Pairs of intermediate subfactors

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Lattices of intermediate subfactors

A fundamental example of a subfactor is the fixed point algebra $N = M^G \subset M$ of an outer action of a finite group G on a factor M. In this case the intermediate subfactors $N \subset P \subset M$ are precisely the fixed point algebras of the subgroups of G.

Subfactor theory - **"noncommutative Galois theory"**.

Watatani asked which finite lattices occur as intermediate subfactor lattices for irreducible finite-index subfactors. Recent progress - Xu.

This talk will focus on the structure associated to 2 intermediate subfactors.

The Jones Index

Let M be a $\mathbf{II}_1$ factor. $L^2(M)$ is the Hilbert space completion of M w.r.t. the unique normalized trace, with $\langle x, y \rangle = tr(y^* x)$.

If $N \subset M$ is a subfactor, let e_N be the projection of $L^2(M)$ onto $L^2(N)$. Let $M_1 =< M, e_N >$. This is the **basic construction**.

The **Jones index** is defined as $[M : N] = tr_{M_1}(e_N)$ if M_1 is a $\mathbf{II}_1$ factor,
$[M : N] = \infty$ otherwise.

Let $N \subset M$ be a subfactor with $[M : N] < \infty$, and let

$$N \subset M \subset M_1$$

be the basic construction. Then $M \subset M_1$ is also a finite-index subfactor, so can iterate the basic construction

$$M \subset M_1 \subset M_2$$

where $M_2 =< M_1, e_M >$.

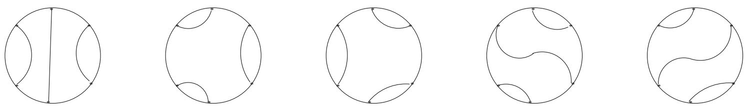
Let $e_1 = e_N, e_2 = e_M$, etc. Get a tower:

$$N \subset M \subset^{e_1} M_1 \subset^{e_2} M_2 ...$$

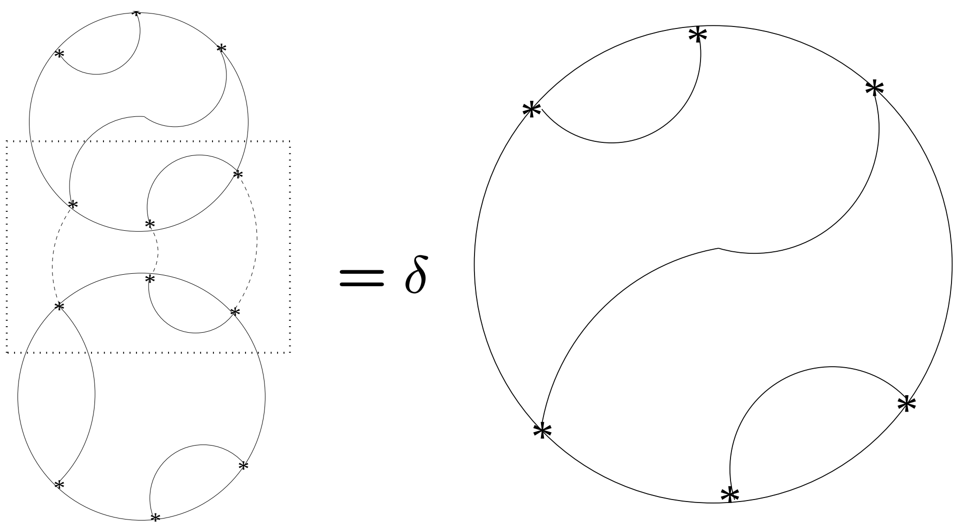
where each $M_{k+1} =< M_k, e_{k+1} >$.

Temperley-Lieb algebra

TL_n is the complex vector space with basis given by planar diagrams on a disk with $2n$ distinguished boundary points. For example: a basis for TL_3

