New maximal functions, commutators, and weighted estimates for the multilinear Calderón-Zygmund theory

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Plan of the talk

- Brief review of multilinear Calderón-Zygmund theory

- Previous multilinear weighted estimates and results for multilinear commutators

- New results

The new results are in collaboration with

A. Lerner, S. Ombrosi, C. Pérez, R. Trujillo

Coifman-Meyer bilinear operators

$$\boxed{T(f,g)(x) = \int m(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix\cdot(\xi+\eta)}\, d\xi d\eta,}$$

where m is a *"suitable symbol of order zero"*.

As with the product we should expect

$$\|T(f,g)\|_{L^r} \le C\|f\|_{L^p}\|g\|_{L^q}$$

for p, q, r related as in Hölder's inequality.

Coifman-Meyer ('70s-'80s) $r > 1$.
Kenig-Stein and Grafakos-T. ('90s–'00s) $r > 1/2$

$$T(f,g)(x) = \int m(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix\cdot(\xi+\eta)}\, d\xi d\eta,$$

$$|\partial^\alpha m(\xi,\eta)| \le C_\alpha(|\xi| + |\eta|)^{-|\alpha|},$$

then

$$\|T(f,g)\|_{L^r} \le C\|f\|_{L^p}\|g\|_{L^q}$$

for $1/p + 1/q = 1/r$, $1 < p.q < \infty$ *and* $1/2 < r < \infty$.

Also,
$$T : L^1 \times L^1 \to L^{1/2,\infty}$$
$$T : L^\infty \times L^\infty \to BMO$$

Formally, if $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$,

$$T(f,g)(x) = \int K(x-y, x-z) f(y) g(z)\, dydz$$

where K is a Calderón-Zygmund kernel in $2n$

$$|\partial^\alpha K(y,z)| \leq C_\alpha(|y| + |z|)^{-(2n+|\alpha|)}$$

One can also consider variable coefficient operators of the form

$$T(f,g)(x) = \int_{\mathbb{R}^{2n}} K(x,y,z) f(y) g(z)\, dydz$$

with

$$|\partial^\alpha K(x,y,z)| \leq C_\alpha(|x-y| + |x-z|)^{-(2n+|\alpha|)}$$

at least when $x \notin \operatorname{supp} f \cap \operatorname{supp} g$.

We say that such a T is a bilinear Calderón-Zygmund operator if it is bounded on some product of L^p spaces. As in the linear case, he boundedness can be characterized with a T1-Theorem.

T1-Theorem for bilinear Calderón-Zygmund operators

$$T : \mathcal{S} \times \mathcal{S} \to \mathcal{S}', \quad \langle T(f_1, f_2), f_3 \rangle = \langle K, f_1 \otimes f_2 \otimes f_3 \rangle$$

$$|\partial^\alpha K(y_0, y_1, y_2)| \lesssim (\sum |y_j - y_k|)^{-(2n+|\alpha|)}, \quad |\alpha| \leq 1$$

Christ-Journé (1987)

$$|\langle K, f_1 \otimes f_2 \otimes f_3 \rangle| \lesssim \|f_j\|_\infty \, \|f_k\|_2 \, \|f_l\|_2$$

iff K satisfies a multilinear WBP and the three distributions $T(1,1)$, $T^{*1}(1,1)$, $T^{*2}(1,1)$ are in BMO.

Grafakos-T. (2002)

$$T : L^4 \times L^4 \to L^2$$

$$\Longleftrightarrow \; T : L^{p_1} \times L^{p_2} \to L^p, \;\; 1/p = 1/p_1 + 1/p_2 < 2$$

$$\Longleftrightarrow \; \sup_{\xi_1, \xi_2} (\|T^{*j}(e^{ix\cdot\xi_1}, e^{ix\cdot\xi_2})\|_{BMO} \leq C$$

($T^{*0} = T$, T^{*1}, T^{*2} are the transposes of T)

Note the p-invariance for the full range of $p > 1/2$.

Remark

There is a recent $T1$-theorem for more singular bilinear operators due to

A. Bényi-C. Demeter-A. Nahmod-C. Thiele-R. T.-P. Villarroya
These are operators with certain modulation invariance like the bilinear Hilbert transform. Essentially,

$$\boxed{T(f_1, f_2)(x) = \int_{\mathbb{R}} K(x,t) f_1(x-t) f_2(x+t)\, dt}$$

where K is a Calderón-Zygmund kernel.

