SPDE and portfolio choice

(joint work with M. Musiela)

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Performance measurement of investment strategies

Market environment

Riskless and risky securities

- $(\Omega, \mathcal{F}, \mathbb{P})$; $W = (W^1, \ldots, W^d)$ standard Brownian Motion

- Traded securities

$$1 \leq i \leq k \qquad \begin{cases} dS_t^i = S_t^i\Big(\mu_t^i dt + \sigma_t^i \cdot dW_t\Big) \ , & S_0^i > 0 \\ dB_t = r_t B_t dt \ , & B_0 = 1 \end{cases}$$

  $\mu_t, r_t \in \mathbb{R}$, $\sigma_t^i \in \mathbb{R}^d$ bounded and $\mathcal{F}_t$-measurable stochastic processes

- Postulate existence of an $\mathcal{F}_t$-measurable stochastic process $\lambda_t \in \mathbb{R}^d$ satisfying

$$\mu_t - r_t \, \mathbb{1} = \sigma_t^T \lambda_t$$

- No assumptions on market completeness

Market environment

- Self-financing investment strategies π_t^0, $\pi_t = (\pi_t^1, \ldots, \pi_t^i, \ldots, \pi_t^k)$

- Present value of this allocation

$$X_t = \sum_{i=0}^{k} \pi_t^i$$

$$dX_t = \sum_{i=1}^{k} \pi_t^i \sigma_t^i \cdot (\lambda_t \, dt + dW_t)$$

$$= \sigma_t \pi_t \cdot (\lambda_t \, dt + dW_t)$$

Traditional framework

A (deterministic) utility datum $u_T(x)$ is assigned at the end of a fixed investment horizon

$$U_T(x) = u_T(x)$$

No market input to the choice of terminal utility

Backwards in time generation of the indirect utility

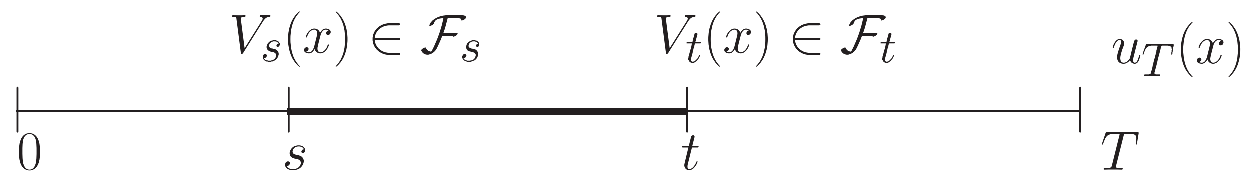
$$V_s(x) = \sup_\pi E_{\mathbb{P}}(u_T(X_T^\pi)|\mathcal{F}_s; X_s^\pi = x)$$

$$V_s(x) = \sup_\pi E_{\mathbb{P}}(V_t(X_t^\pi)|\mathcal{F}_s; X_s^\pi = x) \qquad \text{(DPP)}$$

$$V_s(x) = E_{\mathbb{P}}(V_t(X_t^{\pi^*})|\mathcal{F}_s; X_s^{\pi^*} = x)$$

The value function process becomes the intermediate utility for all $t \in [0, T)$

Investment performance process



- For each self-financing strategy, represented by π, the associated wealth X_t^{π} satisfies

$$E_{\mathbb{P}}(V_t(X_t^{\pi})|\mathcal{F}_s) \leq V_s(X_s^{\pi}) \ , \qquad 0 \leq s \leq t \leq T$$

- There exists a self-financing strategy, represented by π^*, for which the associated wealth $X_t^{\pi^*}$ satisfies

$$E_{\mathbb{P}}(V_t(X_t^{\pi^*})|\mathcal{F}_s) = V_s(X_s^{\pi^*}) \ , \qquad 0 \leq s \leq t \leq T$$

Investment performance process

$$V_{t,T}(x) \in \mathcal{F}_t \ , \qquad 0 \le t \le T$$

- $V_{t,T}(X_t^{\pi})$ is a supermartingale
- $V_{t,T}(X_t^{\pi^*})$ is a martingale
- $V_{t,T}(x)$ is the terminal utility in trading subintervals $\quad [s,t]$, $\quad 0 \le s \le t$

