## Sum-Product Theory in Finite Fields

## Derrick Hart (Joint work with Alex Iosevich)

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$$A + A = \{a + a' : a, a' \in A\}$$
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• A conjecture due to Erdős and Szemeredi says that the answer is no.

#### Conjecture

With the notation above,

$$\max\{|A+A|, |A\cdot A|\} \gtrsim |A|^{2-\epsilon}.$$

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$$\max\{|A+A|, |A\cdot A|\} \gtrsim |A|^{\frac{14}{11}-\epsilon},$$

based partly on the idea to Elekes using the Szemeredi-Trotter Incidence Theorem, who proved the same estimate with a slightly worse exponent  $\frac{5}{4}$ .

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Finite Field Case

Theorem (Bourgain-Katz-Tao)

If  $A \subset \mathbb{Z}_p$ , p a prime, and  $p^{\epsilon} \leq |A| \leq p^{1-\epsilon}$ , for some  $\epsilon > 0$ , then there exists  $\delta > 0$  such that

$$\max\{|A+A|, |A\cdot A|\} \gtrsim |A|^{1+\delta}.$$

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# Explicit Bounds

• Using incidences between points and hyperbolae in the plane the author along with Alex losevich and Joszef Solymosi proved that if  $A \subset \mathbb{F}_q$ , a finite field with q elements, then

 $\max\{|A + A|, |A \cdot A|\} \gtrsim \min\{|A|^{\frac{3}{2}}q^{-\frac{1}{4}}, |A|^{\frac{2}{3}}q^{\frac{1}{3}}\}.$ 

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• The above results yield non-trivial results only in the case that  $|A| > q^{1/2}$  as one would expect with the existence of subfields of size  $q^{1/2}$ . In the case of prime fields however, one may get results in the lower range. The current best result due to Katz and Shen based on an improvement of a method of Garaev yields the for  $|A| < q^{1/2}$ ,

$$\max\{|A+A|,|A\cdot A|\}\gtrsim |A|^{\frac{14}{13}-\epsilon}.$$

## Sum-product basis in Finite Fields

 Let 𝔽<sub>q</sub> be the finite field with q elements. How large does A ⊂ 𝔽<sub>q</sub> need to be so that

$$\mathbb{F}_q = dA^2 = A \cdot A + A \cdot A \cdots + A \cdot A?$$

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• Many results pertaining to this and related questions, under a variety of assumptions, have been published in recent years by Bourgain, Croot, Glibichuk, Konyagin, Shkredov, Tao, Vu and others. For  $d \ge 8$  the problem was solved recently by Glibichuk extending earlier results of Glibichuk and Konyagin for prime fields.

### Theorem (Glibichuk)

If  $A \subset \mathbb{F}_q^*$ , then

$$\mathbb{F}_q = 8A^2 \text{ if } |A| > \sqrt{2}q^{\frac{1}{2}}.$$

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# Short Sum-product basis

- What about when *d* is small?
- Bourgain proved(specifically with d=3) using one-dimensional exponential sums that if q is prime and A ⊂ F<sup>\*</sup><sub>q</sub>, then

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• The author and Alex losevich recently proved the stronger result that if  $A \subset \mathbb{F}_q^*$ , then

$$\mathbb{F}_q^* \subset dA^2 \text{ if } |A| > q^{\frac{1}{2} + \frac{1}{2d}}, \quad \text{and} \quad |dA^2| > \frac{q}{2} \text{ if } |A| > q^{\frac{1}{2} + \frac{1}{2(2d-1)}}.$$

## Sums and products-higher dimensional perspective

• Our idea is to take a higher dimensional perspective. Let  $E \subset \mathbb{F}_q^d$ , the *d*-dimensional vector space over  $\mathbb{F}_q$ . Define

$$\Pi(E) = \{x \cdot y : x, y \in E\}.$$

In this context we ask how large does E need to be to assure that  $\Pi(E)$  is large?

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• Our main result is the following:

#### Theorem

Let  $E \subset \mathbb{F}_q^d$ . Then

$$\mathbb{F}_q^*\subset \Pi(E) \, \, \textit{if} \, \, |E|>q^{rac{d+1}{2}},$$

and if E is a product set,

$$|\Pi(E)| > \frac{q}{2} \text{ if } |E| > q^{\frac{d^2}{2d-1}}.$$

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• Taking  $E = A \times A \dots \times A$  yields the arithmetic result.

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the Radon transform of E.

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In this case:

$$\mathcal{R}: L^2(\mathbb{R}^d) \to L^2_{rac{d-1}{2}}(\mathbb{R}^d)$$

and a suitable analog holds in the finite field setting.

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\vdots &: \\
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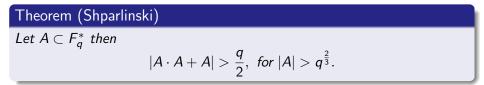
• Simple but important observation: if  $E = A \times \ldots \times A$ ,

$$|E\cap I_k|\leq |A|.$$

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## Open question

## • It is possible to sharpen the positive proportion result. For example



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# Theorem (Shparlinski)Let $A \subset F_q^*$ then $|A \cdot A + A| > \frac{q}{2}$ , for $|A| > q^{\frac{2}{3}}$ .

#### Question

Let  $A \subset F_a^*$  then does there exist an  $1/2 > \epsilon > 0$  such that

$$\mathbb{F}_q^* \subseteq A \cdot A + A$$
, for  $|A| > q^{1-\epsilon}$ .

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