Examples are provided by pseudodifferential operators

$$T(f,g)(x) = \int_{\mathbb{R}^2} \sigma(x, \xi - \eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{ix(\xi+\eta)}\, d\xi d\eta$$

where $\sigma(x,z)$ is a classical linear symbol. For the purposes of this talk we will only consider bilinear (or multilinear) operators of Calderón-Zygmund type.

Typical examples of bilinear Calderón-Zygmund operators

The Riesz transforms in $\mathbb{R}^2$ can be seen as bilinear operators on $\mathbb{R} \times \mathbb{R}$, e.g.

$$\boxed{R_1(f,g)(x) = \text{p.v.} \int_{\mathbb{R}^2} \frac{x-y}{|(x-y,x-z)|^3} f(y)g(z)\, dydz}$$

$$R_1 : L^p(\mathbb{R}) \times L^q(\mathbb{R}) \to L^r(\mathbb{R}),\ 1/p + 1/q = 1/r < 2$$

Bilinear pseudodifferential operators

$$\boxed{T_{\sigma_s}(f,g)(x) = \int \sigma_s(x,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix\cdot(\xi+\eta)}\, d\xi d\eta,}$$

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma_s(x,\xi,\eta)| \le C_{\alpha\beta\gamma}(1+|\xi|+|\eta|)^{s-(|\beta|+|\gamma|)}$$

For order $s = 0$, T_{σ_0} is a Calderón-Zygmund operator

$$T_{\sigma_0} : L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \to L^r(\mathbb{R}^n),\ 1/p + 1/q = 1/r < 2$$

Certainly there are m-linear versions

If

$$|\partial^{\alpha} K(y_0, y_1, \ldots, y_m)| \leq C_{\alpha} \left(\sum_{k,l=0}^{m} |y_k - y_l| \right)^{-mn-|\alpha|}$$

and

$$T : L^{r_1} \times \cdots \times L^{r_m} \to L^{r}$$

for some $1 < r_1, \ldots, r_m < \infty$, $\frac{1}{r_1} + \cdots + \frac{1}{r_m} = \frac{1}{r}$,

then

$$T : L^{p_1} \times \cdots \times L^{p_m} \to L^{p},$$

for all $1 < p_1, \ldots, p_m < \infty$, $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$.

Also

$$T : L^{p_1} \times \cdots \times L^{p_m} \to L^{p,\infty},$$

for all $1 \leq p_1, \ldots, p_m < \infty$, $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$, if at least one $p_j = 1$.

In particular,

$$T : L^{1} \times \cdots \times L^{1} \to L^{1/m,\infty}$$

Weighted estimates (first approach)

Recall the $w \in A_p$, $1 < p < \infty$, if

$$\boxed{\sup_Q \left(\frac{1}{|Q|}\int_Q w\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'}\,dx\right)^{p-1}}$$

is finite; and $w \in A_1$ if

$$\boxed{\frac{1}{|Q|}\int_Q w(y)\,dy \leq C \inf_Q w}$$

The A_p weights are the right ones for linear Calderón-Zygmund operators. So it is then natural to expect:

Grafakos-T. (2002)

Let T be a Calderón-Zygmund operator and let $1 < p_1, \dots, p_m < \infty$ *and* $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$.
If $w \in A_{p_0}$, $p_0 = \min\{p_1, \cdots, p_m\}$, *then*

$$T : L^{p_1}(w) \times \cdots \times L^{p_m}(w) \to L^p(w),$$

If $w \in A_1$, *then*

$$T : L^1(w) \times \cdots \times L^1(w) \to L^{1/m,\infty}(w)$$

Note again the full range $p > 1/m$.

The proof was based on a good-λ estimate involving

$$T(f_1, \ldots, f_m) \qquad \text{and} \qquad \prod_{j=1}^m Mf_j$$

and leading to

$$\int |T(f_1, \ldots, f_m)(x)|^p w(x)\, dx$$

$$\leq C \int \left(\prod_{j=1}^m M(f_j)(x) \right)^p w(x)\, dx.$$

for any A_∞ weight w.

This was simplified later using a pointwise estimate.

