

Discrete fractional integral operators

Lillian Pierce
Princeton University

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Define operators for $0 < \lambda < 1$, by

$$I_\lambda f(n) = \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \frac{f(n - m^2)}{|m|^\lambda}, \quad n \in \mathbb{Z}$$

$$J_\lambda f(n, t) = \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \frac{f(n - m, t - m^2)}{|m|^\lambda}, \quad n, t \in \mathbb{Z}$$

Continuous analogues

$$\mathcal{I}_\lambda(f)(x) = \int f(x - y^2) \frac{dy}{|y|^\lambda}$$

$$\mathcal{J}_\lambda(x, t) = \int f(x - y, t - y^2) \frac{dy}{|y|^\lambda}$$

A (very) brief history

- * Arkhipov and Oskolkov (1987)
- * Bourgain (1988, 1989)
- * Stein and Wainger (1998, 2000, 2002)
- * Oberlin (2001)
- * Ionescu and Wainger (2005)

Operators with quadratic forms

Given Q_1, Q_2 positive definite quadratic forms with integer coefficients, define for $0 < \lambda < 1$

$$I_{Q_1, Q_2, \lambda} f(n) = \sum_{\substack{m \in \mathbb{Z}^k \\ m \neq 0}} \frac{f(n - Q_1(m))}{Q_2(m)^{k\lambda/2}}$$

$$J_{Q_1, Q_2, \lambda} f(n, t) = \sum_{\substack{m \in \mathbb{Z}^k \\ m \neq 0}} \frac{f(n - m, t - Q_1(m))}{Q_2(m)^{k\lambda/2}}$$

Examples

$$Q_1(x) = Q_2(x) = |x|^2$$

$$I_\lambda f(n) = \sum_{\substack{m \in \mathbb{Z}^k \\ m \neq 0}} \frac{f(n - |m|^2)}{|m|^{k\lambda}}$$

$$J_\lambda f(n, t) = \sum_{\substack{m \in \mathbb{Z}^k \\ m \neq 0}} \frac{f(n - m, t - |m|^2)}{|m|^{k\lambda}}$$

Theorem 1. For $k \geq 4$, $I_{Q_1, Q_2, \lambda}$ is a bounded operator from $\ell^p(\mathbb{Z})$ to $\ell^q(\mathbb{Z})$ precisely when $\lambda > (k-2)/k$ and

- (i) $1/q \leq 1/p - \frac{k}{2}(1 - \lambda)$
- (ii) $1/q < 1 - \frac{k}{2}(1 - \lambda)$, $1/p > \frac{k}{2}(1 - \lambda)$.

There exists a constant A_{Q_1, Q_2} such that for $f \in \ell^p(\mathbb{Z})$,

$$\|I_{Q_1, Q_2, \lambda} f\|_{\ell^q} \leq A_{Q_1, Q_2} \|f\|_{\ell^p}.$$

The above also hold for $k = 2, 3$ when $\lambda > 1/2$.

Theorem 2. For $k \geq 2$, the operator $J_{Q_1, Q_2, \lambda}$ is a bounded operator from $\ell^p(\mathbb{Z}^{k+1})$ to $\ell^q(\mathbb{Z}^{k+1})$ when $2/(k+4) < \lambda < 1$ and

- (i) $1/q \leq 1/p - \frac{k}{k+2}(1 - \lambda)$
- (ii) $1/q < \lambda$, $1/p > 1 - \lambda$.

There exists a constant A_{Q_1, Q_2} such that for $f \in \ell^p(\mathbb{Z}^{k+1})$,

$$\|J_{Q_1, Q_2, \lambda} f\|_{\ell^q} \leq A_{Q_1, Q_2} \|f\|_{\ell^p}.$$

Quadratic forms

Q = positive definite quadratic form with integer coefficients:

$$Q(x) = \frac{1}{2}A[x] = \frac{1}{2}x^t Ax$$

A = symmetric positive definite integer matrix, with even diagonal entries

Q^* = adjoint quadratic form $\leftrightarrow A^{-1}$

Comparisons

For Q_1, Q_2 positive definite forms, there exist $c_1, c_2 > 0$ s.t. for all x ,

$$c_1 Q_1(x) \leq Q_2(x) \leq c_2 Q_1(x).$$

At the operator level

$$\|I_{Q,Q_1,\lambda}\|_{p,q} \approx \|I_{Q,Q_2,\lambda}\|_{p,q}$$

$$\|J_{Q,Q_1,\lambda}\|_{p,q} \approx \|J_{Q,Q_2,\lambda}\|_{p,q}$$

Fix Q : consider $I_{Q,\lambda}, J_{Q,\lambda}$.

