WORKSHOP ON HARMONIC ANALYSIS

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Rough and Rougher Singular Integrals

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where the kernel satisfies the *size condition*:

$$|K(x,y)| \le A\,|x-y|^{-n}$$

and the *smoothness condition* (for some $\delta > 0$)

$$|K(x,y) - K(x,y')| \le \frac{|y-y'|^{\delta}}{(|x-y| + |x-y'|)^{n+\delta}}$$

whenever $|x-x'| \le \frac{1}{2}\max(|x-y|, |x'-y|)$. Assume that $K^t(x,y) = K(y,x)$ also satisfies the same condition.

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Then T maps L^1 to weak L^1;
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Same conclusion is valid if smoothness is replaced by the weaker *Hörmander condition*

$$\sup_{y_0 \neq 0} \int_{|x-y| \geq 2|y-y_0|} |K(x,y) - K(x,y_0)| \, dx = A < \infty$$

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Assume that for some $c > 0$

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where K_t is the kernel of TA_t.

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Theorem: If T satisfies this generalized Hörmander condition and T is L^2 bounded, then it is of weak type $(1,1)$.

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(c) Fefferman's “Inequalities for Strongly Singular Convolution Operators" Acta Math. **124** (1970), 9–36.

MULTILINEAR CALDERÓN-ZYGMUND OPERATORS

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$$|K(y_0, y_1, \ldots, y_m) - K(y_0, y_1', \ldots, y_m)| \leq \frac{A|y_1 - y_1'|^{\epsilon}}{(|y_0 - y_1| + \cdots + |y_0 - y_m|)^{mn+\epsilon}}$$

whenever $|y_1 - y_1'| \leq \frac{1}{2} \max\limits_{1 \leq j \leq m} |y_0 - y_j|$

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Theorem: If T is bounded on some product of Lebesgue spaces, then T is of weak type $(1, 1, \ldots, 1/m)$ and bounded on all products of Lebesgue spaces (with R. Torres/Kenig and Stein).

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OPEN QUESTION ON THE SUBJECT:

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OPEN QUESTION ON THE SUBJECT:

Whether the smoothness condition can be replaced by a Hörmander-type condition.

DUONG AND M$^{\mathrm{c}}$INTOSH m-LINEAR SETTING

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Assume there exist A_t, $t > 0$ with kernels $a_t(x,y)$ such that

$$|a_t(x,y)| \leq h_t(x,y) = \frac{t^{-n}}{(1 + t^{-1}|x-y|)^{n+\varepsilon}}$$

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and also if $K_t^{(j)}$ is the kernel of $T \circ_j A_t$

$$\begin{aligned}|K(x, y_1, \ldots, y_m) - K_t^{(j)}(x, y_1, \ldots, y_m)| \\ \leq \frac{A}{(|x-y_1| + \cdots + |x-y_m|)^{mn}} \sum_{\substack{k=1 \\ k \neq j}}^{m} \phi\Big(\frac{|y_j - y_k|}{t}\Big) \\ + \frac{A\, t^{\epsilon}}{(|x-y_1| + \cdots + |x-y_m|)^{mn+\epsilon}}\end{aligned}$$

for some $A > 0$, whenever $t \leq |x - y_j|/2$; supp$(\phi) \subset [-1, 1]$.

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Theorem: (with X. T. Duong and L. X. Yan) Under preceding conditions, T is of weak type $(1, 1, \ldots, 1, 1/m)$ provided it maps $L^{q_1} \times \cdots \times L^{q_m}$ to L^q for *some choice* of indices with $1 \leq q_j \leq \infty$.

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Fundamental example: Commutators of Calderón:

$$\mathcal{C}_{m+1}(f, a_1, \ldots, a_m)(x) = \int_{\mathbb{R}} \Big[\prod_{j=1}^{m} \frac{A_j(x) - A_j(y)}{(x-y)^{m+1}} \Big] f(y)\, dy$$

where $x \in \mathbf{R}$ and $A_j' = a_j$.