Recall the maximal functions

$$\boxed{M_\delta(f) = M(|f|^\delta)^{1/\delta}, \quad \delta > 0}$$

$$\boxed{M^\# f(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q|\, dy,}$$

$$\boxed{M_\delta^\# f = M^\#(|f|^\delta)^{1/\delta}, \ \ \delta > 0}$$

and the estimates

Fefferman-Stein (1972)

Let $0 < p, \delta < \infty$ *and let* $w \in A_\infty$. *Then there exists* $C > 0$ *such that*

$$\int M_\delta(f)(x)^p \, w(x)dx \leq C \int M_\delta^{\#}(f)(x)^p \, w(x)dx. \tag{1}$$

holds for any function f *for which the left hand side is finite.*

Alvarez-Pérez (1994)

Let T *be a linear Calderón-Zygmund operator and let* $0 < \delta < 1$. *Then*

$$M_\delta^{\#}(T(f))(x) \leq C \, M(f)(x) \tag{2}$$

for any bounded function f *with compact support.*

In the multilinear case, using the notation

$$\boxed{\vec{f} = (f_1, \ldots, f_m)}$$

Pérez-T. (2003)

Let T be a linear Calderón-Zygmund operator and let $0 < \delta < 1/m$. Then

$$M_\delta^{\#}(T(\vec{f}\,))(x) \leq C \prod_{j=1}^{m} M f_j(x) \tag{3}$$

for any bounded $\vec{f}$ with compact support.

From (1) and (3),

$$\int |T(\vec{f})(x)|^p w(x)\,dx \leq \int (M_\delta(T(\vec{f}))(x))^p w(x)\,dx$$

$$\leq C \int (M_\delta^{\#}(T(\vec{f}))(x))^p w(x)\,dx$$

$$\leq C \int \left(\prod_{j=1}^{m} M(f_j)(x)\right)^p w(x)\,dx.$$

The weighted estimates easily follow.

An application to (un-weighted) commutators

$$\boxed{[b,T]f(x) = b(x)Tf(x) - T(bf)(x)}$$

Coifman-Rochberg-Weiss (1976)

$$b \in BMO(\mathbb{R}^n) \Rightarrow [b,T] : L^p \longrightarrow L^p,\ 1 < p < \infty$$

Using complex analysis $[b,T]$ can be written as an average of $T_z f = e^{zb}T(e^{-zb}f)$. The boundedness follows from: $b \in BMO$ *and* $\delta > 0$ *small* $\Rightarrow e^{\delta b} \in A_p$.

$[b,T]$ is not of weak-L^1 type but of "weak-$L \log L$" type:

Pérez (1995)

There exists $C > 0$ *(depending on* $\|b\|_{BMO}$*) such that for all* $\lambda > 0$

$$|\{y \in \mathbb{R}^n : |[b,T]f(y)| > \lambda\}| \leq C \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right)dx$$

where

$$\Phi(t) = t(1 + \log^+ t)$$

Similarly, multilinear weighted estimates can be used to study the multilinear commutators.

T m-linear Calderón-Zygmund operator and $\vec{b} = (b_1, \ldots, b_m)$

$$\boxed{T_{\vec{b}}(f_1, \cdots, f_m) = \sum_{j=1}^{m} T_{\vec{b}}^{j}(\vec{f}),}$$

where each term is the commutator of b_j and T in the j-th entry of T, that is,

$$T_{\vec{b}}^{j}(\vec{f}) = b_j T(f_1, \cdots, f_j, \cdots, f_m) - T(f_1, \cdots, b_j f_j, \cdots, f_m)$$

Pérez-T. (2003)

Let $\vec{b} = (b_1, .., b_m) \in BMO^m$*, let* $1 < p < \infty$ *and* $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$*. Then*

$$T_{\vec{b}} : L^{p_1} \times \cdots \times L^{p_m} \longrightarrow L^p$$

The proof is based again on complex analysis. However, it breaks down for $p < 1$ because the proof uses the A_p classes and Minkowski inequality which holds for $p > 1$.

Some afterthoughts/questions

The control of $T(\vec{f})$ by $\prod M(f_j)$ is not optimal.

The same approach will work if we use a smaller maximal operator to control T.

The results were not truly multilinear and a small maximal function could produce a more general class of weights.

Is there a larger class of weights for which multilinear Calderón-Zygmund operators are bounded?

The multilinear weighted theory is not as useful to study commutators as in the linear case.

Based on p-invariance we should expect the result to be true for $p > 1/m$.

Can we get an end-poit estimate and interpolate to obtain the full range of p's?