Necessary conditions

Condition (ii) for $J_{Q,\lambda}$: $1/q < \lambda, 1/p > 1 - \lambda$

Define

$$f(n, t) = \begin{cases} 1 & \text{if } (n, t) = (0, 0) \\ 0 & \text{otherwise.} \end{cases}$$

Then $f \in \ell^p(\mathbb{Z}^{k+1})$ for all $p > 1$.

$$\begin{aligned} \|J_{Q,\lambda} f(n, t)\|_q^q &= \sum_{\substack{n \in \mathbb{Z}^k, t \in \mathbb{Z} \\ Q(n)=t}} Q(n)^{-k\lambda q/2} \\ &= \sum_t t^{-k\lambda q/2} \sum_{\substack{n \\ Q(n)=t}} 1 \\ &= \sum_t t^{-k\lambda q/2} r_{Q,k}(t) \end{aligned}$$

where $r_{Q,k}(n) = \#\{\text{representations of } n \text{ by } Q\}$.

If $r_{Q,k}(n) \approx n^{k/2-1}$ then this converges when

$$q > 1/\lambda.$$

Representations by a quadratic form

$(k \geq 4)$

Let $r_{Q,k}(n) = \# \{ \text{representations of } n \text{ by } Q \}$

For any $n > 0$,

$$r_{Q,k}(n) = \frac{(2\pi)^{k/2} n^{k/2-1}}{\Gamma(k)|A|^{1/2}} \mathfrak{S}(Q, n) + o(n^{k/4-1/4})$$

where $|A| = \det(A)$.

Singular series $\mathfrak{S}(Q, n)$

$$\mathfrak{S}(Q, n) = \sum_{a=1}^{\infty} a^{-k} g_a(Q, n)$$

where

$$g_a(Q, n) = \sum_{d \pmod{a}} \sum_{h \pmod{a}} e^{2\pi i d(Q(h)-n)/a}.$$

When is $\mathfrak{S}(Q, n) \neq 0?$

Often enough: for a positive density of n .

Sufficient conditions: the general idea

Multiplier of $I_{Q,\lambda}$

$$m_{Q,\lambda}(\theta) = \sum_{\substack{m \in \mathbb{Z}^k \\ m \neq 0}} \frac{e^{-2\pi i Q(m)\theta}}{Q(m)^{k\lambda/2}}$$

Goal: show $m_\lambda \in L^{r,\infty}[0, 1]$ for $r = \frac{2}{k(1-\lambda)}$.
 Then $I_{Q,\lambda} : \ell^p \rightarrow \ell^q$ for $1/q = 1/p - 1/r$.

Identity

$$Q(m)^{-k\lambda/2} = c_{k,\lambda} \int_0^\infty e^{2\pi Q(m)y} y^{k\lambda/2-1} dy$$

Restrict attention the multiplier

$$\sum_{j=1}^{\infty} \int_{2^{-j}}^{2^{-j+1}} \Theta(y - i\theta) y^{k\lambda/2-1} dy$$

Theta function

$$\Theta(z) = \sum_{m \in \mathbb{Z}^k} e^{-2\pi Q(m)z}$$

Theta function

$$\Theta(z) = \Theta_Q(z) = \sum_{m \in \mathbb{Z}^k} e^{-2\pi Q(m)z}$$

Jacobi inversion formula

$$\Theta_Q(z) \longleftrightarrow \Theta_{Q^*}(1/z)$$

$$Q \leftrightarrow A$$

$$Q^* \leftrightarrow A^{-1} \text{ adjoint form}$$

Transformation Law

For $\Re(z) > 0$, $m = lq + r$, q a fixed modulus,

$$\begin{aligned} & \sum_{l \in \mathbb{Z}^k} e^{-2\pi Q(lq+r)z} \\ &= \frac{1}{q^k |A|^{1/2} z^{k/2}} \sum_{l \in \mathbb{Z}^k} e^{2\pi i r \cdot l / q} e^{-2\pi Q^*(l)/(q^2 z)} \end{aligned}$$

