Asymptotics of representations of symmetric groups and random matrices

(joint work with Roland Speicher)

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University of Wroclaw

Outline

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Representations of symmetric groups S_n

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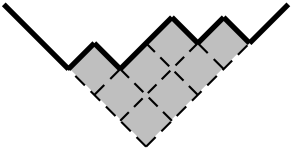

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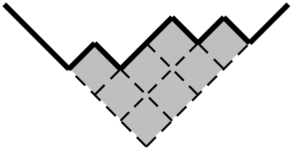

What happens to representations of S_n when $n \to \infty$?

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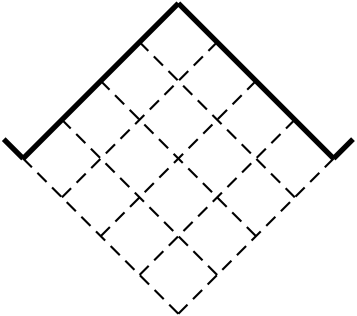
Concrete problem for today

For each n let ρ_n be a representation of S_n and
λ_n be a random Young diagram distributed according to ρ_n.
What are the statistical properties of λ_n in the limit $n \to \infty$?

Example of a concrete problem: Restriction of irreducible representations



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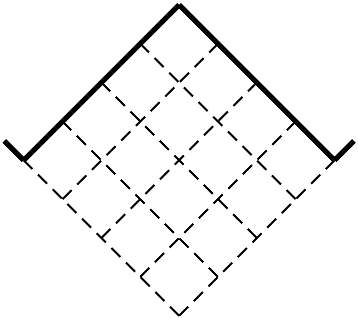

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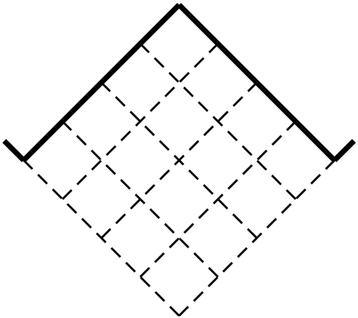
Problem

What is the distribution of the random Young diagram associated to the restriction of the representation ρ^ν to a subgroup $S_{\frac{1}{2}n^2}$?

Alternative description of the problem: Young tableaux



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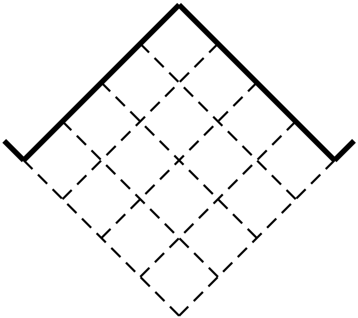

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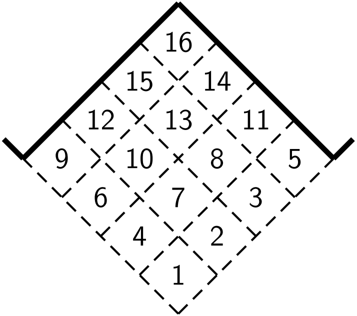

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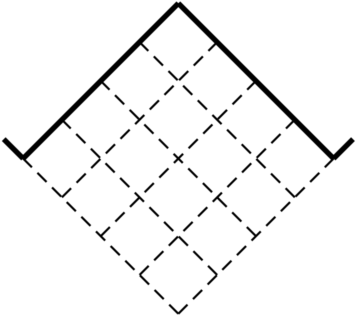

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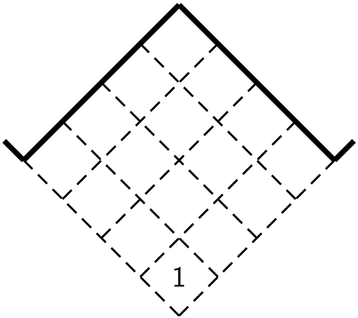

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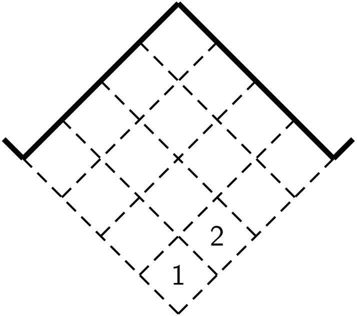

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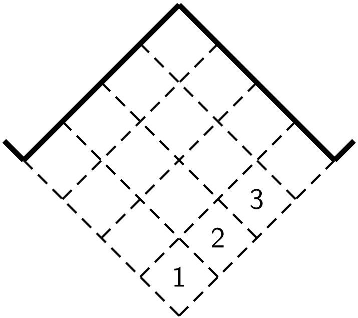

