in L$ is a finite sequence of vertices such that

$x = x_1 \sim x_2 \sim \ldots \sim x_n = y.$

$n :=$ *length* of the path

$d(x, y) :=$ the shortest length of a walk connecting $x$ and $y$.

$d(x,x) = 0$

we always deal with connected graphs, i.e., for any pair of vertices there exists a walk connecting them.

$$\Lambda \subseteq_{\text{fin}} L$$

means that $\Lambda$ is a finite subset of $L$.

the external boundary of $\Lambda$

$$\vec{\partial}\Lambda := \{x \notin \Lambda : \exists\, y \sim x \ , \qquad y \in \Lambda\}$$

the closure of $\Lambda$

$$\overline{\Lambda} := \Lambda \cup \vec{\partial}\Lambda$$

$$\Lambda \subset\subset \Lambda_1$$

means that

$$\overline{\Lambda} \subset \Lambda_1$$

Notice that, by definition

$$\Lambda \cap \vec{\partial}\Lambda = \emptyset$$

$$\vec{\partial}\{x\} =: \partial x = N(x) \setminus \{x\}$$

**Bundles on graphs**

$x \in L \to \mathcal{H}_x$ Hilbert space of dimension $d_{\mathcal{H}}(x) \in \mathbf{N}$

$d := d_{\mathcal{H}}(x) = d_{\mathcal{H}} < +\infty$ (independent of $x$)

Given

$$\Lambda \subseteq_{\mathsf{fin}} L$$

define

$$\mathcal{H}_\Lambda := \otimes_{x \in \Lambda} \mathcal{H}_x$$

fix, $\forall\, x \in L$, an o.n. basis of $\mathcal{H}_x$:

$$(e_j(x)) \equiv e(x) \quad ; \qquad j \in S(x) := \{1, \ldots, d_{\mathcal{H}}\}$$

$\pi_S : S \to L$ bundle whose fibers are the finite sets

$\pi_S^{-1}(x) := S(x)$

the sections of this bundle are the maps:

$$\{\omega_\Lambda : x \in \Lambda \to \omega_\Lambda(x) \in S(x)\} =: \Omega_\Lambda$$

A section $\omega_\Lambda$ is called a *configuration* in the volume $\Lambda$.

For each configuration $\omega_\Lambda$ define

$$e_{\omega_\Lambda} := \otimes_{x\in\Lambda} e_{\omega_\Lambda(x)}(x) \in \mathcal{H}_\Lambda \tag{1}$$

$E_{\omega_\Lambda}$ is the corresponding rank one projection:

$$E_{\omega_\Lambda} := |e_{\omega_\Lambda}\rangle\langle e_{\omega_\Lambda}| = e_{\omega_\Lambda} e^*_{\omega_\Lambda} \tag{2}$$

Then the set

$$\{e_{\omega_\Lambda} \ : \ \omega_\Lambda \in \mathcal{F}(\Lambda, S)\} \tag{3}$$

is an o.n. basis of $\mathcal{H}_\Lambda$.

$$\mathcal{B}_\Lambda := \mathcal{B}(\mathcal{H}_\Lambda) \quad ; \qquad \forall\,\Lambda \subseteq_{\text{fin}} L$$

**Entangled Markov fields on trees**

for trees, the construction of entangled Markov chains proposed in
L. Accardi, F. Fidaleo:
Entangled Markov chains.
Annali di Matematica Pura e Applicata, (2004)
Preprint Volterra N.556 (2003)
can be generalized.

a tree is a connected graph without loops.

This definition implies that any finite subset $\Lambda \subseteq_{fin} L$ enjoys the following fundamental property:

**Property (T)**

*For any* $\Lambda \subseteq_{fin} L$ *and for arbitrary* $x \in \vec{\partial}\Lambda$, *there exists a unique point* $y \in \Lambda$ *such that* $x \sim y$.

$(L, E)$ graph

for each $\{x, y\} \in E$, let be given a transition amplitude $d \times d$ matrix $(\psi_{xy}(i, j))$

i.e. a complex matrix such that the matrix $(|\psi_{xy}(i, j)|^2)$ is bi–stochastic, i.e.

$$\sum_{i=1}^{d} |\psi_{xy}(i, j)|^2 = \sum_{j=1}^{d} |\psi_{xy}(i, j)|^2 = 1$$

$(\psi_{xy}(i, j))$ will be called an *amplitude matrix*
notice that unitarity of the matrices $(\psi_{xy}(x, j))_{i,j}$ is

not required.

Define the vector

$$\psi_{xy} = \sum_{i,j=1}^{d} \psi_{xy}(i,j) \cdot e_i(x) \otimes e_j(y) \in \mathcal{H}_x \otimes \mathcal{H}_y \quad (4)$$

**Algebraic formulation**

**Lemma**

In the notation

$$E_\Lambda := \{\{x,y\} \mid x,y \in \Lambda, x \sim y\}$$

for any $\Lambda \subseteq_{\text{fin}} L$, define the vector $\psi_\Lambda \in \mathcal{H}_\Lambda$ by

$$\psi_\Lambda := \sum_{\omega_\Lambda} \psi_\Lambda(\omega_\Lambda) e_{\omega_\Lambda} \quad (5)$$

$$\psi_\Lambda(\omega_\Lambda) := \prod_{\{x,y\} \in E_\Lambda} \psi_{xy}(\omega_\Lambda(x), \omega_\Lambda(y)) \quad (6)$$