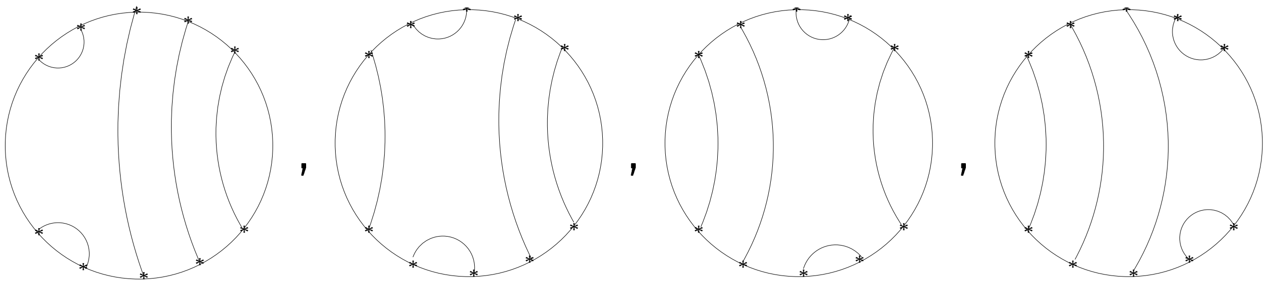


$TL_n(\delta)$, δ a complex parameter, is the algebra with underlying vector space TL_n and multiplication given on basis elements by "concatenation of diagrams", each closed loop contributing a factor of δ. For example:



A set of generators

Generators for $TL_5(\delta)$: E_1, E_2, E_3, E_4



Relations: $E_iE_j = E_jE_i$ if $|i-j| > 1$, $E_iE_{i\pm1}E_i = E_i$.

Let $N \subset M \subset^{e_1} M_1 \subset^{e_2} M_2 ...$ be the Jones tower of a subfactor. The projections $e_1, e_2, e_3, ...$ satisfy the relations: $e_ie_j = e_je_i$ if $|i-j| > 1$, $e_ie_{i\pm1}e_i = \tau e_i$, where $\tau = [M : N]^{-1}$.

There is a surjective map from $TL([M : N]^{\frac{1}{2}})$ to the von Neumann algebra generated by $e_1, e_2, ...$ sending E_i to δe_i. If $[M : N] \geq 4$ then this map is an isomorphism.

Let $N \subset M$ be a subfactor with $[M : N] < \infty$ and Jones tower $N \subset M \subset^{e_1} M_1 \subset^{e_2} M_2 ....$

Each relative commutant $N' \cap M_k$ is f.d., and the lattice:

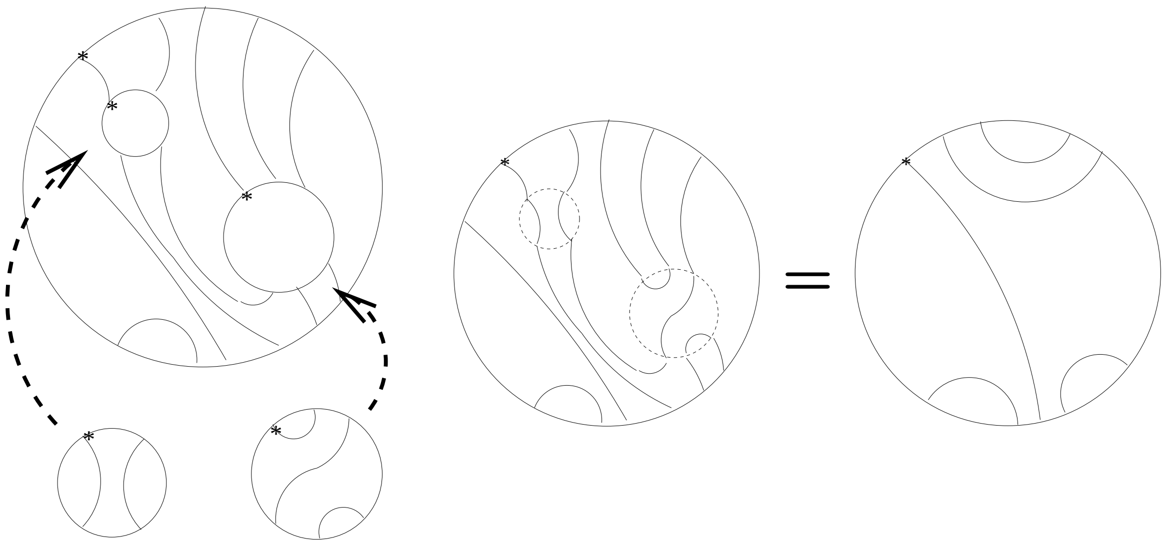
$$\begin{array}{ccccccc} (N' \cap N) & \subset & (N' \cap M) & \subset & (N' \cap M_1) & \subset & ... \\ & & \cup & & \cup & & \\ & & (M' \cap M) & \subset & (M' \cap M_1) & \subset & ... \end{array}$$

is called the **standard invariant**. What is inside the standard invariant?