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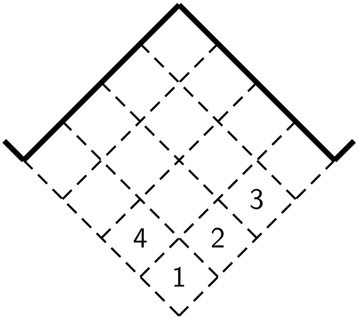

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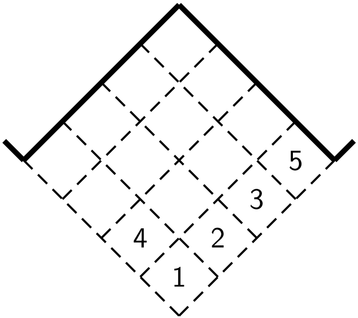

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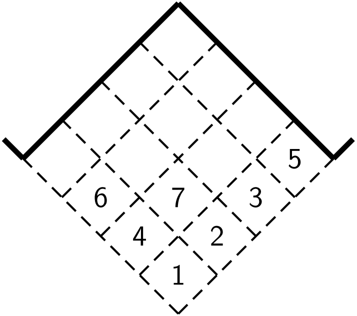

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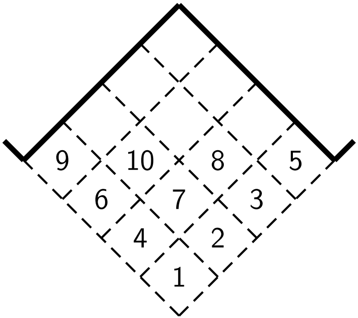

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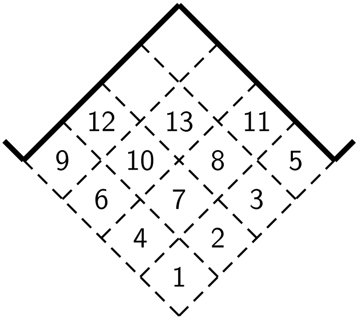

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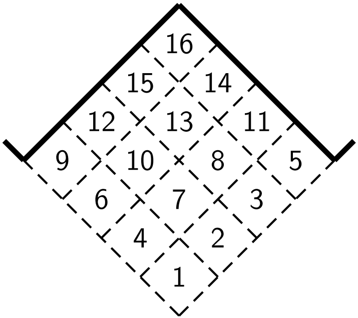

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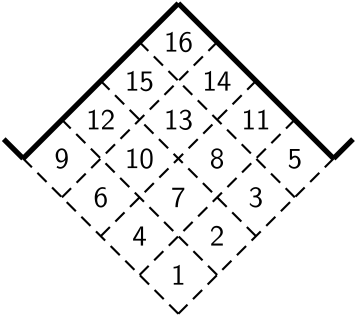

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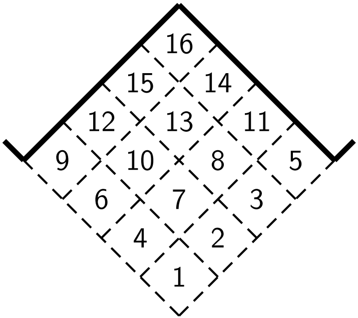

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From a randomly chosen Young tableau

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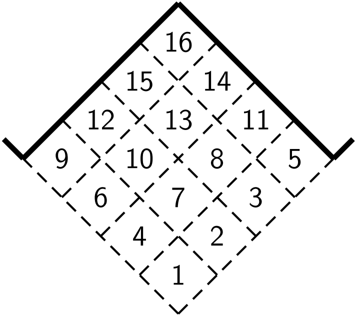

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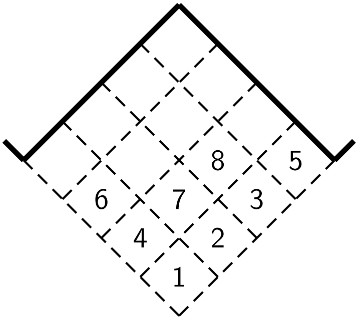

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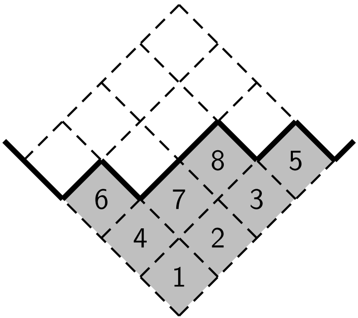

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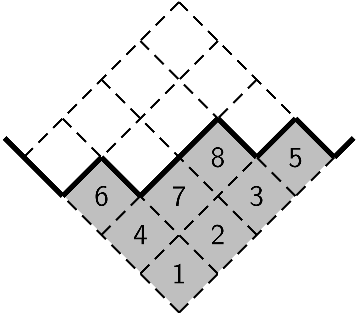

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Empirical eigenvalues distribution

If $M \in \mathcal{M}_d(\mathbb{C})$ is a non-random matrix with eigenvalues $z_1, \dots, z_d \in \mathbb{R}$ we define its eigenvalues distribution as

$$\mu^M = \frac{1}{d} \left(\delta_{z_1} + \cdots + \delta_{z_d} \right).$$

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can be used as an alternative definition of empirical eigenvalues distribution!