If $\Lambda$ enjoys Property T then $\forall x \in \vec{\partial}\Lambda$,

$$\|\psi_{\Lambda \cup \{x\}}\|^2 = \|\psi_\Lambda\|^2$$

*Proof.* Property T $\Rightarrow \forall x \in \vec{\partial}\Lambda$, $\exists! y \in \Lambda$ such that $x \sim y$. Then

$$\|\psi_{\Lambda\cup\{x\}}\|^2 = \sum_{\omega_\Lambda,\omega_x} |\psi_{\Lambda\cup\{x\}}((\omega_\Lambda,\omega_x))|^2$$

$$= \sum_{\omega_{\Lambda\setminus\{y\}},\omega_y,\omega_x} |\psi_{\Lambda\cup\{x\}}((\omega_{\Lambda\setminus\{y\}},\omega_y,\omega_x))|^2$$

$$= \sum_{\omega_{\Lambda\setminus\{y\}},\omega_y} \sum_{\omega_x=1}^{d} |\psi_\Lambda((\omega_{\Lambda\setminus\{y\}},\omega_y))|^2 \cdot |\psi_{xy}(\omega_y,\omega_x)|^2$$

$$= \sum_{\omega_\Lambda} |\psi_\Lambda(\omega_\Lambda)|^2 = \|\psi_\Lambda\|^2$$

which is the thesis.

**Proposition** Suppose that $\Lambda$ enjoys Property T and let

$$\Lambda' \subset\subset \Lambda \subseteq_{\text{fin}} L$$

Then for any $a \in \mathcal{B}_{\Lambda'}$ and $x \in \vec{\partial}\Lambda$ one has:

$$\langle \psi_\Lambda, a\psi_\Lambda \rangle = \langle \psi_{\Lambda\cup\{x\}}, a\psi_{\Lambda\cup\{x\}} \rangle$$

*Proof.* Because of Property T, given $x \in \vec{\partial}\Lambda$, there exists a unique point $y \in \Lambda$ such that $x \sim y$. Then we have

$$
\begin{aligned}
& \langle \psi_{\Lambda\cup\{x\}}, a\psi_{\Lambda\cup\{x\}}\rangle = \\
= & \sum_{\omega_{\Lambda'},\omega'_{\Lambda'}} \sum_{\omega_{\Lambda\setminus\{\Lambda'\cup\{y\}\}}} \sum_{\omega_x,\omega_y} \psi_{\Lambda\cup\{x\}}((\omega_{\Lambda'},\omega_{\Lambda\setminus\{\Lambda'\cup\{y\}\}},\omega_x,\omega_y))^* \\
& \cdot a_{\omega_{\Lambda'}\omega'_{\Lambda'}} \psi_{\Lambda\cup\{x\}}((\omega'_{\Lambda'},\omega_{\Lambda\setminus\{\Lambda'\cup\{y\}\}},\omega_x,\omega_y)) \\
= & \sum_{\omega_{\Lambda'},\omega'_{\Lambda'}} \sum_{\omega_{\Lambda\setminus\{\Lambda'\cup\{y\}\}}} \sum_{\omega_x,\omega_y} \psi_{\Lambda}((\omega_{\Lambda'},\omega_{\Lambda\setminus\{\Lambda'\cup\{y\}\}},\omega_y))^* \\
& a_{\omega_{\Lambda'}\omega'_{\Lambda'}} \psi_{\Lambda}((\omega'_{\Lambda'},\omega_{\Lambda\setminus\{\Lambda'\cup\{y\}\}},\omega_y)) |\psi_{xy}(\omega_x,\omega_y)|^2 \\
= & \sum_{\omega_{\Lambda'},\omega'_{\Lambda'}} \sum_{\omega_{\Lambda\setminus\Lambda'}} \psi_{\Lambda}((\omega_{\Lambda'},\omega_{\Lambda\setminus\Lambda'}))^* a_{\omega_{\Lambda'}\omega'_{\Lambda'}} \psi_{\Lambda}((\omega'_{\Lambda'},\omega_{\Lambda\setminus\Lambda'})) \\
= & \langle \psi_{\Lambda}, a\psi_{\Lambda}\rangle
\end{aligned}
$$

The trouble with Property T is that, if Λ has Property T and $x \in \vec{\partial}\Lambda$, unfortunately it is not true that also $\Lambda\cup x$ has Property T. However trees have a very special property given by the following Lemma.

**Lemma** In a tree every finite subset $\Lambda \subseteq L$ enjoys Property T.

*Proof.* Let $\Lambda \subseteq L$ be a finite subset and let $x \in \vec{\partial}\Lambda$. If there exist $y, z \in \Lambda$ such that $y \sim x$ , $z \sim x$, then since a tree is connected, there is a path between $y$ and $z$ and this would give a loop. Against the definition of tree.

**Corollary** If $(L, E)$ is a tree, and the vector $\psi_\Lambda$ is defined by (5), (6), then, for any $\Lambda \subseteq_{\text{fin}} L$ of cardinality $\geq 2$, one has:

$$\|\psi_\Lambda\|^2 = d \tag{7}$$

and the limit

$$\varphi(a) = \frac{1}{d} \lim_{\Lambda \uparrow L} \langle \psi_\Lambda, a\psi_\Lambda \rangle$$

exists for any $a$ in the local algebra $\mathcal{B}$ and defines a state $\varphi$ on $\mathcal{B}$.