$N' \cap N = \mathbb{C}Id$ - always trivial
$N' \cap M$ may be trivial (**irreducible** subfactor)
$N' \cap M_1 \ni e_1$ - never trivial
...
$N' \cap M_k \supseteq \{e_1, ..., e_k\}$

In general there will be more stuff in the standard invariant. A subfactor has **no extra structure** if $N' \cap M_k = \{e_1, ..., e_k\}''$.

The TL algebra has a planar algebra structure by gluing the basis diagrams into the input discs of a tangle. For example:



This TL planar algebra is present inside any PA as the image of those tangles with no internal discs. For a subfactor PA with $P_k = N' \cap M_{k-1}$, the TL planar algebra is the sub-PA generated by the Jones projections.

Jones projections $\Longleftrightarrow$ TL algebras
standard invariant $\Longleftrightarrow$ planar algebra
no extra structure $\Longleftrightarrow$ PA=TL algebra

The principal graph

$N \subset M$ a subfactor, $[M : N] < \infty$. Let $\rho = {}_N M_M, \bar{\rho} = {}_M M_N$ (bimodules, action by multiplication).

Let $\rho^k = \rho \otimes \bar{\rho} \otimes \rho ...$(k factors). Then $End(\rho^k) \cong N' \cap M_{k-1} = P_k$.

Even vertices $E = \{$ isomorphism classes of irreducible $N - N$ bimodules occuring in the decomposition of ρ^k for some even $k\}$

Odd vertices $O = \{$ isomorphism classes of irreducible $N - M$ bimodules occuring in the decomposition of ρ^k for some odd $k\}$

# of edges connecting $\alpha \in E$ to $\beta \in O$ is the multiplicity of β in $\alpha\rho$

Subfactors with index less than 4

Jones Index Theorem: Let $N \subset M$ be a subfactor with $[M : N] < 4$. Then $[M : N] = 4cos^2\frac{\pi}{k}$ for some $k = 3, 4, \ldots$.

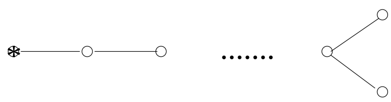
Jones also constructed a subfactor for each k which has no extra structure.

Turns out that subfactors with $[M : N] < 4$ ($\Longleftrightarrow$ planar algebras with $\delta < 2$) are essentially classified by the Coxeter-Dynkin diagrams A_n, D_{2n}, E_6, E_8, except that there are two different planar algebras each for E_6 and E_8. (Ocneanu, Popa).

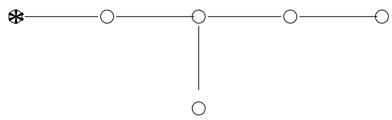
Principal graphs and supertransitivity

A subfactor is **k-supertransitive** if its planar algebra does not contain any nontrivial k-boxes ($\iff$ the initial part of its principal graph looks like A_{k+1}.) Some examples:

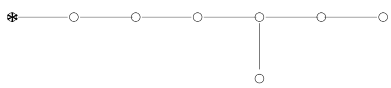
A_n is k-supertransitive for all k.



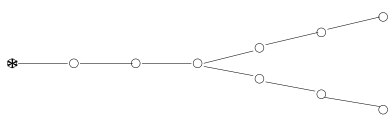
D_{2n} is $2n-3$-supertransitive.



E_6 is 2-supertransitive.



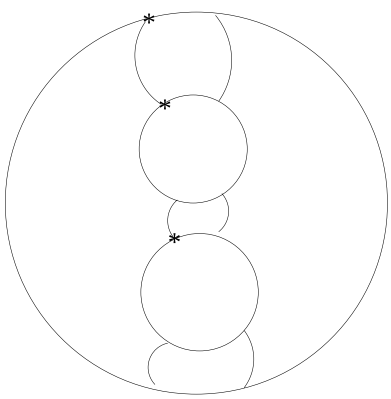
E_8 is 4-supertransitive.



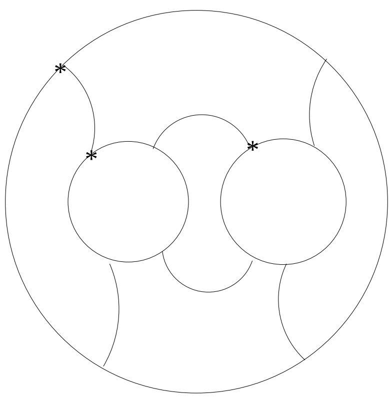
the **Haagerup subfactor** with index $\frac{5+\sqrt{13}}{2}$ is 3-supertransitive.