Observations

- $V_{T,T}(x)$ is chosen exogeneously to the market
- Choice of horizon possibly restrictive
- More realistic to have random terminal data, $\quad V_{T,T}(x,\omega) = U(x,\omega)$

Investment performance process

$$U_t(x) \text{ is an } \mathcal{F}_t\text{-adapted process}, \qquad t \geq 0$$

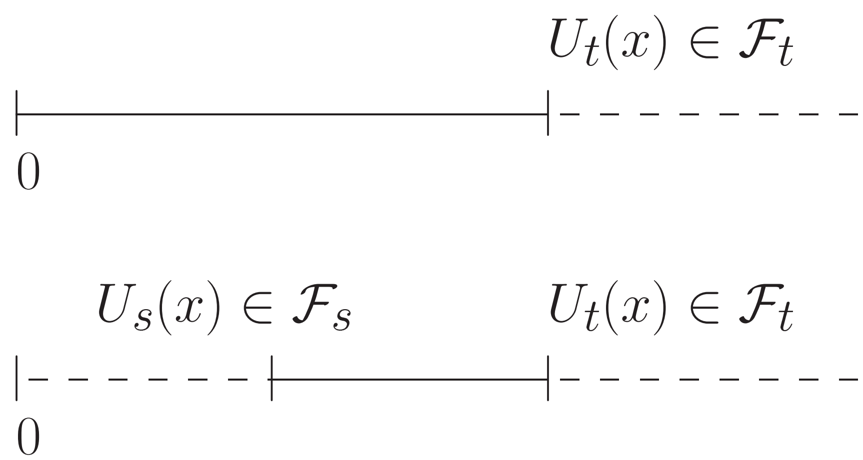
- The mapping $x \to U_t(x)$ is increasing and concave
- For each self-financing strategy, represented by π, the associated (discounted) wealth X_t^π satisfies

$$E_{\mathbb{P}}(U_t(X_t^\pi) \mid \mathcal{F}_s) \leq U_s(X_s^\pi), \qquad 0 \leq s \leq t$$

- There exists a self-financing strategy, represented by π^*, for which the associated (discounted) wealth $X_t^{\pi^*}$ satisfies

$$E_{\mathbb{P}}(U_t(X_t^{\pi^*}) \mid \mathcal{F}_s) = U_s(X_s^{\pi^*}), \qquad 0 \leq s \leq t$$

Optimality across times



$$U_s(x) = \sup_{\mathcal{A}} E(U_t(X_t^{\pi})|\mathcal{F}_s,\ X_s = x)$$

- Does such a process aways exist?
- Is it unique?

Forward performance process

A datum $u_0(x)$ is assigned at the beginning of

the trading horizon, $t = 0$

$$U_0(x) = u_0(x)$$

Forward in time generation of optimal performance

$$E_{\mathbb{P}}(U_t(X_t^{\pi})|\mathcal{F}_s) \leq U_s(X_s^{\pi}), \qquad 0 \leq s \leq t$$

$$E_{\mathbb{P}}(U_t(X_t^{\pi^*})|\mathcal{F}_s) = U_s(X_s^{\pi^*}), \qquad 0 \leq s \leq t$$

Many difficulties due to "inverse in time"

nature of the problem

The stochastic PDE of the forward performance process

The forward performance SPDE

Let $U(x,t)$ be an $\mathcal{F}_t-$measurable process such that the mapping $x \longrightarrow U(x,t)$ is increasing and concave. Let also $U = U(x,t)$ be the solution of the stochastic partial differential equation

$$dU = \frac{1}{2}\frac{\left|\sigma\sigma^{+}\mathcal{A}\left(U\lambda + a\right)\right|^{2}}{\mathcal{A}^{2}U}dt + a \cdot dW$$

where $a = a(x,t)$ is an $\mathcal{F}_t-$adapted process, while $\mathcal{A} = \frac{\partial}{\partial x}$.