*Proof.* The first statement follows by induction because, if $\Lambda = \{x, y\}$, then we get

$$\|\psi_{xy}\|^2 = \sum_{i,j} |\psi_{xy}(i,j)|^2 = d$$

The second statement follows from the first one

## Algebraic formulation

The simplification coming from considering trees rather than general graphs manifests itself in the fact that the analogue of the basic isometries, used in the construction of [AcFi03], in this case commute.

**Proposition** For $\Lambda \subseteq_{\mathsf{fin}} L$, $x \in \vec{\partial}\Lambda$ and $z \in \Lambda$, define $V_{(z|x)} : \mathcal{H}_z \to \mathcal{H}_z \otimes \mathcal{H}_x$ by

$$V_{(z|x)} e_{i_z} = \sum_{i_x} \psi_{xz}(i_x, i_z) e_{i_x} \otimes e_{i_z} \tag{8}$$

and extend it naturally to a map $\mathcal{H}_{\Lambda_0} \to \mathcal{H}_{\Lambda_0}$ for any $\Lambda_0$ containing $\overline{\Lambda}$. Then $\forall x, y \in \vec{\partial}\Lambda$, $z \in \Lambda$ with $x \sim z$, $y \sim z$, $V_{(z|x)}$ and $V_{(z|y)}$ are isometries satisfying:

$$V_{(z|x)} \psi_\Lambda = \psi_{\Lambda \cup \{x\}}$$

$$V_{(z|x)} V_{(z|y)} = V_{(z|y)} V_{(z|x)}$$

*Proof.* We have

$$\begin{aligned} \langle V_{(z|x)} e_{i_z}, V_{(z|x)} e_{j_z} \rangle &= \delta_{i_z, j_z} \sum_{i_x, j_x} \langle \psi_{xz}(i_x, i_z) e_{i_x}, \psi_{xz}(j_x, i_z) e_{j} \\ &= \delta_{i_z, j_z} \sum_{i_x} |\psi_{xz}(i_z, i_x)|^2 = \delta_{i_z, j_z} = \langle e_{i_z} \end{aligned}$$

Therefore any $V_{(z|x)}$ is an isometry. Next, we get

$$\begin{aligned}
V_{(z|x)}\psi_\Lambda &= V_{(z|x)}(\sum_{\omega_{\Lambda\setminus\{z\}},i_z}\psi_\Lambda((\omega_{\Lambda\setminus\{z\}},i_z))e_{\omega_{\Lambda\setminus\{z\}}}\otimes e_{i_z}) \\
&= \sum_{\omega_{\Lambda\setminus\{z\}},i_z}\psi_\Lambda((\omega_{\Lambda\setminus\{z\}},i_z))(\sum_{i_x}\psi_{xz}(i_x,i_z)e_{\omega_{\Lambda\setminus\{z\}}}\otimes \\
&= \sum_{\omega_{\Lambda\setminus\{z\}},i_x,i_z}\psi_{\Lambda\cup\{x\}}((\omega_{\Lambda\setminus\{z\}},i_x,i_z))e_{\omega_{\Lambda\setminus\{z\}}}\otimes e_{i_x}\otimes \\
&= \psi_{\Lambda\cup\{x\}}.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
V_{(z|x)}V_{(z|y)}e_{i_z} &= V_{(z|x)}(\sum_{i_y}\psi_{yz}(i_y,i_z)e_{i_y}\otimes e_{i_z}) \\
&= \sum_{i_x,i_y}\psi_{xz}(i_x,i_z)\psi_{yz}(i_y,i_z)e_{i_x}\otimes e_{i_y}\otimes e_{i_z} \\
&= V_{(z|y)}(\sum_{i_x}\psi_{xz}(i_x,i_z)e_{i_x}\otimes e_{i_z}) \\
&= V_{(z|y)}V_{(z|x)}e_{i_z}.
\end{aligned}$$

**Proposition** Define the transition expectation $E_{(z|x)}$ : $\mathcal{B}_x\otimes\mathcal{B}_z\to\mathcal{B}_z$ by

$$E_{(z|x)}(a_x\otimes a_z) = V^*_{(z|x)}(a_x\otimes a_z)V_{(z|x)}$$

and extend it to $\mathcal{B}$ in the usual way. Let $x_o$ be any (initial) point in $L$ and denote

$$\varphi_0 = \frac{1}{d}\langle \sum_{i_{x_0}=1}^{d} e_{i_{x_o}}, \cdot \sum_{j_{x_0}=1}^{d} e_{j_{x_o}}\rangle$$

the $e(x_o)$–maximally entangled state on $\mathcal{B}_{x_0}$. Define inductively $L_0 = \{x_0\}$ and

$$L_n = \bar{L}_{n-1}$$

$$E_{L_n} := \prod\{E_{(x|y)} \ : \ x \in L_n \ , \ y \in \vec{\partial} L_n \ , \ x \sim y\}$$

where the product is well-defined because the factors commute (being implemented by commuting isometries). Then for any $\Lambda \subseteq L_n$ and any $a_\Lambda \in \mathcal{B}_\Lambda$:

$$\varphi(a_\Lambda) = \varphi_0 \circ E_{L_0} \circ \cdots \circ E_{L_n}(a_\Lambda)$$

and $\varphi$ is a QMF.