We will now address all these issues in a very explicit way.

Weighted estimates (new approach)

The rest of the talk is about joint work with

A. Lerner, S. Ombrosi, C. Pérez, R. Trujillo

The main object of study in this approach is

$$\boxed{\mathcal{M}(\vec{f})(x) = \sup_{x\in Q} \prod_{i=1}^{m} \frac{1}{|Q|}\int_Q |f_i(y)|dy}$$

Simple observations

1) $\mathcal{M}(\vec{f})(x) \leq \displaystyle\prod_{j=1}^{m} Mf_j(x)$

2) $\mathcal{M} : L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$

3) $\mathcal{M} : L^1(\mathbb{R}^n) \times \cdots \times L^1(\mathbb{R}^n) \to L^{1/m,\infty}(\mathbb{R}^n)$

Pointwise estimate

T m-linear Calderón-Zygmund operator.

Theorem 1 *For* $0 < \delta < \frac{1}{m}$,

$$M_\delta^{\#}(T(\vec{f}))(x) \le C\,\mathcal{M}(\vec{f})(x)$$

As a corollary,

Theorem 2 *Let* $0 < p < \infty$ *and* $w \in A_\infty$, *then*

$$\int_{\mathbb{R}^n} |T(\vec{f})|^p\, w\, dx \le C \int_{\mathbb{R}^n} (\mathcal{M}(\vec{f})(x))^p\, w\, dx$$

To prove Theorem 1 with $\mathcal{M}(\vec{f})$ in place of $\prod M(f_j)$, a more careful analysis is needed; but Theorem 2 follows then in the same way.

To take advantage of Theorem 2 we need to exploit the fact that $\mathcal{M}$ is really smaller than a product of maximal functions.

Towards a multilinear A_p theory of weights

We want a Muckenhoupt's theorem for the multilinear case.

We start with the characterization of the two-weight weak-type problem for $\mathcal{M}$.

Consider vector weights

$$\vec{w} = (w_1, \ldots, w_m)$$

For m exponents $p_1, \cdots, p_m$ and p such that

$$\boxed{\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}}$$

we will write

$$\boxed{\vec{P} = (p_1, \cdots, p_m)}$$

$$\boxed{\vec{1} = (1, \cdots, 1)}$$

and define

$$\boxed{\vec{P} \geq \vec{Q} \text{ if } p_i \geq q_i, \quad i = 1, \cdots, m}$$

The two-weight case

Theorem 3 *Let $\vec{P} \geq \vec{1}$ and $\vec{w} = (w_1, \ldots, w_m)$ and u be weights. Then the inequality*

$$\|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(u)} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}$$

holds if and only if the expression

$$\sup_Q \left(\frac{1}{|Q|}\int_Q u\right)^{\frac{1}{p}} \prod_{j=1}^{m} \left(\frac{1}{|Q|}\int_Q w_j^{1-p_j'}\right)^{\frac{1}{p_j'}}$$

is finite.

It is understood that

$$\left(\frac{1}{|Q|}\int_Q w_j^{1-p_j'}\right)^{\frac{1}{p_j'}} = (\inf_Q w_j)^{-1}$$

when $p_j = 1$

Another way to write the two-weight condition is

$$\boxed{\prod_{j=1}^{m} \sup_{Q} \left(\frac{1}{|Q|}\int_Q u\right)^{\frac{1}{p_j}} \left(\frac{1}{|Q|}\int_Q w_j^{1-p_j'}\right)^{\frac{1}{p_j'}}}$$

because

$$\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$$

Simple observations

1) If $m = 1$ is just the two-weight A_p condition.

2) By differentiation,

$$u(x) \leq c \prod_{j=1}^{m} w_j(x)^{p/p_j}$$

This suggests a way to define a multilinear analogue of the one-weight $A_{\vec{P}}$ condition.

The one-weight multilinear $A_{\vec{P}}$ condition

Given $\vec{w} = (w_1, \ldots, w_m)$, let

$$\boxed{u_{\vec{w}} = \prod_{j=1}^{m} w_j^{p/p_j}}$$

For $\vec{P} \geq \vec{1}$, we say that $\vec{w}$ satisfies the $A_{\vec{P}}$ condition if

$$\boxed{\sup_Q \left(\frac{1}{|Q|}\int_Q u_{\vec{w}}\right)^{\frac{1}{p}} \prod_{j=1}^{m} \left(\frac{1}{|Q|}\int_Q w_j^{1-p_j'}\right)^{\frac{1}{p_j'}} \text{ is finite.}}$$

Observations:

1) Again if $m = 1$ is the classical one-weight A_p condition.

