Extension Theorems for Paraboloids in the Finite Field Setting

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Extension problems in the Euclidean Setting

- S : a subset of $\mathbb{R}^d$.
  $d\sigma$: a positive measure supported on S.

Question

What are the exponents p and r such that the following estimate holds?

$$\|(fd\sigma)^{\vee}\|_{L^r(\mathbb{R}^d)} \leq C_{p,r}\|f\|_{L^p(S,d\sigma)} \quad \text{for all} \quad f \in L^p(S, d\sigma),$$

where $(fd\sigma)^{\vee}$ is given by the formula

$$(fd\sigma)^{\vee}(\xi) = \int_{x\in S} e^{2\pi i x\cdot\xi} d\sigma(x).$$

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- 2.Definition of Extension Theorems in the Finite Field Setting
- 3.Main Results and the best Possible Results
- Overview of the Proof of the Main Results

Notation and Fourier Analysis Machinary in the Finite Field setting

- $\mathbb{F}_q$: a finite field with q elements with $Char(\mathbb{F}_q) > 2$.
  $\mathbb{F}_q^d$: $d-$dimensional vector space over $\mathbb{F}_q$.

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Definition

We define the paraboloid S in $\mathbb{F}_q^d$ by the set

$$S = \{(\underline{x}, x_d) : \underline{x} \in \mathbb{F}_q^{d-1}, x_d = \underline{x} \cdot \underline{x} \in \mathbb{F}_q\}.$$

Let χ be a nontrivial additive character of $\mathbb{F}_q$. We denote by $d\sigma$ the normalized surface measure on S. Given $f : \mathbb{F}_q^d \to \mathbb{C}$, we define the inverse Fourier Transforms of f and $fd\sigma$ as follows.

$f^\vee(m) = q^{-d} \sum_{x \in \mathbb{F}_q^d} \chi(x \cdot m) f(x)$.

$(fd\sigma)^\vee(m) = \frac{1}{|S|} \sum_{x \in S} \chi(x \cdot m) f(x)$,

where $|S|$ denotes the number of elements in S.

Norms Related to Function Spaces and Frequency Spaces

- We denote by $(\mathbb{F}_q^d, dx)$ a d-dimensional vector space with the normalized counting measure dx, and by $(\mathbb{F}_q^d, dm)$ the dual space with the counting measure dm.

Definition

For each $1 \leq p, r < \infty$, we define

$$\|f\|_{L^p(\mathbb{F}_q^d, dx)}^p = q^{-d} \sum_{x \in \mathbb{F}_q^d} |f(x)|^p,$$

$$\|f^\vee\|_{L^r(\mathbb{F}_q^d, dm)}^r = \sum_{m \in \mathbb{F}_q^d} |f^\vee(m)|^r$$

$$\|f\|_{L^p(S, d\sigma)}^p = \frac{1}{|S|} \sum_{x \in S} |f(x)|^p.$$

We denote by $\|f\|_{L^\infty}$ the maximum value of f.

Definition of Extension Theorems in the Finite Field Setting

- Let $1 \le p, r \le \infty$.

Definition

We define $R^*(p \to r)$ to be the best constant such that the extension estimate

$$\|(fd\sigma)^{\vee}\|_{L^r(\mathbb{F}_q^d, dm)} \le R^*(p \to r)\|f\|_{L^p(S, d\sigma)}$$

holds for all functions f on S.

Question

What is the set of exponents p and r such that $R^*(p \to r) \lesssim 1$?

Results by Mochenhaupt and Tao

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  in which the exponents are known as the standard Stein-Tomas exponents.
- In particular, in the case when $d = 3$, and -1 is not a square number in $\mathbb{F}_q$, they obtained the "p" index improvement of the Stein-Tomas exponents $R^*(2 \to 4)$ by showing that $R^*(\frac{8}{5} \to 4) \lessapprox 1$.

- In higher even dimensions $d \geq 4$, the following theorems give better exponents than the Stein-Tomas exponents.