Kerov's transition measure

Define Biane's matrix J

$$J = \begin{bmatrix} 0 & (1,2) & \dots & (1,n) & 1 \\ (2,1) & 0 & \dots & (2,n) & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (n,1) & (n,2) & \dots & 0 & 1 \\ 1 & 1 & \dots & 1 & 0 \end{bmatrix} \in \mathcal{M}_{n+1} \otimes \mathbb{C}[S_n]$$

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Kerov's transition measure μ^ρ corresponding to ρ is defined as the empirical eigenvalues distribution of J.

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Normalized Biane's matrix $\widetilde{J} = \frac{1}{\sqrt{n}} J$ has spectral measure equal to $\frac{1}{\sqrt{n}}\mu^{\lambda}$, supported on $[-C, C]$. Corresponds to rescaled Young diagram $\frac{1}{\sqrt{n}}\lambda$.

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Namely, what can we say about the random variables

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- Bad news: we have to find the relation between the characters of S_n and moments of the entries of J.
- Strange news: fluctuations are **almost** the same as for unitarily invariant random matrices, higher order freeness,...

Outline

1. Problem: Representations of symmetric groups S_n
2. Tools: Permutationally invariant random matrices
   - Permutationally invariant random matrices
   - Partitions and genus expansion
3. Main result: Gaussian fluctuations of Young diagrams

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Very weak condition!

Permutationally invariant random matrices

Let $[M_{ij}]_{1\leq i,j\leq d}$ be a random matrix (possibly, with non-commuting entries).

We say that M is permutationally invariant if for any permutation $\pi \in S_d$

$$\mathbb{E}\left[M_{i_1 i_2} M_{i_2 i_3} \cdots M_{i_{k-1} i_k} M_{i_k i_1}\right] = \mathbb{E}\left[M_{\pi(i_1)\pi(i_2)} \cdots M_{\pi(i_{k-1})\pi(i_k)} M_{\pi(i_k)\pi(i_1)}\right].$$

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Example

$$J = \begin{bmatrix} 0 & (1,2) & \dots & (1,n) & 1 \\ (2,1) & 0 & \dots & (2,n) & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (n,1) & (n,2) & \dots & 0 & 1 \\ 1 & 1 & \dots & 1 & 0 \end{bmatrix}$$ is permutationally invariant.

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Example

For Biane's matrix J

$$\operatorname{tr}_1 J^k = \operatorname{tr} J^k.$$

Macroscopic vs microscopic quantities

What is the relation between

- macroscopic: distribution and fluctuations of eigenvalues,

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micro: ρ is the left-regular representation $\longleftrightarrow$ joint distribution of entries in Biane's matrix is specified
macro: What are the fluctuations of a random irreducible component. $\longleftrightarrow$ what are the fluctuations of eigenvalues of Biane's matrix?

Partitions

If p is a partition of $\{1, \dots, k\}$ we define

$$\Sigma_p(M) = \Sigma_p = \sum_{\substack{\mathbf{i}=(i_1,\dots,i_k) \\ \mathbf{i}\sim p \\ i_1=1}} M_{i_1 i_2} M_{i_2 i_3} \cdots M_{i_{k-1} i_k} M_{i_k i_1}$$

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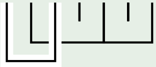

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Normalized conjugacy classes

For integers $k_1, \dots, k_m \geq 1$ we define the conjugacy class $C_{k_1,\dots,k_m} \in \mathbb{C}[S_n]$ as a linear combination of all permutations which in a cycle decomposition have cycles of length $k_1, \dots, k_m$.

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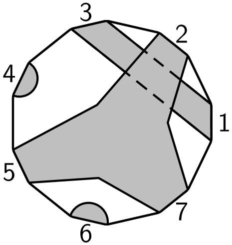
Formula for $\Sigma_\pi(J)$ for Biane's matrix



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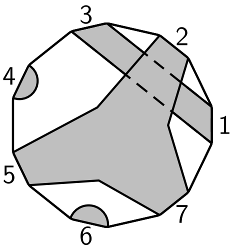