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$$K(y_0,\ldots,y_{m+1}) = \frac{(-1)^{m\chi_{y_{m+1}>y_0}}}{(y_0-y_{m+1})^{m+1}}\prod_{\ell=1}^{m}\chi_{\left(\min(y_0,y_{m+1}),\max(y_0,y_{m+1})\right)}(y_\ell)$$

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Let $\phi \in C^\infty(\mathbb{R})$ be even, $0 \le \phi \le 1$, $\phi(0) = 1$ and supp$(\phi) \subset [-1,1]$. We set $\Phi = \phi'$ and $\Phi_t(x) = t^{-1}\Phi(x/t)$. Define,

$$A_t(f)(x) \quad = \quad \int_{\mathbb{R}} a_t(x,y)f(y)dy \quad \text{where} \quad a_t(x,y) = \Phi_t(x-y)\chi_{(x,\infty)}(y)$$

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Theorem: The operator $\mathcal{C}_m$ maps

(a) $L^1 \times \cdots \times L^1$ to weak $L^{1/m}$.

(b) $L^{p_1} \times \cdots \times L^{p_m}$ to weak L^p if some $p_j = 1$.

(c) $L^{p_1} \times \cdots \times L^{p_m}$ to L^p if all $p_j > 1$.

Here, $1 \leq p_j \leq \infty$ and

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History: $m = 1$: C. Calderón
History: $m = 2, 3$: Coifman and Meyer

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We want to estimate

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Apply the CZ decomposition on each f_j at height $(\alpha\lambda)^{1/m}$ for some α to be chosen.

Write for $j = 1, \ldots, m$

$$f_j = g_j + b_j$$

and assume that $\|f_j\|_{L^1} = 1$ (by scaling).

ESTIMATES TO PROVE

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\begin{aligned}
E_\lambda^{(1)} &= \left\{x \in \mathbb{R}^n : |T(g_1, g_2, ..., g_m)(x)| > \lambda/2^m\right\} \\
E_\lambda^{(2)} &= \left\{x \in \mathbb{R}^n : |T(b_1, g_2, ..., g_m)(x)| > \lambda/2^m\right\} \\
E_\lambda^{(3)} &= \left\{x \in \mathbb{R}^n : |T(g_1, b_2, ..., g_m)(x)| > \lambda/2^m\right\} \\
& \qquad \cdots\cdots\cdots\cdots\cdots \\
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We need to show that $|E_\lambda^{(r)}| \leq C\,(A + B)\,\lambda^{-1/m}$.

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The idea is better presented when $m = 2$.

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For linear operators, one writes $T(f) = T(g) + T(b)$

$$\begin{aligned} T(g) &= \text{good term} \\ T(b) &= \text{bad term} \end{aligned}$$

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Here (say in the case $m = 2$)

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But there is also the

$$T(b_1, b_2) = \text{ugly}$$

THE UGLY TERM

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Write $T(b_1,\ b_2) = T\Big(\sum_k b_{1,k},\ \sum_j b_{2,j}\Big) = \sum_{i=1}^{5} T_i(b_1, b_2)\,,$ where

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$$T_1(b_1,b_2) \;=\; T\Big(\sum_k A_{t_{1,k}}(b_{1,k})\,,\ \sum_j A_{t_{2,j}}(b_{2,j})\Big)$$

$$T_2(b_1,b_2) \;=\; \sum_k \sum_{j:\,\ell(Q_{1,k})\le\ell(Q_{2,j})} T\Big(A_{t_{1,k}}(b_{1,k})\,,\ b_{2,j} - A_{t_{2,j}}(b_{2,j})\Big)$$

$$T_3(b_1,b_2) \;=\; \sum_k \sum_{j:\,\ell(Q_{1,k})\le\ell(Q_{2,j})} T\Big(b_{1,k} - A_{t_{1,k}}(b_{1,k})\,,\ b_{2,j}\Big)$$

$$T_4(b_1,b_2) \;=\; \sum_k \sum_{j:\,\ell(Q_{1,k})>\ell(Q_{2,j})} T\Big(b_{1,k} - A_{t_{1,k}}(b_{1,k})\,,\ A_{t_{2,j}}(b_{2,j})\Big)$$

$$T_5(b_1,b_2) \;=\; \sum_k \sum_{j:\,\ell(Q_{1,k})>\ell(Q_{2,j})} T\Big(b_{1,k}\,,\ b_{2,j} - A_{t_{2,j}}(b_{2,j})\Big),$$

where $t_{1,k} = \ell(Q_{1,k})$ and $b_{i,k}$ is supported in $Q_{i,k}$, $i = 1, 2$.