*Proof*. From Proposition **??**, we have

$$\begin{aligned} & \varphi_0 \circ E_{L_0} \circ \cdots \circ E_{L_n}(a_\Lambda) \\ = \ & \frac{1}{d}\langle \sum_{i_{x_0}=1}^{d} \prod_{x\in L_1} V_{(x_0|x)} e_{i_{x_0}}, E_{L_1} \circ \cdots \circ E_{L_n}(a_\Lambda) \prod_{x\in L_1} V_{(x_0|x} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{d}\langle \psi_{L_1}, E_{L_1}\circ\cdots\circ E_{L_n}(a_\Lambda)\psi_{L_1}\rangle \\
&\vdots \\
&= \frac{1}{d}\langle \psi_{L_{n+1}}, a_\Lambda \psi_{L_{n+1}}\rangle \\
&= \varphi(a_\Lambda)
\end{aligned}
$$

Now, we prove that $\varphi$ is a QMF. For each $\Lambda \subseteq_{\text{fin}} L$ and $\omega_{\vec{\partial}\Lambda} \in \Omega_{\vec{\partial}\Lambda}$, we define

$$\psi_{\omega_{\vec{\partial}\Lambda}} = \sum_{\omega_\Lambda} \psi(\omega_\Lambda, \omega_{\vec{\partial}\Lambda}) \cdot e_{\omega_\Lambda} \otimes e_{\omega_{\vec{\partial}\Lambda}}$$

and $V_{\vec{\partial}\Lambda,\bar{\Lambda}} : \mathcal{H}_{\vec{\partial}\Lambda} \to \mathcal{H}_{\bar{\Lambda}}$ by

$$V_{\vec{\partial}\Lambda,\bar{\Lambda}}(e_{\omega_{\vec{\partial}\Lambda}}) = \|\psi_{\omega_{\vec{\partial}\Lambda}}\|^{-1}\psi_{\omega_{\vec{\partial}\Lambda}}$$

Then $V_{\vec{\partial}\Lambda,\bar{\Lambda}}$ is well defined because $\|\psi_{\omega_{\vec{\partial}\Lambda}}\| \neq 0$ for each $\omega_{\vec{\partial}\Lambda}$ (otherwise $\psi(\omega_\Lambda, \omega_{\vec{\partial}\Lambda}) = 0$ for each $\omega_\Lambda$, contradicting (7) with $\Lambda$ replaced by $\overline{\Lambda}$). Moreover, since the $\psi_{\omega_{\vec{\partial}\Lambda}}$ are mutually orthogonal, $V_{\vec{\partial}\Lambda,\bar{\Lambda}}$ is an isometry. If we put

$$\psi_{\vec{\partial}\Lambda} = \sum_{\omega_{\vec{\partial}\Lambda}} \|\psi_{\omega_{\vec{\partial}\Lambda}}\| e_{\omega_{\vec{\partial}\Lambda}}$$

then we have

$$V_{\vec{\partial}\Lambda,\bar{\Lambda}}(\psi_{\vec{\partial}\Lambda}) = \sum_{\omega_{\vec{\partial}\Lambda},\omega_\Lambda} \psi(\omega_\Lambda, \omega_{\vec{\partial}\Lambda}) e_{\omega_\Lambda} \otimes e_{\omega_{\vec{\partial}\Lambda}} = \psi_{\bar{\Lambda}}$$

Since $V_\Lambda$ is an isometry, we get $\|\psi_{\vec{\partial}\Lambda}\|^2 = d$. Denoting

$$\varphi_{\vec{\partial}\Lambda} := d^{-1}\langle\psi_{\vec{\partial}\Lambda}, \cdot\,\psi_{\vec{\partial}\Lambda}\rangle$$

$$\mathcal{E}_{\Lambda^c}(a_{\bar{\Lambda}}) := V^*_{\vec{\partial}\Lambda,\bar{\Lambda}} a_{\bar{\Lambda}} V_{\vec{\partial}\Lambda,\bar{\Lambda}} \qquad , \qquad E_{\Lambda^c} := \mathcal{E}_{\Lambda^c} \otimes \mathrm{id}_{\mathcal{B}_{\bar{\Lambda}^c}}$$

for each $a_{\bar{\Lambda}} \in \mathcal{B}_{\bar{\Lambda}}$, we see that

$$\varphi_{\vec{\partial}\Lambda} E_{\Lambda^c}(a_\Lambda) = d^{-1}\langle\psi_{\vec{\partial}\Lambda}, V^*_{\vec{\partial}\Lambda,\bar{\Lambda}} a_\Lambda V_{\vec{\partial}\Lambda,\bar{\Lambda}}\rangle = d^{-1}\langle\psi_{\bar{\Lambda}}, a_\Lambda \psi_{\bar{\Lambda}}\rangle =$$

for each $a_\Lambda \in \mathcal{B}_\Lambda$. Hence $\varphi$ is a QMF.

## Maximally Entangled Markov fields on general graphs

for maximally entangled amplitude matrices,
the construction of the previous section
can be carried over to general graphs.
Origin of the difficulties: loops (trees have no loops)

if $x \sim y \sim z \sim u$ is a path in $L$ ($x, y, z, u$ mutually $\neq$)
then the bi–stochasticity of $|\psi_{xy}(i_x, i_y)|^2$ implies that

$$\sum_{i_x,i_y,i_z} |\psi_{xy}(i_x, i_y)\psi_{yz}(i_y, i_z)\psi_{zx}(i_z, i_u)|^2 =$$

$$(\sum_{i_x} |\psi_{xy}(i_x, i_y)|^2) \sum_{i_y, i_z} |\psi_{yz}(i_y, i_z)\psi_{zx}(i_z, i_u)|^2 =$$

$$(\sum_{i_y} |\psi_{yz}(i_y, i_z)|^2) \sum_{i_z} |\psi_{zx}(i_z, i_u)|^2 = \sum_{i_z} |\psi_{zx}(i_z, i_u)|^2 = 1$$

if $x \sim y \sim z \sim x$ is a loop in $L$, then

$$\sum_{i_x, i_y, i_z} |\psi_{xy}(i_x, i_y)\psi_{yz}(i_y, i_z)\psi_{zx}(i_z, i_x)|^2$$

is equal to the trace of a bi–stochastic matrix, which can be any number in $[0, d]$.