Then $U(x,t)$ is a forward performance process.

The process a may depend on t, x, U, its spatial derivatives etc.

At the optimum

- The optimal portfolio vector π^* is given in the feedback form

$$\pi_t^* = \pi^*\left(X_t^*, t\right) = -\sigma^+ \frac{\mathcal{A}\left(U\lambda + a\right)}{\mathcal{A}^2 U}\left(X_t^*, t\right)$$

- The optimal wealth process X^* solves

$$dX_t^* = -\sigma\sigma^+ \frac{\mathcal{A}\left(U\lambda + a\right)}{\mathcal{A}^2 U}\left(X_t^*, t\right)\left(\lambda dt + dW_t\right)$$

Intuition for the structure of the forward performance process

- Assume that $U = U(x,t)$ solves
$$dU(x,t) = b(x,t)\,dt + a(x,t)\cdot dW_t$$
where b, a are $\mathcal{F}_t-$measurable processes.

- Recall that for an arbitrary admissible portfolio π, the associated wealth process, X^π, solves
$$dX_t^\pi = \sigma_t \pi_t \left(\lambda_t dt + dW_t\right)$$

- Apply the Ito-Ventzell formula to $U(X_t^\pi, t)$ we obtain
$$dU(X_t^\pi, t) = b(X_t^\pi, t)\,dt + a(X_t^\pi, t)\cdot dW_t$$
$$+U_x(X_t^\pi, t)\,dX_t^\pi + \frac{1}{2}U_{xx}(X_t^\pi, t)\,d\langle X^\pi\rangle_t + a_x(X_t^\pi, t)\,d\langle W, X^\pi\rangle_t$$
$$= \left(b(X_t^\pi, t) + U_x(X_t^\pi, t)\,\sigma_t\pi_t\cdot\lambda_t + \sigma_t\pi_t\cdot a_x(X_t^\pi, t) + \frac{1}{2}U_{xx}(X_t^\pi, t)\left|\sigma_t\pi_t\right|^2\right)dt$$
$$+\left(a(X_t^\pi, t) + U_x(X_t^\pi, t)\,\sigma_t\pi_t\right)\cdot dW_t$$

Intuition (continued)

- By the monotonicity and concavity assumptions, the quantity

$$\sup_{\pi}\left(U_x\left(X_t^{\pi},t\right)\sigma_t\pi_t\cdot\lambda_t+\sigma_t\pi_t\cdot a_x(X_t^{\pi},t)+\frac{1}{2}U_{xx}\left(X_t^{\pi},t\right)\left|\sigma_t\pi_t\right|^2\right)$$

  is well defined.

- Calculating the optimum π^* yields

$$\pi_t^*=-\sigma_t^+\frac{U_x\left(X_t^{\pi^*},t\right)\lambda_t+a_x\left(X_t^{\pi^*},t\right)}{U_{xx}\left(X_t^{\pi^*},t\right)}$$

- Deduce that the above supremum is given by

$$M^*\left(X_t^{\pi^*},t\right)=-\frac{\left|\sigma_t\sigma_t^+\left(U_x\left(X_t^{\pi^*},t\right)\lambda_t+a_x\left(X_t^{\pi^*},t\right)\right)\right|^2}{2U_{xx}\left(X_t^{\pi^*},t\right)}$$

- Choose the drift coefficient

$$b\left(x,t\right)=-M^*\left(x,t\right)$$

Solutions to the forward performance SPDE

$$dU = \frac{1}{2}\frac{\left|\sigma\sigma^{+}\mathcal{A}\left(U\lambda + a\right)\right|^{2}}{\mathcal{A}^{2}U}dt + a \cdot dW$$

Local differential coefficients

$$a\left(x,t\right) = F\left(x,t,U\left(x,t\right),U_{x}\left(x,t\right)\right)$$

Difficulties

- The equation is fully nonlinear
- The diffusion coefficient depends, in general, on U_x and U_{xx}
- The equation is not (degenerate) elliptic