THE MARCINKIEWICZ FUNCTION

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The Marcinkiewicz function associated with $\{Q_{i,k}\}_k$.

$$\mathcal{J}_{i,\epsilon}(x) = \sum_k \frac{\ell(Q_{i,k})^{n+\epsilon}}{(\ell(Q_{i,k}) + |x - x_{Q_{i,k}}|)^{n+\epsilon}}, \qquad i = 1, 2,$$

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where $x_{Q_{i,k}}$ is the center of $Q_{i,k}$.

Fefferman and Stein proved that for $n/(n+\epsilon) < p < \infty$ we have

$$\|\mathcal{J}_{i,\epsilon}\|_{L^p(\mathbb{R}^n)} \leq C_{n,\varepsilon,p}(\sum_k |Q_{i,k}|)^{1/p} \leq C_{n,\varepsilon,p}\,(\alpha\lambda)^{-1/2p}.$$

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The Marcinkiewicz function comes appears as follows:

$$
\begin{aligned}
|\sum_k A_{t_{1,k}}(b_{1,k})(x)| &\leq C\sum_k \int_{\mathbb{R}^n} \frac{\ell(Q_{1,k})^\epsilon}{(\ell(Q_{1,k}) + |x-y|)^{n+\epsilon}} |b_{1,k}(y)|dy \\
&\leq C\sum_k \frac{\ell(Q_{1,k})^\epsilon}{(\ell(Q_{1,k}) + |x-x_{Q_{1,k}}|)^{n+\epsilon}} \|b_{1,k}\|_{L^1(\mathbb{R}^n)} \\
&\leq C(\alpha\lambda)^{1/2}\mathcal{J}_{1,\epsilon}(x),
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\end{aligned}
$$

This allows us to control terms of the form

$$
T\Big(\sum_k A_{t_{1,k}}(b_{1,k}), g_2\Big), \quad T\Big(g_1, \sum_k A_{t_{2,k}}(b_{2,k})\Big)
$$

and

$$
T\Big(\sum_k A_{t_{1,k}}(b_{1,k}), \sum_k A_{t_{2,k}}(b_{2,k})\Big)
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\begin{aligned}
& \left|\left\{x \in \mathbb{R}^n : \left|T\Big(\sum_k A_{t_{1,k}}(b_{1,k})\,,\; g_2\Big)(x)\right| > \frac{\lambda}{4}\right\}\right| \\
\leq \quad & (4B)^q \lambda^{-q} \Big\| \sum_k A_{t_{1,k}}(b_{1,k}) \Big\|^q_{L^{q_1}(\mathbb{R}^n)} \|g_2\|^q_{L^{q_2}(\mathbb{R}^n)} \\
\leq \quad & C \lambda^{-q} B^q (\alpha\lambda)^{q/2} \|\mathcal{J}_{1,\epsilon}\|^q_{L^{q_1}(\mathbb{R}^n)} (\alpha\lambda)^{\frac{q}{2}(1-\frac{1}{q_2})} \\
\leq \quad & C \lambda^{-q} B^q (\alpha\lambda)^{\frac{q}{2}(1-\frac{1}{q_1})} (\alpha\lambda)^{\frac{q}{2}(1-\frac{1}{q_2})} \\
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Note that one needs here that $q_1 < \infty$ (analogously $q_2 < \infty$).

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\begin{aligned}
&\left|\left\{x \in \mathbb{R}^n : \left|T\Big(\sum_k A_{t_{1,k}}(b_{1,k})\,,\; g_2\Big)(x)\right| > \frac{\lambda}{4}\right\}\right| \\
\le \quad & (4B)^q \lambda^{-q} \Big\| \sum_k A_{t_{1,k}}(b_{1,k}) \Big\|^q_{L^{q_1}(\mathbb{R}^n)} \|g_2\|^q_{L^{q_2}(\mathbb{R}^n)} \\
\le \quad & C\lambda^{-q} B^q (\alpha\lambda)^{q/2} \|\mathcal{J}_{1,\epsilon}\|^q_{L^{q_1}(\mathbb{R}^n)} (\alpha\lambda)^{\frac{q}{2}(1-\frac{1}{q_2})} \\
\le \quad & C\lambda^{-q} B^q (\alpha\lambda)^{\frac{q}{2}(1-\frac{1}{q_1})} (\alpha\lambda)^{\frac{q}{2}(1-\frac{1}{q_2})} \\
\le \quad & C' B^q \lambda^{-1/2} \alpha^{q-\frac{1}{2}}.
\end{aligned}
$$

Note that one needs here that $q_1 < \infty$ (analogously $q_2 < \infty$).