A taste of the circle method

Dirichlet approximation:

For every $\theta \in [0, 1]$ there exist $1 \leq q \leq 2^{j/2}$, and $1 \leq a \leq q$ with $(a, q) = 1$, such that

$$|\theta - a/q| \leq 1/(q2^{j/2}).$$

Major arcs: θ “close” to “small” q

For $1 \leq q \leq 2^{j/2-10}$, define

$$\mathfrak{M}_j(a; q) = \{\theta : |\theta - a/q| \leq 1/(q2^{j/2})\}$$

* disjoint if $(a_1, q_1) \neq (a_2, q_2)$

Minor arcs: θ “close” to “large” q

$$2^{j/2-10} < q \leq 2^{j/2}$$

Gauss sum

$$S_Q(a; q) = \sum_{r \pmod{q^k}} e^{2\pi i Q(r)a/q}$$

Classical bound

$$|S_Q(a; q)| \leq B_Q q^{k/2}$$

Approximate Identity for Θ

For $2^{-j} \leq y \leq 2^{-j+1}$ and θ, a, q as above:

$$\Theta(y - i\theta) = \frac{S_Q(a; q)^k}{q^k |A|^{1/2} (y - i(\theta - a/q))^{k/2}} + O(y^{-k/4})$$

Three types of contribution to the multiplier

- * Main term of $\Theta(y - i\theta)$ for θ in major arcs
- * Main term of $\Theta(y - i\theta)$ for θ in minor arcs
- * Remainder term of $\Theta(y - i\theta)$

Main term on minor arcs

$$\sum_{j=1}^{\infty} 2^{jk/4} \int_{2^{-j}}^{2^{-j+1}} y^{k\lambda/2-1} dy = O(1) \text{ if } \lambda > 1/2$$

Main term on major arcs

$$\sum_{s=0}^{\infty} \sum_{2^s \leq q < 2^{s+1}} \sum_{\substack{1 \leq a \leq q \\ (a,q)=1}} \frac{|S(a; q)|^k}{q^k} |\theta - a/q|^{-\frac{k}{2}(1-\lambda)} \chi_{a/q}(\theta)$$

The operator $J_{Q,\lambda}$: $\lambda < 1/2$

Multiplier $m_{Q,\lambda}(\theta, \phi)$ for $\theta \in [0, 1]$, $\phi \in [0, 1]^k$:

$$m_{Q,\lambda}(\theta, \phi) = \sum_{\substack{m \in \mathbb{Z}^k \\ m \neq 0}} \frac{e^{2\pi i(Q(m)\theta + m \cdot \phi)}}{Q(m)^{k\lambda/2}}$$

Reduce consideration to multiplier

$$\sum_{j=1}^{\infty} \int_{2^{-j}}^{2^{-j+1}} \Theta(y - i\theta, \phi) y^{k\lambda/2 - 1} dy$$

Theta function

$$\Theta(z, \phi) = \sum_{m \in \mathbb{Z}^k} e^{-2\pi Q(m)z} e^{2\pi i m \cdot \phi}$$

Gauss sum

$$S_Q(a, b; q) = \sum_{r \pmod{q}^k} e^{2\pi i Q(r)a/q} e^{2\pi i r \cdot b/q}$$

Classical bound

$$|S_Q(a, b; q)| \leq B_Q q^{k/2}$$

Approximate identity for Θ :

For (θ, ϕ) , pick a, b, q such that $|\theta - a/q| \leq 1/(q2^{j/2})$, $(a, q) = 1$, and $|\phi - b/q| \leq 1/(2q)$, where $1 \leq q \leq 2^{j/2}$. Then if $2^{-j} \leq y \leq 2^{-j+1}$,

$$\Theta(y - i\theta, \phi) = \frac{S_Q(a, b; q)}{q^k |A|^{1/2}} \cdot \frac{e^{-2\pi Q^*(\beta)/(y - i\alpha)}}{(y - i\alpha)^{\frac{k}{2}}} + R_\lambda.$$