Choices of volatility coefficient

- The deterministic case: $a(x,t) = 0$

The forward performance SPDE simplifies to

$$dU = \frac{1}{2} \frac{\left| \sigma \sigma^{+} \mathcal{A}(U \lambda) \right|^{2}}{\mathcal{A}^{2} U} dt$$

The process

$$U(x,t) = u(x, A_t) \quad \text{with} \quad A_t = \int_0^t \left| \sigma_s \sigma_s^{+} \lambda_s \right|^2 ds$$

with $u : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$, increasing and concave with respect to x, and solving

$$u_t u_{xx} = \frac{1}{2} u_x^2$$

is a solution.

MZ (2006)

Berrier, Rogers and Tehranchi (2007)

- $a(x,t) = 0$

  σ, λ constants and u separable (in space and time)

The forward performance process reduces to a deterministic function.

$$U(x,t) = u(x,t)$$

$$u(x,t) = -e^{-x+\frac{t}{2}} \qquad \text{or} \qquad u(x,t) = \frac{1}{\gamma} x^{\gamma} e^{-\frac{\gamma}{2(1-\gamma)}\lambda^2 t}$$

Horizon-unbiased utilities

Henderson-Hobson (2006)

- $a(x,t) = k\ , \quad k \in \mathbb{R}$

$$U(x,t) = u(x, A_t) + kW_t$$

The "market-view" case

$a = U\phi, \qquad \phi$ is a $d-$dim $\mathcal{F}_t-$adapted process

- The forward performance SPDE becomes

$$dU = \frac{1}{2}\frac{\left|\sigma\sigma^{+}\mathcal{A}U\left(\lambda+\phi\right)\right|^{2}}{\mathcal{A}^{2}U}dt + U\phi\cdot dW$$

- Define the processes Z and A by

$$dZ = Z\phi\cdot dW \qquad \text{and} \qquad Z_0 = 1$$

and

$$A_t = \int_0^t \left|\sigma_s\sigma_s^{+}(\lambda_s+\phi_s)\right|^2 ds$$

- The process $U = U\left(x,t\right)$

$$U\left(x,t\right) = u\left(x,A_t\right)Z_t$$

with u solving

$$u_t u_{xx} = \frac{1}{2}u_x^2$$

is a solution

The "benchmark" case

$$a(x,t) = -xU(x,t)\,\delta, \quad \delta \text{ is a } d\text{--dim } \mathcal{F}_t\text{--adapted process}$$

- The forward performance SPDE becomes

$$dU(x,t) = \frac{1}{2}\frac{\left|\sigma_t\sigma_t^+\left(U_x(x,t)(\lambda_t - \delta_t) - xU_{xx}(x,t)\right)\right|^2}{U_{xx}(x,t)}dt - xU_x(x,t)\,\delta_t \cdot dW_t$$

- Define the processes Y and A by

$$dY_t = Y_t\delta_t(\lambda_t dt + dW_t) \quad \text{with} \quad Y_0 = 1$$

and

$$A_t = \int_0^t \left|\sigma_s\sigma_s^+\lambda_s - \delta_s\right|^2 ds.$$

- Assume $\sigma\sigma^+\delta = \delta$

- The process

$$U = U(x,t) = u\left(\frac{x}{Y_t}, A_t\right)$$

with u as before is a forward performance.

A general case

$$a(x,t) = -xU_x(x,t)\,\delta + U(x,t)\,\phi$$

- The forward performance SPDE becomes

$$dU(x,t) = \frac{1}{2}\frac{\left|\sigma_t\sigma_t^{+}\left(U_x(x,t)\left((\lambda_t+\phi_t)-\delta_t\right)-xU_{xx}(x,t)\,\delta_t\right)\right|^2}{U_{xx}(x)}dt$$

$$+\left(-xU_x(x,t)\,\delta_t + U(x,t)\,\phi_t\right)\cdot dW_t$$

- Recall the "benchmark" and "market view processes"

$$dY_t = Y_t\delta_t\left(\lambda_t dt + dW_t\right) \quad \text{with} \quad Y = 1$$

and

$$dZ_t = Z_t\phi_t \cdot dW_t \quad \text{with} \quad Z = 1$$

- Define the process

$$A_t = \int_0^t \left| \sigma_s \sigma_s^+ (\lambda_s + \phi_s) - \delta_s \right|^2 ds$$