Note that $\alpha = (A+B)^{-1}$ yields the claimed constant.

TERMS CONTAINING $\sum_k \left(b_{1,k} - A_{t_{1,k}}(b_{1,k})\right)$

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Recall

$$
\begin{aligned}
&|K(x, y_1, \ldots, y_m) - K_t^{(j)}(x, y_1, \ldots, y_m)| \\
&\qquad \le \frac{A\, t^\epsilon}{(|x - y_1| + \cdots + |x - y_m|)^{mn+\epsilon}} \\
&\qquad\quad + \frac{A}{(|x - y_1| + \cdots + |x - y_m|)^{mn}} \sum_{\substack{k=1 \\ k \neq j}}^{m} \phi\Big(\frac{|y_j - y_k|}{t}\Big)
\end{aligned}
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for some $A > 0$, whenever $t \le |x - y_j|/2$; supp$(\phi) \subset [-1, 1]$.

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for some $A>0$, whenever $t \le |x-y_j|/2$; supp$(\phi) \subset [-1,1]$.

Take $x \notin \bigcup_k (Q_{1,k})^* \cup \bigcup_k (Q_{2,k})^* \; =$ exceptional set.

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\begin{aligned}
&\sum_k \int_{(Q^*_{1,k})^c} \left|T(b_{1,k} - A_{t_{1,k}}(b_{1,k})\,,\ g_2)(x)\right| dx \\
\le &\sum_k \int_{(\mathbb{R}^n)^2} \Big[\int_{(Q^*_{1,k})^c} \left|K(x,y_1,y_2) - K^{(1)}_{t_{1,k}}(x,y_1,y_2)\right| dx \Big] |b_{1,k}(y_1)||g_2(y_2)| dy_1 dy_2 \\
\le &CA \sum_k \int_{(\mathbb{R}^n)^2} \Big[\int_{\mathbb{R}^n} \frac{\ell(Q_{1,k})^\epsilon dx}{(|x-y_1| + \ell(Q_{1,k}) + |y_1 - y_2|)^{2n+\epsilon}} \Big] |b_{1,k}(y_1)||g_2(y_2)| dy_1 dy_2 \\
+ &CA \sum_k \int_{(\mathbb{R}^n)^3} \frac{1}{(|x-y_2| + \ell(Q_{1,k}))^{2n}} \phi\Big(\frac{|y_1 - y_2|}{\ell(Q_{1,k})} \Big) |b_{1,k}(y_1)||g_2(y_2)| dy_1 dy_2 dx \\
= &I + II.
\end{aligned}
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I &\le \sum_k \int_{(\mathbb{R}^n)^2} \frac{CA\,\ell(Q_{1,k})^\epsilon}{(\ell(Q_{1,k}) + |y_1 - y_2|)^{n+\epsilon}} |b_{1,k}(y_1)||g_2(y_2)|dy_1 dy_2 \\
&\le \sum_k \int_{(\mathbb{R}^n)^2} \frac{CA\,\ell(Q_{1,k})^\epsilon}{(\ell(Q_{1,k}) + |x_{Q_{1,k}} - y_2|)^{n+\epsilon}} |b_{1,k}(y_1)||g_2(y_2)|dy_1 dy_2 \\
&\le CA(\alpha\lambda)^{1/2} \int_{\mathbb{R}^n} |g_2(y_2)|\mathcal{J}_{1,\epsilon}(y_2) dy_2 \le CA(\alpha\lambda)^{1/2}.
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II &\leq CA\sum_k \int_{(\mathbb{R}^n)^2} \frac{|b_{1,k}(y_1)|}{(\ell(Q_{1,k})+|x-y_2|)^{2n}} |g_2(y_2)|\phi\Big(\frac{|y_1-y_2|}{\ell(Q_{1,k})}\Big)dy_1dy_2dx \\
&\leq \sum_k \int_{(\mathbb{R}^n)^2} \frac{CA(\alpha\lambda)^{1/2}\ell(Q_{1,k})^n}{(\ell(Q_{1,k})+|x-y_2|)^{2n}} |g_2(y_2)|\mathbb{1}_{(Q_{1,k})^*}(y_2)dy_2dx \\
&\leq CA(\alpha\lambda)^{1/2} \int_{\mathbb{R}^n} |g_2(y_2)|\mathbb{1}_{\cup_k (Q_{1,k})^*}(y_2)dy_2 \\
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RETURN TO THE UGLY PART