Hence, we need more assumptions. Now, we assume

$$\psi_{xy}(i_x, i_y) = \frac{1}{\sqrt{d}} e^{i\theta_{xy}(i_x, i_y)} \tag{9}$$

where $\theta_{xy}(i_x, i_y) \in \mathbf{R}$.

For $\Lambda \subseteq_{\text{fin}} L$ define
– $v_\Lambda$ the number of vertices in $\Lambda$
– $\epsilon_\Lambda$ the number of edges in $\Lambda$

$$\alpha_\Lambda := v_\Lambda - \epsilon_\Lambda \tag{10}$$

is a numerical invariant of the graph.

**Lemma** For $\Lambda \subseteq_{\text{fin}} L$ let be defined by

$$\psi_\Lambda(\omega_\Lambda) := \prod_{\{x,y\}\in E_\Lambda} \psi_{xy}(\omega_\Lambda(x), \omega_\Lambda(y))$$

$$\psi_\Lambda := \sum_{\omega_\Lambda} \psi_\Lambda(\omega_\Lambda) e_{\omega_\Lambda}$$

$$\psi_{xy}(i_x, i_y) = \frac{1}{\sqrt{d}} e^{i\theta_{xy}(i_x, i_y)}$$

Then

$$\|\psi_\Lambda\|^2 = d^{\alpha_\Lambda}$$

*Proof.*

For each $\omega_\Lambda \in \Omega_\Lambda$ (9) and (6) imply that

$$|\psi_\Lambda(\omega_\Lambda)|^2 = \prod_{\{x,y\}\in E_\Lambda} |\psi_{xy}(\omega_\Lambda(x), \omega_\Lambda(y))|^2 = (\frac{1}{d})^{\epsilon_\Lambda} = d^{-\epsilon_\Lambda}$$

Since the number of configurations is $d^{v_\Lambda}$, we obtain

$$\|\psi_\Lambda\|^2 = \sum_{\omega_\Lambda} |\psi_\Lambda(\omega_\Lambda)|^2 = d^{-\epsilon_\Lambda} d^{v_\Lambda} = d^{\alpha_\Lambda}$$

**Proposition** Let $\Lambda' \subset\subset \Lambda \subseteq_{\text{fin}} L$. Then for any

$a \in \mathcal{B}_{\Lambda'}$ and $x \in \vec{\partial}\Lambda$

$$d^{-\alpha_\Lambda}\langle \psi_\Lambda, a\psi_\Lambda\rangle = d^{-\alpha_{\Lambda\cup\{x\}}}\langle \psi_{\Lambda\cup\{x\}}, a\psi_{\Lambda\cup\{x\}}\rangle$$

*Proof.* Denoting

$$\partial_\Lambda x := \vec{\partial}x \cap \Lambda$$

we find

$$d^{-\alpha_{\Lambda\cup\{x\}}}\langle \psi_{\Lambda\cup\{x\}}, a\psi_{\Lambda\cup\{x\}}\rangle =$$

$$= d^{-\alpha_{\Lambda\cup\{x\}}} \sum_{\omega_{\Lambda'},\omega'_{\Lambda'}} \sum_{\omega_{\Lambda\setminus\{\Lambda'\cup\partial_\Lambda x\}}} \sum_{\omega_{\partial_\Lambda x}} \sum_{i_x}$$

$$\psi_{\Lambda\cup\{x\}}((\omega_{\Lambda'}, \omega_{\Lambda\setminus\{\Lambda'\cup\partial_\Lambda x\}}, \omega_{\partial_\Lambda x}, i_x))^*$$

$$\cdot a_{\omega_{\Lambda'}\omega'_{\Lambda'}}\psi_{\Lambda\cup\{x\}}((\omega'_{\Lambda'}, \omega_{\Lambda\setminus\{\Lambda'\cup\partial_\Lambda x\}}, \omega_{\partial_\Lambda x}, i_x))$$

$$= d^{-\alpha_{\Lambda\cup\{x\}}} \sum_{\omega_{\Lambda'},\omega'_{\Lambda'}} \sum_{\omega_{\Lambda\setminus\{\Lambda'\cup\partial_\Lambda x\}}} \sum_{\omega_{\partial_\Lambda x}} \sum_{i_x}$$

$$\psi_\Lambda((\omega_{\Lambda'}, \omega_{\Lambda\setminus\{\Lambda'\cup\partial_\Lambda x\}}, \omega_{\partial_\Lambda x}))^*$$

$$a_{\omega_{\Lambda'}\omega'_{\Lambda'}}\psi_\Lambda(\omega'_{\Lambda'}, \omega_{\Lambda\setminus\{\Lambda'\cup\partial_\Lambda x\}}, \omega_{\partial_\Lambda x})) \prod_{y\in\partial_\Lambda x} |\psi_{xy}(i_x, \omega_{\partial_\Lambda x}(y))|^2$$