- The process

$$U = U(x,t) = u\left(\frac{x}{Y_t}, A_t\right) Z_t$$

is a forward performance

MZ (2006, 2007)

The u-pde

An important differential object is the fully non-linear pde

$$u_t u_{xx} = \frac{1}{2} u_x^2 \qquad t > 0,$$

with $u_0(x) = U(x, 0)$.

The local risk tolerance

A quantity that enters in the explicit representation of the optimal portfolios

$$r = -\frac{u_x}{u_{xx}}$$

Modelling considerations

Three related pdes

- Fast diffusion equation for risk tolerance

$$\begin{cases} r_t + \frac{1}{2} r^2 r_{xx} = 0 \\ r(x,0) = r_0(x) \end{cases} \qquad \text{(FDE)}$$

Conductivity : $\quad r^2$

- The transport equation

$$u_t + \frac{1}{2} r u_x = 0$$

with u_0 such that $r_0 = r\left(x,0\right) = -\frac{u_0'(x)}{u_0''(x)}$

- Porous medium equation for risk aversion $\quad \gamma = r^{-1}$

$$\gamma_t = \frac{1}{2} F\left(\gamma\right)_{xx} \qquad \text{with} \qquad F\left(\gamma\right) = \gamma^{-1}$$

Difficulties

- Differential input equation: $u_t \, u_{xx} = \frac{1}{2}u_x^2$

  Inverse problem and fully nonlinear

- Transport equation: $u_t + \frac{1}{2}r u_x = 0$

  Shocks, solutions past singularities

- Fast diffusion equation: $r_t + \frac{1}{2}r^2 r_{xx} = 0$

  Inverse problem and backward parabolic, solutions might not exist, locally integrable data might not produce locally bounded slns in finite time

- Porous medium equation: $\gamma_t = \frac{1}{2}(\frac{1}{\gamma})_{xx}$
  Majority of results for (PME), $\gamma_t = (\gamma^m)_{xx}$, are for $m > 1$, partial results for $-1 < m < 0$

An example of local risk tolerance

(MZ (2006) and Z-Zhou (2007))

$$r(x,t;\alpha,\beta)=\sqrt{\alpha x^2+\beta e^{-\alpha t}} \qquad \alpha,\beta>0$$

(Very) special cases

$$r(x,t;0,\beta)=\sqrt{\beta} \quad \longrightarrow \quad u(x,t)=-e^{-\frac{x}{\sqrt{\beta}}+\frac{t}{2}}, \quad x\in R$$

$$r(x,t;1,0)=|x| \quad \longrightarrow \quad u(x,t)=\log x-\frac{t}{2}, \quad x>0$$

$$r(x,t;\alpha,0)=\sqrt{\alpha}\ |x| \ \longrightarrow \ u(x,t)=\frac{1}{\gamma}x^{\gamma}e^{-\frac{\gamma}{2(1-\gamma)}t}, \ x\geq 0, \ \gamma=\frac{\sqrt{\alpha}-1}{\sqrt{\alpha}}$$

Optimal allocations

Optimal portfolio vector

- The SPDE for the forward performance process

$$dU = \frac{1}{2}\frac{\left|\sigma\sigma^{+}\mathcal{A}\left(U\lambda + a\right)\right|^{2}}{\mathcal{A}^{2}U}dt + a \cdot dW$$

- The optimal portfolio vector

$$\pi_t^* = \pi^*\left(t, X_t^*\right) = -\sigma^{+}\frac{\mathcal{A}\left(U\lambda + a\right)}{\mathcal{A}^{2}U}\left(X_t^*, t\right)$$

- The optimal wealth process

$$dX_t^* = -\sigma\sigma^{+}\frac{\mathcal{A}\left(U\lambda + a\right)}{\mathcal{A}^{2}U}\left(X_t^*, t\right)\left(\lambda dt + dW_t\right)$$