Major arcs: $\mathfrak{M}_j(a, b; q)$ for $1 \leq q \leq 2^{j/2-10}$

Minor arcs: $2^{j/2-10} < q \leq 2^{j/2}$

Notation: $\alpha = \theta - a/q$ $\beta = \phi - b/q$

Multiplier for main term on major arcs:

$$M_\lambda(\theta, \phi) = \sum_{s=0}^{\infty} B_\lambda(s; \theta, \phi)$$

with

$$B_\lambda(s; \theta, \phi) = \sum_{2^s \leq q < 2^{s+1}} \sum_{\substack{a=1 \\ (a,q)=1}}^q \sum_{b \pmod{q}^k} \frac{S_Q(a, b; q)}{q^k |A|^{1/2}} \cdot \int_{\rho(s, \alpha)}^{2^{-2s-19}} \frac{y^{k\lambda/2-1} e^{-2\pi Q^*(\beta)/(y - i\alpha)}}{(y - i\alpha)^{\frac{k}{2}}} dy$$

Define family of operators $\mathcal{B}_{\lambda,s}$

$$(\mathcal{B}_{\lambda,s}f)^{\wedge}(\theta, \phi) = B_{\lambda}(s; \theta, \phi)\widehat{f}(\theta, \phi).$$

Sufficient to show:

For $2/(k+4) < \lambda < 1/2$, and $1/p + 1/p' = 1$
with $1/p' = 1/p - \frac{k}{k+2}(1-\lambda)$,

$$\|\mathcal{B}_{\lambda,s}f\|_{\ell^{p'}} \leq C 2^{-\delta(\lambda)s} \|f\|_{\ell^p}, \quad \delta(\lambda) > 0.$$

Two interpolation estimates

For $\lambda = 1 + i\gamma$:

$$\|\mathcal{B}_{\lambda,s}f\|_{\ell^2} \leq A 2^{-a(\lambda)s} \|f\|_{\ell^2}.$$

- * direct bound for the integral in $B_{\lambda}(s; \theta, \phi)$
- * decay both from Gauss sum and a
“double decomposition” of the integral

For $\lambda = -2/k + i\gamma$:

$$\|\mathcal{B}_{\lambda,s}f\|_{\ell^\infty} \leq B 2^{+b(\lambda)s} \|f\|_{\ell^1}.$$

- * bound of the *Fourier coefficients* of the multiplier
- * growth “not too bad”

A difficulty

Multiplier:

$$L_\lambda(\alpha, \beta) = \chi(c2^{2s}\alpha) \cdot \int_{\rho(s, \alpha)}^{2^{-2s-19}} \frac{y^{k\lambda/2-1} e^{-2\pi Q^*(\beta)/(y-i\alpha)}}{(y-i\alpha)^{\frac{k}{2}}} dy$$

where $\rho(s, \alpha) \approx 2^{2s}\alpha^2$.

Fourier transform with respect to β :

$$L_\lambda^{\widehat{\beta}}(\alpha, \eta) = |A|^{1/2} \chi(c2^{2s}\alpha) \cdot \int_{2^{2s}\alpha^2}^{2^{-2s-19}} y^{k\lambda/2-1} e^{-2\pi Q(\eta)(y-i\alpha)} dy.$$

Fourier transform with respect to α :

Show $L_\lambda^{\widehat{\beta}}(\alpha, \eta)$ is integrable.

For $\Re(\lambda) = -2/k$,

$$|L_\lambda^{\widehat{\beta}}(\alpha, \eta)| \approx |A|^{1/2} \chi(c2^{2s}\alpha) \alpha^{-2} 2^{-2s}.$$

A Remedy

$\rho(s, \alpha) = \text{quadratic in } \alpha \rightarrow \text{linear in } \alpha$

Goal: integral $\approx \alpha^{-1+i\gamma}$, where $\gamma = \Im(\lambda) \neq 0$.