$$= d^{-\alpha_{\Lambda \cup \{x\}}} d^{\epsilon_\Lambda - \epsilon_{\Lambda \cup \{x\}} + 1} \sum_{\omega_{\Lambda'}, \omega'_{\Lambda'}} \sum_{\omega_{\Lambda \setminus \Lambda'}}$$

$$\psi_\Lambda((\omega_{\Lambda'}, \omega_{\Lambda \setminus \Lambda'}))^* a_{\omega_{\Lambda'} \omega'_{\Lambda'}} \psi_\Lambda((\omega'_{\Lambda'}, \omega_{\Lambda \setminus \Lambda'}))$$

$$= d^{-\alpha_\Lambda} \langle \psi_\Lambda, a \psi_\Lambda \rangle$$

**Construction of the purely generated transition expectations**

From the above it follows that the limit

$$\varphi(a) = \lim_{\Lambda \uparrow L} d^{-\alpha_\Lambda} \langle \psi_\Lambda, a\psi_\Lambda \rangle$$

exists finitely for any $a$ in the local algebra and defines a state $\varphi$ on $\mathcal{B}$

We want to prove that this state is a generalized quantum Markov chain.

From the definition of $\varphi$, we have

$$\varphi(E_{\omega_\Lambda}) = d^{-v_\Lambda}$$

For any $\Lambda \subseteq_{\mathrm{fin}} L$, we define the operator $V_{\vec{\partial}\Lambda, \bar{\Lambda}} : \mathcal{H}_{\vec{\partial}\Lambda} \to \mathcal{H}_{\bar{\Lambda}}$ by

$$V_{\vec{\partial}\Lambda, \bar{\Lambda}} e_{\omega_{\vec{\partial}\Lambda}} = d^{\frac{\beta_\Lambda}{2}} \sum_{\omega_\Lambda} \psi_\Lambda(\omega_\Lambda, \omega_{\vec{\partial}\Lambda}) e_{\omega_\Lambda} \otimes e_{\omega_{\vec{\partial}\Lambda}}$$

where $\beta_\Lambda = |\Lambda_E| - v_\Lambda$ and

$$\psi_\Lambda(\omega_\Lambda, \omega_{\vec{\partial}\Lambda}) = \prod_{\{x,y\} \in \Lambda_E} \psi_{xy}((\omega_\Lambda, \omega_{\vec{\partial}\Lambda})(x), (\omega_\Lambda, \omega_{\vec{\partial}\Lambda})(y))$$

**Lemma** For any $\Lambda \subseteq_{\text{fin}} L$, the operator $V_{\vec{\partial}\Lambda,\bar{\Lambda}}$ is isometry. Moreover, for sufficiently large $\Lambda' \subseteq_{\text{fin}} L$, we obtain

$$V_{\vec{\partial}\Lambda,\bar{\Lambda}} d^{-\frac{\alpha_{\Lambda'\setminus\Lambda}}{2}} \psi_{\Lambda'\setminus\Lambda} = d^{-\frac{\alpha_{\Lambda'}}{2}} \psi_{\Lambda'}$$

*Proof.* For each orthonormal basis $e_{\omega_{\vec{\partial}\Lambda}}, e_{\omega'_{\vec{\partial}\Lambda}} \in \mathcal{H}_{\vec{\partial}\Lambda}$, we have

$$\begin{aligned}
& \langle V_{\vec{\partial}\Lambda} e_{\omega_{\vec{\partial}\Lambda}}, V_{\vec{\partial}\Lambda} e_{\omega'_{\vec{\partial}\Lambda}} \rangle \\
= & \langle d^{\frac{\beta_\Lambda}{2}} \sum_{\omega_\Lambda} \psi_\Lambda(\omega_\Lambda, \omega_{\vec{\partial}\Lambda}) e_{\omega_\Lambda} \otimes e_{\omega_{\vec{\partial}\Lambda}}, d^{\frac{\beta_\Lambda}{2}} \sum_{\omega'_\Lambda} \psi_\Lambda(\omega'_\Lambda, \omega'_{\vec{\partial}\Lambda}) e_{\omega'_\Lambda} \\
= & \delta_{\omega_{\vec{\partial}\Lambda}, \omega'_{\vec{\partial}\Lambda}} d^{\beta_\Lambda} \langle \sum_{\omega_\Lambda} \psi_\Lambda(\omega_\Lambda, \omega_{\vec{\partial}\Lambda}) e_{\omega_\Lambda}, \sum_{\omega'_\Lambda} \psi_\Lambda(\omega'_\Lambda, \omega_{\vec{\partial}\Lambda}) e_{\omega'_\Lambda} \rangle \\
= & \delta_{\omega_{\vec{\partial}\Lambda}, \omega'_{\vec{\partial}\Lambda}} d^{\beta_\Lambda} \sum_{\omega_\Lambda} |\psi_\Lambda(\omega_\Lambda, \omega_{\vec{\partial}\Lambda})|^2 \\
= & \delta_{\omega_{\vec{\partial}\Lambda}, \omega'_{\vec{\partial}\Lambda}} d^{\beta_\Lambda} d^{v_\Lambda} d^{-|\Lambda_E|} = \delta_{\omega_{\vec{\partial}\Lambda}, \omega'_{\vec{\partial}\Lambda}}
\end{aligned}$$