Comultiplication in planar algebras

Multiplication of 2-boxes is defined as vertical contraction via the tangle



In a similar way, one can define a "comultiplication" as horizontal contraction via the tangle



Intermediate subfactors and biprojections

Let $N \subset M$ be a subfactor with $N' \cap M = \mathbb{C}Id$ and planar algebra $\cup_i P_i$.

Theorem (Bisch): The intermediate subfactors $N \subset P \subset M$ are in one-to-one correspondence with elements of P_2 which are (up to a scalar) projections with repect to both multiplication and comultiplication, and the two adjoints.

intermediate subfactors $\Longleftrightarrow$ biprojections

Bisch and Jones constructed the planar algebra generated by a biprojection, giving a generic construction of an intermediate subfactor $N \subset P \subset M$ such that $N \subset P$ and $P \subset M$ have no extra structure. This **Fuss-Catalan** algebra is a free product of Temperley-Lieb algebras.

Two intermediate subfactors

A **quadrilateral** of subfactors is a diagram $\begin{matrix} P & \subset & M \\ \cup & & \cup \\ N & \subset & Q \end{matrix}$ such that $P \vee Q = M$ and $P \wedge Q = N$. (and $N' \cap M = \mathbb{C}Id$.) Every quadrilateral has a **dual quadrilateral** of commutants $\begin{matrix} P' & \subset & N' \\ \cup & & \cup \\ M' & \subset & Q' \end{matrix}$.

A quadrilateral **commutes** if $e_P e_Q = e_Q e_P$. It **cocommutes** if its dual commutes.

Sano and Watatani studied **angles** between subfactors: $Ang(P, Q) = spec(cos^{-1}(e_P e_Q e_P))$, a numerical invariant which measures the noncommutativity of P and Q.

Pairs of intermediate subfactors with no extra structure

A tensor product gives an easy construction of commuting, cocommuting quadrilaterals with no extra structure, and there is no obstruction to this. Constructing quadrilaterals with nontrivial angles is harder.

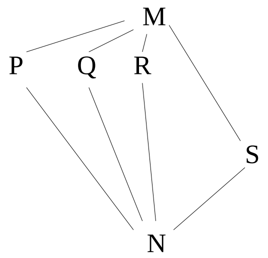
Example: Let G be the symmetric group S_3 acting as outer automorphisms of a factor M, and let H and K be distinct order 2 subgroups.

Then
$$\begin{tikzcd}[row sep=small, column sep=small]
& M & \\
M^H \arrow[ur, phantom, sloped, "\subset"] & & M^K \arrow[ul, phantom, sloped, "\supset"] \\
& M^G \arrow[ul, phantom, sloped, "\supset"] \arrow[ur, phantom, sloped, "\subset"] &
\end{tikzcd}$$
is a quadrilateral which does not commute since $HK \neq KH$. It does however cocommute.

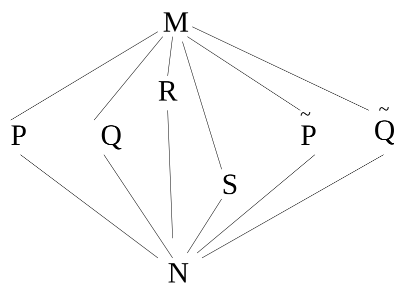
It turns out that there is no generic construction of noncommuting pairs of intermediate subfactors.

Theorem (G-Jones): Let $\begin{matrix} P & \subset & M \\ \cup & & \cup \\ N & \subset & Q \end{matrix}$ be a noncommuting quadrilateral such that the elementary subfactors $N \subset P, P \subset M, N \subset Q$ $Q \subset M$ have no extra structure. Then either N is the fixed-point algebra of an outer action of the symmetric group S_3 on M, or $[M : P] = [P : N] = 2 + \sqrt{2}$. In either case the planar algebra for $N \subset M$ is uniquely determined.

Remark: The original proof used 6-supertransitivity of the elementary subfactors- a subsequent proof by Izumi relaxed the hypothesis to 4-supertransitivity.

The S_3 quadrilateral is cocommuting and we have $[M : P] = [M : Q] = 2$ and $[P : N] = [Q : N] = 3$. The full intermediate subfactor lattice is 

and the angle between P and Q is $\pi/3$.