2) $A_{(1,\cdots,1)} \subset A_{\vec{P}}$ for each $\vec{P}$, however in general $A_{\vec{P}}$ is not included in $A_{\vec{Q}}$ for $\vec{P} \leq \vec{Q}$.

Exercise: For $n = 1$, $\vec{P} = (p_1, p_2) = (2, 2)$, and $\vec{w} = (w_1, w_2) = (|x|^{-5/3}, 1)$. Since $w_1^{1/2} \in A_1$ it is easy to see that $\vec{w} \in A_{(2,2)}$. It is also easy to show that $\vec{w} \notin A_{\vec{Q}}$ if $\vec{Q}$ is large.

3) If each w_j is in A_{p_j}, then

$$\sup_Q \left(\frac{1}{|Q|}\int_Q u_{\vec{w}}\right)^{1/p} \prod_{j=1}^{m} \left(\frac{1}{|Q|}\int_Q w_j^{1-p_j'}\right)^{1/p_j'}$$

$$\leq \sup_Q \prod_{j=1}^{m} \left(\frac{1}{|Q|}\int_Q w_j\right)^{1/p_j} \left(\frac{1}{|Q|}\int_Q w_j^{1-p_j'}\right)^{1/p_j'} < \infty,$$

so

$$\prod_{j=1}^{m} A_{p_j} \subset A_{\vec{P}}.$$

This inclusion is strict.

Exercise:

$$w_1 = \frac{\chi_{[0,2]}(x)}{|x-1|} + \chi_{\mathbb{R}/[0,2]}(x)$$

and $w_j(x) = \frac{1}{|x|}$ for $j = 2, ..., m$. Using the definition it is not difficult to check that $u_{\vec{w}} \in A_1$. Also $\inf_Q u_{\vec{w}} \sim \prod_{j=1}^{m} \inf_Q w_j^{p/p_j}$. These last two facts together imply that $\vec{w} \in A_{\vec{P}}$.

4) Again using Hölder's inequality

$$\sup_Q \left(\frac{1}{|Q|}\int_Q u_{\vec{w}}\right)^{1/mp}\left(\frac{1}{|Q|}\int_Q u_{\vec{w}}^{-\frac{1}{mp-1}}\right)^{(mp-1)/mp}$$

$$\leq \sup_Q \prod_{j=1}^{m}\left(\frac{1}{|Q|}\int_Q w_j\right)^{1/mp_j}\left(\frac{1}{|Q|}\int_Q w_j^{-\frac{1}{p_j-1}}\right)^{(p_j-1)/mp_j}$$

$$< \infty,$$

where we have used that $m - 1/p = \sum(p_j - 1)/p_j$. It follows that $u_{\vec{w}}$ is in A_{mp} if $w_j \in A_{p_j}$.

It turns out that something more general happens.

Theorem 4 $\vec{w} \in A_{\vec{P}}$ *if and only if*

$w_j^{1-p'_j} \in A_{mp'_j}$ *and* $u_{\vec{w}} \in A_{mp}$

This can be used to prove the following strong-type characterization.

Theorem 5 *Let $\vec{w} = (w_1, \ldots, w_m)$ be a vector of weights and $\vec{P} > \vec{1}$.*
Then the inequality

$$\|\mathcal{M}(\vec{f})\|_{L^p(u_{\vec{w}})} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)} \tag{4}$$

holds if and only if $\vec{w}$ satisfies the $A_{\vec{P}}$ condition.

Remarks:

The condition $\vec{P} > \vec{1}$ is crucial.

There are counterexamples that show the $\mathcal{M}(\vec{f})$ can not be replaced by $\prod_{j=1}^{m} M f_j$.

The distribution functions of M and the dyadic maximal function M^d are comparable, but in the multilinear case $\mathcal{M}$ and its dyadic version $\mathcal{M}^d$ are not.

Take for instance $m = 2$ and $n = 1$ and set $f_1 = \chi_{(-1,0)}$ and $f_2 = \chi_{(0,1)}$.

Then, $\mathcal{M}^d(\vec{f}) \equiv 0$ but $\mathcal{M}(\vec{f})(x) > 0$ everywhere.