Therefore, $V_{\vec{\partial}\Lambda,\bar{\Lambda}}$ is isometry. Furthermore, $\Lambda_E \cup (\Lambda'\setminus\Lambda)_e = E'_\Lambda$. Now, we have

$$V_{\vec{\partial}\Lambda,\bar{\Lambda}} d^{-\frac{\alpha_{\Lambda'\setminus\Lambda}}{2}} \psi_{\Lambda'\setminus\Lambda}$$

$$
\begin{aligned}
&= V_{\vec{\partial}\Lambda,\bar{\Lambda}} d^{-\frac{\alpha_{\Lambda'\setminus\Lambda}}{2}} \sum_{\omega_{\Lambda'\setminus\Lambda}} \psi_{\Lambda'\setminus\Lambda}(\omega_{\Lambda'\setminus\Lambda}) e_{\omega_{\Lambda'\setminus\Lambda}} \\
&= d^{-\frac{\alpha_{\Lambda'\setminus\Lambda}}{2}} d^{\frac{\beta_\Lambda}{2}} \sum_{\omega_{\Lambda'\setminus\bar{\Lambda}},\omega_\Lambda,\omega_{\vec{\partial}\Lambda}} \psi_{\Lambda'\setminus\Lambda}((\omega_{\Lambda'\setminus\bar{\Lambda}},\omega_{\vec{\partial}\Lambda}))\psi_\Lambda(\omega_\Lambda,\omega_{\vec{\partial}\Lambda}) \\
&= d^{-\frac{\alpha_{\Lambda'}}{2}} \sum_{\omega_{\Lambda'}} \psi(\omega_{\Lambda'}) e_{\omega_{\Lambda'}} = d^{-\frac{\alpha_{\Lambda'}}{2}} \psi_{\Lambda'}
\end{aligned}
$$

For each $\Lambda \subseteq_{\text{fin}} L$, we define the transition maps $\mathcal{E}_{\Lambda^c} : \mathcal{B}_{\bar{\Lambda}} \to \mathcal{B}_{\vec{\partial}\Lambda}$ by

$$\mathcal{E}_{\Lambda^c}(a_{\bar{\Lambda}}) = V^*_{\vec{\partial}\Lambda,\bar{\Lambda}} a_{\bar{\Lambda}} V_{\vec{\partial}\Lambda,\bar{\Lambda}}$$

for any $a_{\bar{\Lambda}} \in \mathcal{B}_{\bar{\Lambda}}$, and quasi-conditional expectations $E_{\Lambda^c} : \mathcal{B} \to \mathcal{B}$ by

$$E_{\Lambda^c}(a_{\bar{\Lambda}} \otimes a_{\bar{\Lambda}^c}) = I_{\mathcal{B}_\Lambda} \otimes \mathcal{E}_{\Lambda^c}(a_{\bar{\Lambda}}) \otimes a_{\bar{\Lambda}^c}$$

for any $a_{\bar{\Lambda}} \in \mathcal{B}_{\bar{\Lambda}}$ and $a_{\bar{\Lambda}^c} \in \mathcal{B}_{\bar{\Lambda}^c}$. By definition, we have

$$\mathcal{E}_{\Lambda^c}(e_{\omega_\Lambda,\omega'_\Lambda}) = \sum_{\omega_{\vec{\partial}\Lambda}} d^{\beta_\Lambda} \psi_\Lambda(\omega_\Lambda,\omega_{\vec{\partial}\Lambda})^* \psi_\Lambda(\omega'_\Lambda,\omega_{\vec{\partial}\Lambda}) E_{\omega_{\vec{\partial}\Lambda}}$$

In particular,

$$\mathcal{E}_{\Lambda^c}(E_{\omega_\Lambda}) = d^{-v_\Lambda} \cdot 1$$

**Lemma** For any $\Lambda_0 \subset\subset \Lambda \subseteq_{\text{fin}} L$ and $a_{\Lambda_0} \in \mathcal{B}_{\Lambda_0}$, we have

$$E_{\Lambda^c}(a_{\Lambda_0}) = \varphi(a_{\Lambda_0})$$

In particular the family $\{E_{\Lambda^c}\}$ is weakly projective and $\varphi$ is the unique weakly invariant state.