The other quadrilateral is noncocommuting, all of the elementary subfactors have index $2 + \sqrt{2}$, the full intermediate subfactor lattice is 

and the angle between P and Q is $\cos^{-1}(\sqrt{2} - 1)$.

$\tilde{P}$ and $\tilde{Q}$ generate an isomorphic quadrilateral, and the planar algebra for $N \subset M$ is isomorphic to its dual.

Subfactors with index = 4 have principal graph: $A_\infty, A_{-\infty,\infty}, D_\infty, A_n^{(1)}, D_n^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$. There are very few noncommuting quadrilaterals with small index.

Theorem (G-Izumi): Let $\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ N & \subset & Q \end{array}$ be a noncommuting quadrilateral such that $[M:P]$, $[M:Q], [P:N], [Q:N] \leq 4$.

Then the principal graphs $(G_{N\subset P}, G_{P\subset M}) = (G_{N\subset Q}, G_{Q\subset M})$ are one of the following pairs:

$$(A_7, A_7), \quad (E_7^{(1)}, E_7^{(1)})$$

$$(A_5, A_3), \quad (D_6, A_4), \quad (E_7^{(1)}, A_5), \quad (E_6^{(1)}, D_4)$$

$$(D_6^{(1)}, A_3)$$

There is a unique planar algebra corresponding to each configuration.

Case 1: $(G_{N\subset P}, G_{P\subset M}) = (A_7, A_7)$
Noncocommuting and $[M : P] = [P : N] = 4cos^2\frac{\pi}{8} = 2+\sqrt{2}$.

Case 2: $(E_7^{(1)}, E_7^{(1)})$
Noncocommuting and $[M : P] = [P : N] = 4$.

Case 3: (A_5, A_3)
Cocommuting and $[M : P] = 2, [P : N] = 3$.

Case 4: (D_6, A_4)
Cocommuting and $[M : P] = 4cos^2\frac{\pi}{5} = \frac{3+\sqrt{5\ominus}}{2}$,
$[P : N] = 4cos^2\frac{\pi}{10} = \frac{5+\sqrt{5}}{2}$.

Cases 5-6: $(E_7^{(1)}, A_5)$, $(E_6^{(1)}, D_4)$
Both cocommuting, $[M : P] = 3, [P : N] = 4$.

Case 7: $(D_6^{(1)}, A_3)$ $[M : P] = 2, [P : N] = 4$.

Theorem (G-Izumi): Let $\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ N & \subset & Q \end{array}$ be a noncommuting quadrilateral such that $[M:P]$, $[M:Q], [P:N], [Q:N]$ are all 3-supertransitive. Then either the quadrilateral cocommutes and $[M:P] = [P:N] - 1$ or the quadrilateral does not cocommute and $[M:P] = [P:N]$.

In the first case $Gal(M/N)$, the group of automorphisms of M which fix N pointwise, is a subgroup of S_3.

Remark: Consider the symmetric groups S_{n+2} on $S = \{1, ..., n+2\}$, S^1_{n+1} on $\{1, ..., n, n+1\}$, S^2_{n+1} on $\{1, ..., n, n+2\}$, and S_n on $\{1, .., n\}$ and an outer action of S_{n+2} on a factor R. Then letting $M = R^{S_n}, P = R^{S^1_{n+1}}, Q = R^{S^2_{n+1}}, N = R^{S_{n+2}}$ gives a quadrilateral of the first type.

The (A_5, A_3) and $(E_7^{(1)}, A_5)$ quadrilaterals are of this form for $n = 1, 2$ respectively.

Angles and Second Cohomolgy

A key element of the proof is the notion of second cohomology for subfactors, introduced by Izumi-Kosaki, which counts inner conjugacy classes of subfactors sharing the same basic construction (as a bimodule class). Second cohomolgy is also closely related to angles. In particular I-K showed that any 3-supertransitive subfactor has trivial second cohomolgy.

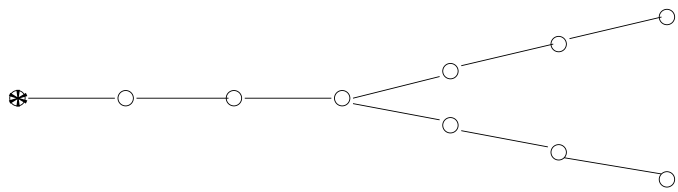
Theorem (G-Izumi): Let $\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ N & \subset & Q \end{array}$

be a noncommuting quadrilateral such that $N \subset P, N \subset Q$ are 3 supertransitive. Then the unique nontrivial angle value is always $cos^{-1}\dfrac{1}{[P:N]-1}$.

