A critical-exponent Balian–Low theorem

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Gabor Systems

Operators on $L^2(\mathbb{R})$:

- Modulation $\mathrm{M}_y f = e^{2\pi i y \cdot} f$
- Translation $\mathrm{T}_x f = f(\cdot - x)$

For $(x, y) \in \mathbb{R}^2$, define

$$\rho(x, y) = \mathrm{M}_y \mathrm{T}_x \in \mathcal{U}(L^2(\mathbb{R})).$$

Definition

For $\Lambda \subset \mathbb{R}^2$ and $f \in L^2(\mathbb{R})$, the associated Gabor system is

$$\mathcal{G}(f, \Lambda) := \rho(\Lambda) f \subset L^2(\mathbb{R}).$$

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Special case: $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$,

$$\begin{aligned} \mathcal{G}(f, \alpha\mathbb{Z} \times \beta\mathbb{Z}) &= \{e^{2\pi i n\beta \cdot} f(\cdot - m\alpha)\}_{m,n\in\mathbb{Z}} \\ &= \{\mathrm{M}_{n\beta}\,\mathrm{T}_{m\alpha}\, f\} \end{aligned}$$

Question

Given α, β, for which f is $\mathcal{G}(f, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ a frame for $L^2(\mathbb{R})$?:

$$A^2\|g\|_2^2 \leq \sum_{m,n\in\mathbb{Z}} |\langle g, \mathrm{M}_{n\beta}\,\mathrm{T}_{m\alpha}\, f\rangle|^2 \leq B^2\|g\|_2^2,$$

some $A, B > 0$.

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Gabor Systems

Algebraic structure makes this especially tractable:

$$\mathrm{M}_y\,\mathrm{T}_x = e^{2\pi i x y}\,\mathrm{T}_x\,\mathrm{M}_y$$

- Basic von Neumann algebra methods $\Rightarrow$ For $\alpha\beta > 1$, $\mathcal{G}(f, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is *never* a frame for $f \in L^2$.
- For $\alpha\beta < 1$, one can find $f \in C_c^\infty(\mathbb{R})$ for which $\mathcal{G}(f, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is a frame for L^2 (in fact an orthonormal basis).

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Interesting case:

$$\alpha = \beta = 1$$

Easy example: $\alpha = \beta = 1$, $f = \chi_{[0,1]}$

$$\begin{aligned} \mathcal{G}(\chi_{[0,1]}, \mathbb{Z} \times \mathbb{Z}) &= \{\mathrm{M}_n \mathrm{T}_m \chi_{[0,1]}\} \\ &= \{e^{2\pi i n \cdot} \chi_{[m,m+1]}\} \\ &=: \{e_{m,n}\} \end{aligned}$$

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The Balian–Low Theorem

Uncertainty principle for Gabor frames:

Theorem (Balian–Low–Coifman–Semmes)

If $f \in H^1(\mathbb{R})$ and $\hat{f} \in H^1(\mathbb{R})$, then $\mathcal{G}(f, \mathbb{Z} \times \mathbb{Z})$ is not a frame for $L^2(\mathbb{R})$.

So if f is suitably well-localized in phase space, it cannot generate a Gabor frame.

Altering time-frequency regularity conditions

From now on, we consider $f \in H^{p/2} \cap \mathcal{F}H^{q/2}(\mathbb{R})$.

Theorem ($\sim$ Gröchenig '96)

If $\frac{1}{p} + \frac{1}{q} < 1$, then $\mathcal{G}(f, \mathbb{Z} \times \mathbb{Z})$ is not a frame for $L^2(\mathbb{R})$.

Theorem (Benedetto–Czaja–Gadziński–Powell '03)

If $\frac{1}{p} + \frac{1}{q} > 1$ then f may generate a Gabor frame.

Question

What about $\frac{1}{p} + \frac{1}{q} = 1$? ($p = 2$ is the Balian–Low Theorem, so we expect an obstruction result.)

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Main Theorem

Theorem (G '07)

Let $1 < p < \infty$. If $f \in H^{p/2} \cap \mathcal{F}H^{p'/2}(\mathbb{R})$, then $\mathcal{G}(f, \mathbb{Z} \times \mathbb{Z})$ is not a frame for $L^2(\mathbb{R})$.

Zak Transform

Key tool for detecting Gabor frames:
"Zak transform" $Z : L^2(\mathbb{R}) \to L^2_{\text{loc}}(\mathbb{R}^2)$

Definition

$$Z f(x, y) = \sum_{\ell \in \mathbb{Z}} e^{2\pi i \ell y} f(x - \ell)$$

"Quasiperiodicity" (algebraic structure!)