Optimal portfolios in the MZ example

The structure of optimal portfolios

$$dX_t^* = \sigma_t \pi_t^* \cdot (\lambda_t \, dt + dW_t)$$

Stochastic input
Market

$(Y_t,\ Z_t)$
$\lambda_t,\ \sigma_t,\ \delta_t,\ \phi_t$
A_t

Differential input
Individual

wealth x
risk tolerance $r(x,t)$
$r_t + \frac{1}{2} r^2 r_{xx} = 0$

$$U(x,t) = u\left(\frac{x}{Y_t}, A_t\right) Z_t$$

$\frac{1}{Y_t}\pi_t^*$ is a *linear* combination

of (benchmarked) optimal wealth

and subordinated (benchmarked) risk tolerance

Optimal asset allocation

- Let X_t^* be the optimal wealth, Y_t the benchmark and A_t the time-rescaling processes

$$dX_t^* = \sigma_t \pi_t^* \cdot (\lambda_t dt + dW_t)$$

$$dY_t = Y_t \delta_t \cdot (\lambda_t dt + dW_t)$$

$$dA_t = |\sigma_t \sigma_t^+ (\lambda_t + \phi_t) - \delta_t|^2 dt$$

- Define

$$\widetilde{X}_t^* \triangleq \frac{X_t^*}{Y_t} \qquad \text{and} \qquad \widetilde{R}_t^* \triangleq r(\widetilde{X}_t^*, A_t)$$

Optimal (benchmarked) portfolios

$$\boxed{\begin{array}{c} \hat{\pi}_t^* \triangleq \dfrac{1}{Y_t} \pi_t^* = m_t \widetilde{X}_t^* + n_t \widetilde{R}_t^* \\[2em] m_t = \sigma_t^+ \delta_t \qquad n_t = \sigma_t^+ (\lambda_t + \phi_t - \delta_t) \end{array}}$$

Stochastic evolution of wealth-risk tolerance
Explicit construction of optimal processes

A system of SDEs at the optimum

$$\widetilde{X}_t^* = \frac{X_t^*}{Y_t} \quad \text{and} \quad \widetilde{R}_t^* = r(\widetilde{X}_t^*, A_t)$$

$$\begin{cases} d\widetilde{X}_t^* = r(\widetilde{X}_t^*, A_t)(\sigma_t \sigma_t^+(\lambda_t + \phi_t) - \delta_t) \cdot ((\lambda_t - \delta_t)\, dt + dW_t) \\ d\widetilde{R}_t^* = r_x(\widetilde{X}_t^*, A_t) d\widetilde{X}_t^* \end{cases}$$

The optimal wealth and portfolios are explicitly constructed

if the function $r(x,t)$ is known

Solutions of the fast diffusion risk tolerance pde

$$r_t + \frac{1}{2} r^2 r_{xx} = 0$$

Positive and increasing space-time harmonic functions

- Assume that $h(x,t)$ is positive, increasing in x, and satisfies

$$h_t + \frac{1}{2} h_{xx} = 0$$

- Then, it follows from Widder's theorem, that there exists a finite positive Borel measure such that

$$h(x,t) = \int_0^\infty e^{yx - \frac{1}{2}y^2 t} \nu(dy)$$

Risk tolerance function

- Take a positive and increasing space time harmonic function $h(x,t)$
- Define the risk tolerance function $r(x,t)$ by