New estimates for Calderón-Zygmund operators

Let T be any m-linear Calderón-Zygmund operator.

Since for $\vec{w}$ in $A_{\vec{P}}$, we have that $u_{\vec{w}}$ is in A_∞. Then, Theorem 2 gives

Theorem 6 *Let $\vec{w} \in A_{\vec{P}}$.*

(i) If $\vec{P} > \vec{1}$, then

$$\|T(\vec{f})\|_{L^p(u_{\vec{w}})} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}.$$

(ii)

$$\|T(\vec{f})\|_{L^{1/m,\infty}(u_{\vec{w}})} \leq C \prod_{j=1}^{m} \|f_j\|_{L^1(w_j)}.$$

The classes $A_{\vec{P}}$ are also characterize by the m-linear Calderón-Zygmund operators.

For $i = 1, \cdots, n$, the m-linear i-th Riesz transform is defined by

$$\boxed{R_i(\vec{f})\,(x) = \text{p.v.} \int_{(\mathbb{R}^n)^m} \frac{\sum_{j=1}^m (x_i - (y_j)_i)}{(\sum_{j=1}^m \left|x - y_j\right|^2)^{\frac{nm+1}{2}}} \vec{f}(\vec{y})d\vec{y},}$$

where $(y_j)_i$ denotes the i-th coordinate of y_j.

Theorem 7 *If either estimate in the previous theorem holds for each of the m-linear Riesz transforms $R_i(\vec{f})$, then $\vec{w} \in A_{\vec{P}}$.*

Back to commutators with BMO functions

Theorem 8 *Let* $\overrightarrow{b} \in BMO^m$. *Then there exits a constant* $C > 0$, *such that, for any* $t > 0$,

$$|\{x \in \mathbb{R}^n : |T_{\overrightarrow{b}}(\vec{f})(x)| > t^m\}|$$

$$\leq C \prod_{j=1}^{m} \left(\int_{\mathbb{R}^n} \Phi(\frac{|f_j(x)|}{t}) dx \right)^{1/m}$$

where $\Phi(t) = t(1 + \log^+ t)$.

Furthermore, this estimate is sharp in the sense that none of the Φ's can be replaced by $\Phi(t) = t$.

Also we can replace the Lebesgue measure by weights $\vec{w} \in A_{(1,\cdots,1)}$.

Logartihmic multilinear maximal function

The proof is based on the relationship with the following maximal function

$$\boxed{\mathcal{M}_{L(logL)}(\vec{f})(x) = \sup_{x\in Q} \prod_{j=1}^{m} \|f_j\|_{L(logL),Q}}$$

Let $\vec{b} \in BMO^m$. The following pointwise estimate holds.

Theorem 9 *Let* $0 < \delta < 1/m$ *and* $0 < \varepsilon$. *Then there exists a constant* $C > 0$ *such that*

$$M_\delta^{\#}(T_{\vec{b}}(\vec{f}))(x)$$

$$\leq C\|\vec{b}\|_{BMO^m} \left[\mathcal{M}_{L(logL)}(\vec{f})(x) + M_{\delta+\varepsilon}(T(\vec{f}))(x)\right]$$

Unlike in the linear case, we do not how to interpolate from the end-point estimate in Theorem 8. We can get, nevertheless, a more general result directly from Theorem 9.

Full range of estimates for the commutators

Theorem 10 *Let* $0 < p < \infty$ *and* $w \in A_\infty$, *then*

$$\int_{\mathbb{R}^n} |T_{\vec{b}}(\vec{f})|^p \, w \, dx$$

$$\leq C \|\vec{b}\|_{BMO^m} \int_{\mathbb{R}^n} (\mathcal{M}_{L(logL)}(\vec{f})(x))^p \, w \, dx$$

From this we obtain (using further properties of the $A_{\vec{P}}$ classes)

If $\vec{w} \in A_{\vec{P}}$ with $\vec{P} > \vec{1}$, then

$$\|T_{\vec{b}}(\vec{f})\|_{L^p(u_{\vec{w}})} \leq C \|\vec{b}\|_{BMO^m} \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}$$

In particular,

$$T_{\vec{b}} : L^{p_1} \times \cdots \times L^{p_m} \to L^p,$$

for all $1 < p_1, \ldots, p_m < \infty$, $\quad \frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$.

That is, we get the full range $p > 1/m$!!!