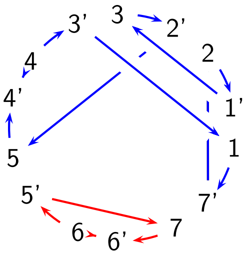

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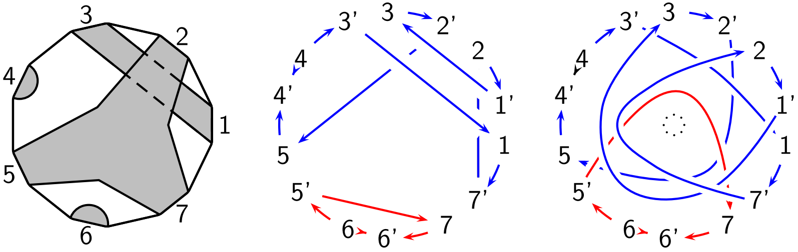

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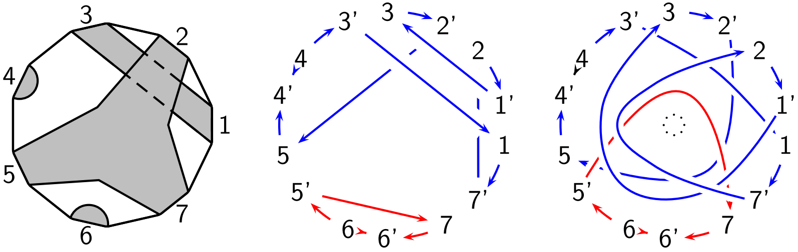

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$$\left(\frac{\text{number of visited vertices}}{2} - \text{number of winds} \right);$$

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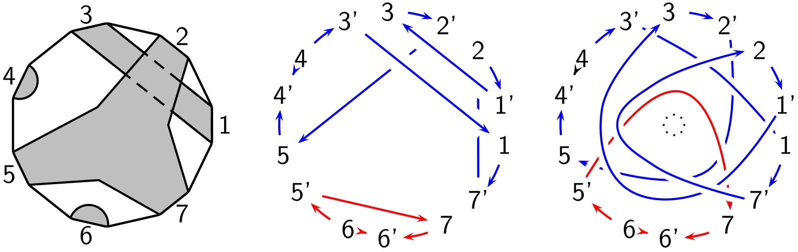

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In our example $\Sigma_\pi(J) = C_{\frac{10}{2}-3, \frac{4}{2}-1} = C_{2,1}$.

Genus

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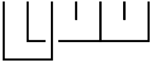

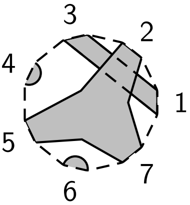

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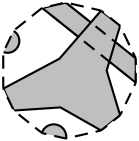

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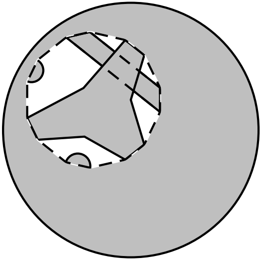

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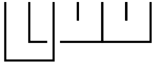

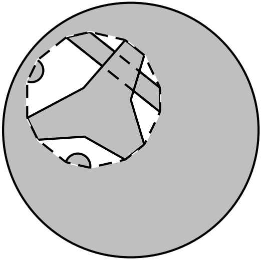

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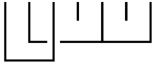

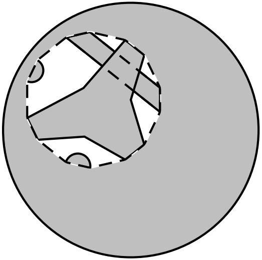

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Claim

Under mild assumptions (for example, $M = \frac{1}{\sqrt{n}} J$, balanced Young diagrams)

$$\mathbb{E}\Sigma_p(M) = O\left(d^{-\operatorname{genus}(p)}\right)$$

Outline

Quick decay of cumulants

For *nice* random matrices we can expect quick decay of cumulants:

$$k\big(\Sigma_{p_1}(\widetilde{J}),\dots,\Sigma_{p_l}(\widetilde{J})\big) = O\left(d^{-\operatorname{genus}(p_1)-\cdots-\operatorname{genus}(p_l)-(l-1)}\right).$$

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We call this property approximate factorization of characters:

$$\chi(\pi_1 \cdots \pi_l) \approx \chi(\pi_1) \cdots \chi(\pi_l)$$

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Theorem (law of large numbers, Philippe Biane 1998)

The sequence of rescaled random Young diagrams $(\frac{1}{\sqrt{n}}\lambda_n)$ converges in probability to some (generalized) Young diagram λ. The shape of this limit can be described by the free probability theory.

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Theorem (central limit theorem, Piotr Śniady 2005)

The sequence of the fluctuations $(\frac{1}{\sqrt{n}}\lambda_n - \lambda)$, after some additional rescaling, converges in distribution to a Gaussian process. The covariance of this process can be described by second-order free probability theory (with a small correction).

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Corollary

What happens if we remove half of the boxes from a random Young tableau? The answer for this problem is given by a certain Gaussian process.

Other connections between representations of S_n and random matrices

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- and much more

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