$$r(x,t) = h_x(h^{-1}(x,t),t)$$

- Then $r(x,t)$ solves the FDE

$$r_t + \frac{1}{2}r^2 r_{xx} = 0\ , \qquad r(0,t) = 0$$

The differential input function u

- Define the function

$$u(x,t) = \int_0^x \exp\left(-h^{-1}(y,t) + \frac{1}{2}t\right)\, dy$$

- Then u solves

$$u_t u_{xx} = \frac{1}{2} u_x^2$$

- Alternatively, use $r(x,t) = h_x(h^{-1}(x,t),t)$ and the transport equation

$$u_t + \frac{1}{2} r u_x = 0$$

Example

- Consider the case when the positive Borel measure is a Dirac delta, i.e.,

$$\nu = \delta_\gamma \ , \qquad \gamma > 0$$

- Then

$$h(x,t) = e^{\gamma x - \frac{1}{2}\gamma^2 t} \ ,$$

$$h^{-1}(x,t) = \frac{1}{\gamma}\left(\log x + \frac{1}{2}\gamma^2 t\right) \ ,$$

$$r(x,t) = \lambda x \ ,$$

$$u(x,t) = \frac{\gamma}{\gamma - 1} x^{\frac{\gamma-1}{\gamma}} e^{-\frac{1}{2}(\gamma-1)t}$$

Globally defined solutions to the u-pde and the FDE

- Assume that for a finite positive Borel measure on $\mathbb{R}$

$$\int_{\mathbb{R}} e^{-yx} \left(1 + |y| + \frac{1}{|y|}\right) \nu(dy) < \infty$$

- Assume that the equation below has a solution

$$b'(t) = -\frac{1}{2} \frac{\int_{\mathbb{R}} e^{-yb(t)-\frac{1}{2}y^2 t} \; \nu(dy)}{\int_{\mathbb{R}} e^{-yb(t)-\frac{1}{2}y^2 t} \; y\nu(dy)}, \qquad b(0) = b_0$$

Increasing space-time harmonic functions

- Define the function

$$h(x,t) = \int_{\mathbb{R}} \left(-e^{-yx-\frac{1}{2}y^2 t} + e^{-yb(t)-\frac{1}{2}y^2 t} \right) \frac{1}{y} v(dy)$$

- The above function satisfies

$$h_t + \frac{1}{2} h_{xx} = 0 \ ,$$

$$h(b(t),t) = 0 \quad \Longleftrightarrow \quad b(t) = h^{-1}(0,t)$$

Risk tolerance function

- The solution to the fast diffusion risk tolerance pde is given by

$$r(x,t) = h_x(h^{-1}(x,t),t)$$

Example

- For positive constants a and b define

$$h(x,t) = \frac{b}{a}\exp\left(-\frac{1}{2}a^2 t\right)\sinh(ax)$$

- Observe that

$$r(x,t) = \sqrt{a^2x^2 + b^2\exp(-a^2 t)}$$

- The corresponding $u(x,t)$ function can be calculated explicitly

- The above class covers the classical exponential, logarithmic and power cases

Notice that $r(x,t)$ is globally defined

Optimal wealth and risk tolerance processes

- Define the process

$$M_t = \int_0^t \sigma\sigma^+ \lambda_s dW_s$$

- Note that

$$A_t = \langle M \rangle_t$$

- Optimal wealth process

$$X_t^{x,*} = h(h^{-1}(x,0) + A_t + M_t, A_t)$$

- Risk tolerance process

$$R_t^{x,*} = r(X_t^{x,*}, A_t) = h_x(h^{-1}(x,0) + A_t + M_t, A_t)$$

Construction

- Initial data $u_0(x)$, or $r_0(x)$, yields $h(x,0)$

- Backward heat equation for h

- Solution $h(x,t)$

- Risk tolerance function $r(x,t) = h_x(h^{-1}(x,t),t)$

- Market input $M_t = \int_0^t \sigma_s \sigma_s^+ \lambda_s dW_s$

Construction

- Optimal wealth $X_t^{x,*} = h\left(h^{-1}(x,0) + \langle M \rangle_t + M_t, \langle M \rangle_t\right)$

- Optimal risk tolerance $r(X_t^{x,*}, t) = h_x\left(h^{-1}(X_t^{x,*}, t), t\right)$

- Optimal portfolio

$$\pi_t^* = k_t^1 X_t^{x,*} + k_t^2 r(X_t^{x,*}, t)$$

- Distributional properties of optimal wealth

- Specification of initial data $h(x,0)$?

- Inference of initial data from the investor's wish list.