*Proof*. From the above calculation, we obtain

$$\begin{aligned}
& E_{\Lambda^c}(e_{\omega_{\Lambda_0},\omega'_{\Lambda_0}}) \\
= & \sum_{\omega_{\Lambda\setminus\Lambda_0},\omega_{\vec{\partial}\Lambda}} d^{\beta_\Lambda}\psi_\Lambda((\omega_{\Lambda\setminus\Lambda_0},\omega_{\Lambda_0}),\omega_{\vec{\partial}\Lambda})^*\psi_\Lambda((\omega_{\Lambda\setminus\Lambda_0},\omega'_{\Lambda_0}),\omega \\
= & \sum_{\omega_{\vec{\partial}\Lambda},\omega_{\vec{\partial}\Lambda_0}} d^{|\Lambda_{0E}|-v_{\bar{\Lambda}_0}}\psi_{\Lambda_0}(\omega_{\Lambda_0},\omega_{\vec{\partial}\Lambda_0})^*\psi_{\Lambda_0}(\omega'_{\Lambda_0},\omega_{\vec{\partial}\Lambda_0})E_{\omega_{\vec{\partial}\Lambda}} \\
= & \sum_{\omega_{\vec{\partial}\Lambda_0}} d^{|\Lambda_{0E}|-v_{\bar{\Lambda}_0}}\psi_{\Lambda_0}(\omega_{\Lambda_0},\omega_{\vec{\partial}\Lambda_0})^*\psi_{\Lambda_0}(\omega'_{\Lambda_0},\omega_{\vec{\partial}\Lambda_0})\cdot I
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\varphi(e_{\omega_{\Lambda_0},\omega'_{\Lambda_0}}) &= d^{-\alpha_{\bar{\Lambda}_0}}\langle\psi_{\bar{\Lambda}_0}, e_{\omega_{\Lambda_0},\omega'_{\Lambda_0}}\psi_{\bar{\Lambda}_0}\rangle \\
&= d^{-\alpha_{\bar{\Lambda}_0}}\sum_{\omega_{\vec{\partial}\Lambda_0}}\psi_{\bar{\Lambda}_0}((\omega_{\vec{\partial}\Lambda_0},\omega_{\Lambda_0}))^*\psi_{\bar{\Lambda}_0}((\omega_{\vec{\partial}\Lambda_0},\omega'_{\Lambda_0} \\
&= d^{|\Lambda_{0E}|-v_{\bar{\Lambda}_0}}\sum_{\omega_{\vec{\partial}\Lambda_0}}\psi_{\Lambda_0}(\omega_{\Lambda_0},\omega_{\vec{\partial}\Lambda_0})^*\psi_{\Lambda_0}(\omega'_{\Lambda_0},\omega_{\vec{\partial}
\end{aligned}$$

**Lemma** The family $\{E_{\Lambda^c}\}$ is not projective. In particular, $\varphi$ is not a Markov state.

*Proof*. For arbitrary $e_{\omega_{\bar{\Lambda}},\omega'_{\bar{\Lambda}}} \in \mathcal{B}_{\bar{\Lambda}}$, we get

$$\mathcal{E}_{\Lambda^c}(e_{\omega_{\bar{\Lambda}},\omega'_{\bar{\Lambda}}}) = d^{\beta_\Lambda}\psi_\Lambda(\omega_\Lambda,\omega_{\vec{\partial}\Lambda})^*\psi_\Lambda(\omega'_\Lambda,\omega'_{\vec{\partial}\Lambda})e_{\omega_{\vec{\partial}\Lambda},\omega'_{\vec{\partial}\Lambda}}$$

where $\omega_{\bar{\Lambda}} = (\omega_\Lambda,\omega_{\vec{\partial}\Lambda})$ and $\omega'_{\bar{\Lambda}} = (\omega'_\Lambda,\omega'_{\vec{\partial}\Lambda})$. From the proof of Lemma (**??**) we have that

$$E_{\Lambda'^c}E_{\Lambda^c} \neq E_{\Lambda'^c}$$

for $\Lambda \subset\subset \Lambda' \subseteq_{\mathrm{fin}} L$.

**Interpretation**

In many models used in statistical mechanics, the vertices $x \in L$ are identified to particles, the Hilbert space $\mathcal{H}(x)$ to their state space, the basis $(e_j(x))$ to the eigenvectors of some non degenerate observable $A(x)$ and the index set $S(x)$ to the eigenvalues of this observable, say

$$S(x) = \{1,\ldots,d_{\mathcal{H}}\} \equiv \{a_1(x),\ldots,a_{d_{\mathcal{H}}}(x)\}$$

With these identifications the section $\omega_{\Lambda}$ is identified to the event or configuration:

$$\omega_{\Lambda} \equiv \{[A(x) = a(x)\omega_{\Lambda}(x)] \ ; \quad \forall\, x \in \Lambda\}$$

aaa

Abstract Toronto

In the past 30 years the theory of quantum Markov chains (QMC) has undergone several developments.

The attempt to give an intrinsic operator–theoretical characterization of QMC produced a deep analysis, due to C. Cecchini, of various notions of quantum Markovianity.

The notion of Markovianity on CAR algebras, and the corresponding structure theorems, revealed a surprisingly richer structure than in the infinite tensor product case.

Moreover, as it often happens in quantum probability, the efforts to better understand the quantum case has lead to question some deeply rooted beliefs concerning classical Markov processes.

Applications to physics have proliferated in several different directions, ranging from the interesting theoretical results of Fannes, Nachtergaele, Werner, Matsui, Mohari, Mukhammedov, Ohno,... to numerical simulations related to the Bethe approximation.

these authors gave important contributions to the theory of quantum Markov chains with finite state space

The results of Lindblad, Alicki and Fannes on the notion of quantum dynamical entropy and the subsequent extension by Ohya, Watanabe and others, have established a connection between QMC and the theory of quantum chaos.

Finally Petz and his school has shown that QMC play a relevant role also in quantum information, notably in the problems related to the capacity of quantum channels.

Now the boundary of this line of research is the extension of the above results to Markov fields (i.e. processes with multidimensional index set).

What is needed is not so much an abstract theory (many variants are possible and some of them already published) as a new nontrivial, class of concrete examples which could play for fields a role analogue to that, played in the 1–dimensional case, by the QMC, i.e. a benchmark on which to test the power of different theoretical proposals.

Now such class of examples has begun to be developed.

I will use this class to illustrate some points of the abstract theory and some interesting open problems.

Sibiu Abstract

The notion of Markovianity on CAR algebras, and the corresponding structure theorems, revealed a surprisingly richer structure than in the infinite tensor product case.