The Haagerup subfactor

The Haagerup subfactor with index $\dfrac{5+\sqrt{13}}{2}$ has the smallest index above 4 of any finite depth subfactor. This subfactor gives an exotic tensor category which is not known to appear in any other context.

Its (dual) principal graph is:

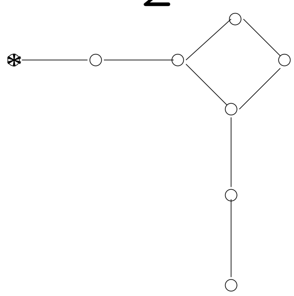


Note that it is 3-supertransitive.

Quadrilaterals of Haagerup subfactors

There is a noncommuting, noncocommuting quadrilateral $\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ N & \subset & Q \end{array}$ such that $[M:P] = [M:Q] = [P:N] = [Q:N] = \dfrac{5+\sqrt{13}}{2}$. $N \subset P$ and $N \subset Q$ are each the Haagerup subfactor and $P \subset M$ and $Q \subset M$ are each the dual Haagerup subfactor.

There is also a noncommuting but cocommuting quadrilateral such that $P \subset M$ and $Q \subset M$ are each the Haagerup subfactor and $[P:N] = [Q:N] = \dfrac{7+\sqrt{13}}{2}$. The principal graph of $N \subset P$ is



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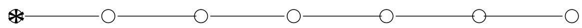
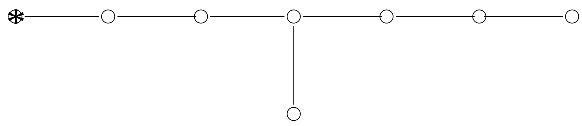
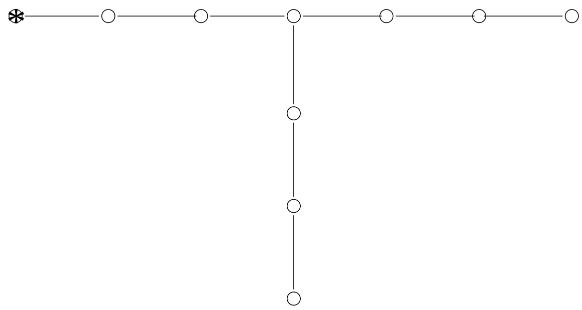
There are 3 known examples of noncommuting, noncocommuting quadrilaterals of 3-supertransitive subfactors. The principal graphs of their elementary subfactors are, respectively, A_7,$E_7^{(1)}$, and the Haagerup graph.



A_7



$E_7^{(1)}$



Haagerup graph

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The construction uses the symmetry of the third and fifth vertices. It is unknown whether there are any additional examples.

Quadrilaterals whose lower subfactors have no extra structure

Theorem: Let $\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ N & \subset & Q \end{array}$ be a noncommuting quadrilateral such that the principal graphs of $N \subset P$ and $N \subset Q$ are A_n. Then n is odd. There exists such a quadrilateral for each odd $n \geq 3$, unique up to isomorphism of the planar algebra.

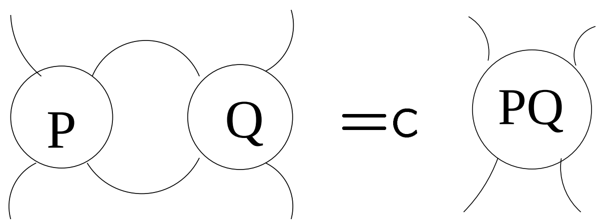
The two quadrilaterals with no extra structure are the first two members of this series, corresponding to $n = 3, 5$. However when $n \geq 7$ the upper subfactors have extra structure.

The proof uses a result of Evans-Gould on algebras associated to T-shaped graphs.

It is unknown whether there exist any noncommuting quadrilaterals whose lower subfactors have no extra structure and index greater than 4.

Landau's PQ relation

Let P and Q be biprojections in an irreducible planar algebra. Then



However additional relations between two biprojections are not known.

Conclusion

The rigidity imposed by the presence of multiple intermediate subfactors suggests a rich structure to intermediate subfactor lattices.

The planar algebra interpretation of this rigidity is not yet understood.