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Consider for instance $T_3(b_1, b_2)(x)$ for $x \notin$ the exceptional set:

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\begin{aligned}
&|T_3(b_1, b_2)(x)| \\
\leq &\sum_k \sum_{j:\ell(Q_{1,k})\leq \ell(Q_{2,j})} \int_{(\mathbb{R}^n)^2} \Big| \Big[K(x, y_1, y_2) - K^{(1)}_{t_{1,k}}(x, y_1, y_2) \Big] b_{1,k}(y_1) b_{2,j}(y_2) \Big| dy_1 dy_2 \\
\leq &CA \sum_k \sum_{j:\ell(Q_{1,k})\leq \ell(Q_{2,j})} \int_{(\mathbb{R}^n)^2} \frac{\ell(Q_{1,k})^\epsilon}{(|x - y_1| + |x - y_2|)^{2n+\epsilon}} |b_{1,k}(y_1) b_{2,j}(y_2)| dy_1 dy_2 \\
&+ CA \sum_k \sum_{j:\ell(Q_{1,k})\leq \ell(Q_{2,j})} \int_{(\mathbb{R}^n)^2} \frac{|b_{1,k}(y_1) b_{2,j}(y_2)|}{(|x - y_1| + |x - y_2|)^{2n}} \phi\Big(\frac{|y_1 - y_2|}{\ell(Q_{1,k})} \Big) dy_1 dy_2 \\
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\leq\ & \sum_j \int_{\mathbb{R}^n} \frac{CA(\alpha\lambda)^{1/2}\ell(Q_{2,j})^n}{(\ell(Q_{2,j}) + |x-y_2|)^{2n}} |b_{2,j}(y_2)| \mathbb{1}_{\cup_k (Q_{1,k})^*}(y_2)dy_2 \\
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This gives

$$
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&\int_{(\cup_{i=1}^{2}\Omega_i^*)^c} |T_{32}(b_1, b_2)(x)| dx \\
\leq\ & CA(\alpha\lambda)^{1/2} \sum_j \int_{(\mathbb{R}^n)^2} \frac{\ell(Q_{2,j})^n}{(\ell(Q_{2,j}) + |x - y_2|)^{2n}} |b_{2,j}(y_2)| dy_2 dx \\
\leq\ & CA(\alpha\lambda)^{1/2} \sum_j \int_{\mathbb{R}^n} |b_{2,j}(y_2)| dy_2 \\
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This term is uglier!

HIGHER-DIMENSIONAL COMMUTATORS (Rougher)

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The higher dimensional commutator is defined as

$$\mathcal{C}_2^{(n)}(f,a)(x) = \int_{\mathbb{R}^n} K(x-y)\left(\int_0^1 a((1-t)x+ty)\,dt\right) f(y)\,dy$$

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Christ and Journé proved that $\mathcal{C}_2^{(n)}$ is bounded from $L^p(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

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We discuss some off-diagonal bounds $L^p \times L^q \to L^r$, whenever $1/p + 1/q = 1/r$ and $1 < p, q, r < \infty$.

USE BILINEAR HILBERT TRANSFORMS

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$$H_{\alpha,\beta}(f,g)(t) = \text{p.v.} \int_{-\infty}^{+\infty} f(t-\alpha s)g(t-\beta s)\frac{ds}{s} \qquad t \in \mathbb{R}$$

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It is easy to see that with $e_1 = (1, 0, \dots, 0)$

$$\mathcal{H}^{e_1}_{\alpha,\beta}(f,g)(x) = \text{p.v.} \int_{-\infty}^{+\infty} f(x-\alpha s\, e_1)g(x-\beta s\, e_1)\frac{ds}{s} \qquad x \in \mathbb{R}^n$$

defined for functions f, g on $\mathbb{R}^n$ is bounded in the same range.
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It follows that for all $\theta \in \mathbb{S}^{n-1}$

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Here we use that rotation by a matrix of the form $\begin{pmatrix} M & O \\ O & M \end{pmatrix}$ preserves boundedness.

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What if $\theta_1, \theta_2 \in \mathbb{S}^{n-1}$, and $\theta_1 \neq \theta_2$?

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This is not known (to me).

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This is not known (to me).

However the case $\theta_1 = \theta_2$ is OK and suffices to treat the higher-dimensional commutators.