Theorem

Let S be the paraboloid in $\mathbb{F}_q^d$. If $d \geq 4$ is even, then we have

$$R^*(p \to 4) \lessapprox 1 \quad \text{for all} \quad p \geq \frac{4d}{3d-2},$$

$$R^*(2 \to r) \lessapprox 1 \quad \text{for all} \quad r \geq \frac{2d^2}{d^2-2d+2}.$$

Main Results

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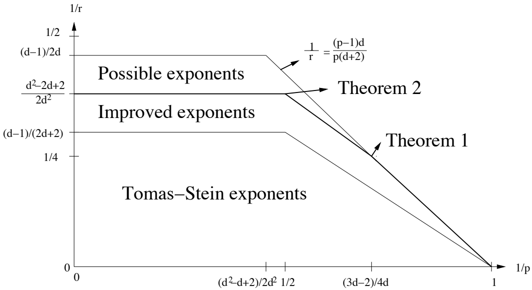
- The following theorem enables us to extend the work by Mockenhaupt and Tao to higher odd dimensions $d \geq 7$.

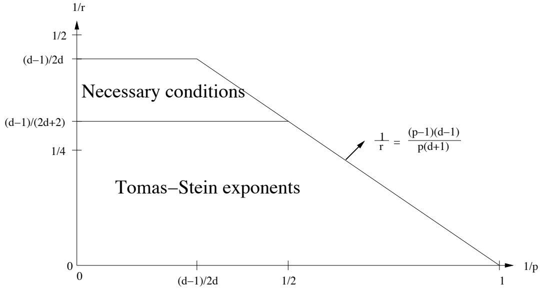
Theorem

If -1 is not a square number in $\mathbb{F}_q$ and the dimensions $d = 4k+3$ for some $k \in \mathbb{N}$, then the conclusions in above theorem hold.

In even dimensions $d \geq 4$, exponents for $R^*(p \to r)$ bound

- 

In Odd Dimensions $d \geq 3$, the Necessary Conditions for $R^*(p \to r)$ bound



Overview of the Proof of $L^p - L^4$ results

- We shall write $E(x)$ for $\chi_E(x)$. Let $S \subset \mathbb{F}_q^d$ be the paraboloid. The usual dyadic pigeonholing argument , it is enough to show that

$$\|(Ed\sigma)^\vee\|_{L^4(\mathbb{F}_q^d, dm)} \lesssim \|E\|_{L^{p_0}(S, d\sigma)}, \quad \text{for all} \quad E \subset S, \tag{1}$$

where $p_0 = \frac{4d}{3d-2}$. Expanding both sizes in (1) and using $|S| = q^{d-1}$, it suffices to show that

$$\Lambda_4(E) \lesssim |E|^{\frac{4}{p_0}} q^{3d-4} q^{\frac{-4d+4}{p_0}} \quad \text{for all} \quad E \subset S, \tag{2}$$

where $\Lambda_4(E) = \sum\limits_{\substack{x,y,z,w \in E \\ : x+y=z+w}} 1$.

- We shall need the following estimate.

Lemma

Let S be the paraboloid in $(\mathbb{F}_q^d, dx)$. In addition, we assume that the dimension of $\mathbb{F}_q^d, d \geq 4$, is even. If E is any subset of S then we have

$$\Lambda_4(E) \lesssim \min\{|E|^3, q^{-1}|E|^3 + q^{\frac{d-2}{4}}|E|^{\frac{5}{2}} + q^{\frac{d-2}{2}}|E|^2\}.$$

Note that above Lemma implies that if $d \geq 4$ is even and E is any subset of the paraboloid S, then

$$\Lambda_4(E) \lesssim \begin{cases} q^{-1}|E|^3 & \text{if} \quad q^{\frac{d+2}{2}} \lesssim |E| \lesssim q^{d-1} \\ q^{\frac{d-2}{4}}|E|^{\frac{5}{2}} & \text{if} \quad q^{\frac{d-2}{2}} \lesssim |E| \lesssim q^{\frac{d+2}{2}} \\ |E|^3 & \text{if} \quad 1 \lesssim |E| \lesssim q^{\frac{d-2}{2}}. \end{cases}$$

Using these upper bounds of $\Lambda_4(E)$ depending on the size of the subset of S, the inequality in (2) follows by the direct calculation.