- $Z f(x, y + 1) = Z f(x, y)$
- $Z f(x + 1, y) = e^{2\pi i y} Z f(x, y)$

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Zak Transform

$Z : L^2(\mathbb{R}) \to L^2(\mathbb{T}^2)$ unitary isomorphism:

$$\begin{tikzcd} & L^2(\mathbb{Z}^2) \arrow[dr, "\mathcal{F}"] & \\ L^2(\mathbb{R}) \arrow[ur, "\langle \cdot\,, e_{m,n} \rangle"] \arrow[rr, "Z"'] & & L^2(\mathbb{T}^2) \end{tikzcd}$$

Key properties of the Zak transform

- $\mathcal{G}(f, \mathbb{Z} \times \mathbb{Z})$ is an (A, B)-frame for L^2 if and only if $A \leq |\operatorname{Z} f| \leq B$ a.e.
- $\operatorname{Z} f$ continuous $\Rightarrow \operatorname{Z} f$ automatically has a zero.
  - Quasiperiodicity $\Rightarrow \operatorname{Z} f\left(\partial([0,1]^2)\right)$ has nonzero winding number about 0.

To get a Gabor frame obstruction result, it suffices to show

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$$f \in H^1(\mathbb{R}) \cap \mathcal{F}H^1(\mathbb{R})$$

Balian–Low "proof" ('81 / '85)

$Z f \in H^1_{\text{loc}}(\mathbb{R}^2) \subseteq C(\mathbb{R}^2)$

Endpoint Soblev embedding: $H^1(\mathbb{R}^2) \subset \text{VMO}(\mathbb{R}^2)$, $\|f\|_{\text{BMO}} \lesssim \|f\|_{H^1(\mathbb{R}^2)}$.

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Degree Theory and VMO

The Coifman–Semmes argument predicts a more general phenomenon:

General Principle (Brezis–Nirenberg, mid-'90s)

"Degree theory works for VMO maps."

$$F_\varepsilon(x) := \fint_{B_\varepsilon(x)} F,$$

$$\deg_{\mathrm{VMO}}(F, p) := \deg(F_\varepsilon, p) \text{ for } \varepsilon \text{ small.}$$

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Proof of the Main Theorem

Gröchenig's argument shows $Z : H^{p/2} \cap \mathcal{F}H^{q/2}(\mathbb{R}) \to C(\mathbb{R}^2)$ for $\frac{1}{p} + \frac{1}{q} < 1$; might hope for VMO when $q = p'$.

Take the "Sobolev embedding" route as above:

Modified Sobolev space

$$\|f\|_{S_{p,q}}^2 := \int_{\mathbb{R}^2} |\hat{f}(\xi,\eta)|^2 (1 + |\xi|^p + |\eta|^q) d\xi d\eta$$

("$H^{p/2}$ in x-direction, $H^{q/2}$ in y." $S_{p,p} = H^{p/2}$.)

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Theorem (à la endpoint Sobolev embedding)

For $1 < p < \infty$,

$$\|f\|_{\mathrm{BMO}} \lesssim_p \|f\|_{S_{p,p'}}.$$

(Use Littlewood-Paley decompositions.)

So if $f \in H^{p/2}(\mathbb{R}) \cap \mathcal{F}H^{p'/2}(\mathbb{R})$, then $Zf \in \mathrm{VMO}(\mathbb{R}^2)$, and we can run a winding number argument to show $\operatorname{ess\,inf}|Zf| = 0$. So $\mathcal{G}(f, \mathbb{Z} \times \mathbb{Z})$ is *not* a frame for $L^2(\mathbb{R})$.

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$p = 1$ Endpoint Results

Theorem (Benedetto–Czaja–Powell–Sterbenz '06)

If $f \in H^{1/2}(\mathbb{R})$ and $\operatorname{supp}(f) \subseteq [-1, 1]$, then $\mathcal{G}(f, \mathbb{Z} \times \mathbb{Z})$ is not a frame for $L^2(\mathbb{R})$.

Theorem (G '07)

If $f \in H^{1/2}(\mathbb{R})$ has compact support, then $\mathcal{G}(f, \mathbb{Z} \times \mathbb{Z})$ is not a frame.

(Compact support implies $Z f(x, y) = \sum e^{2\pi i\ell y} f(x - \ell)$ lies in the algebraic tensor product

$$\mathrm{VMO} \cap L^\infty(\mathbb{R}) \otimes \mathrm{VMO} \cap L^\infty(\mathbb{R}) \subset \mathrm{VMO}(\mathbb{R}^2)$$

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