RELATION WITH nth DIM. COMMUTATOR

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For K homogeneous of degree $-n$ and odd we can write

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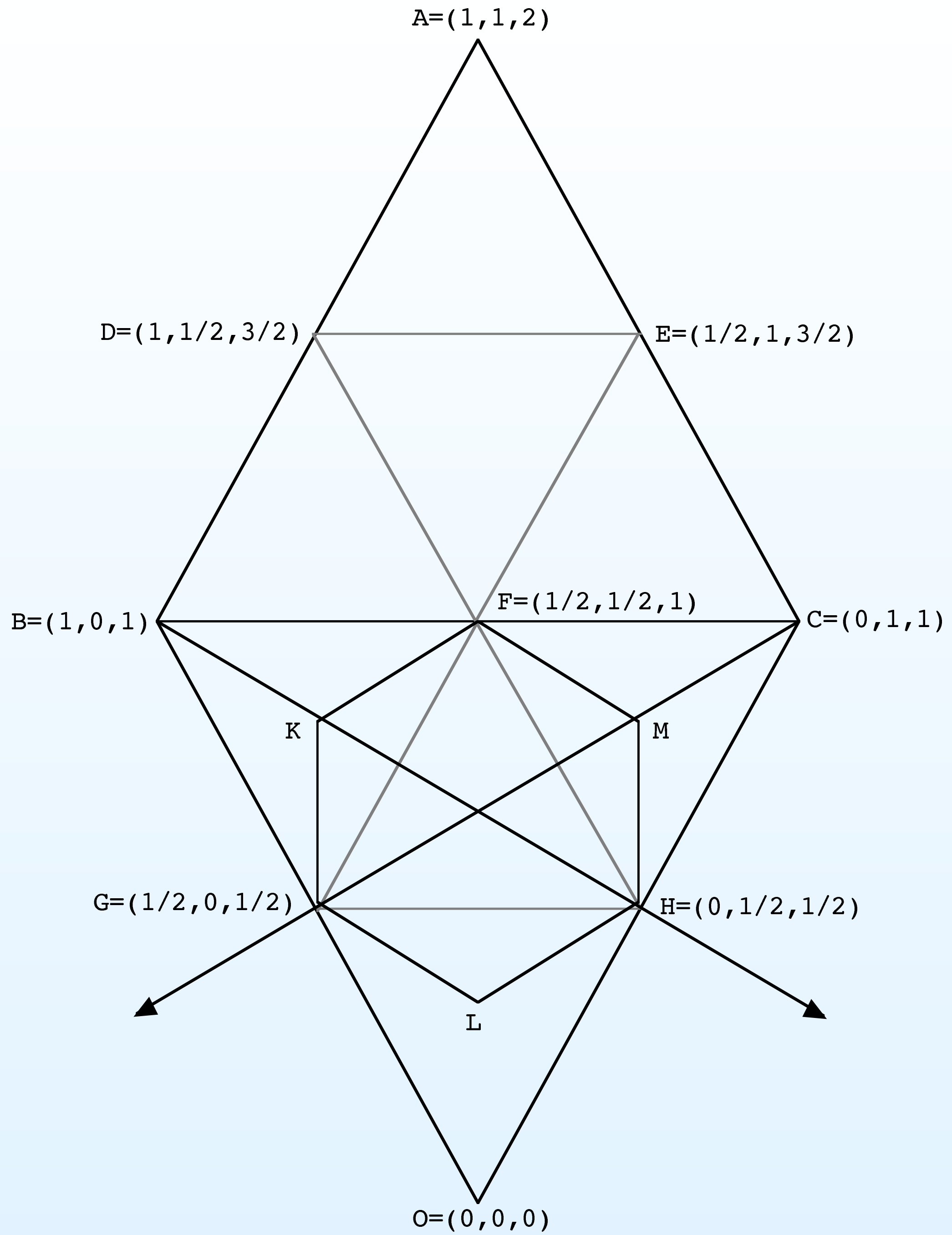
Thus $C_2^{(n)}(f,a)$ is bounded whenever $H_{\alpha,\beta}$ are uniformly bounded from $L^{p_1}\times L^{p_2}\longrightarrow L^p$.

Use uniform bounds for $H_{\alpha,\beta}$ in a hexagonal region that contains the local L^2 triangle (X. Li).

PICTURE



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Suitable adaptation of Fefferman's counterexample based on the Besicovitch construction of a Kakeya set.

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Step 5: use duality.