# Arbitrage-free joint models for assets and derivatives

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based on joint work with **Johannes Wissel** (ETH Zürich)

# A digression

**Call for Papers for a**

**Special Issue of**

# Finance and Stochastics

**"Computational Methods in Finance"**

http://www.math.ethz.ch/∼finasto

# The problem

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**Basic goal:** Construct a joint dynamic model for

- a stock $S$
- a bond ($\equiv 1$ in discounted units)
- **several call options** $C(K, T)$ on $S$

in such a way that the model

- is **arbitrage-free,**
- gives **explicit joint dynamics** of stock and calls,
- is (perhaps) **practically usable.**

**Where is the problem ?**

# Martingale models ?

- Write down model only for $S$, directly under a pricing measure $Q$.
- Define $C_t(K,T) := E_Q[(S_T - K)^+ | \mathcal{F}_t]$.
- This **martingale model** is obviously arbitrage-free.
- But ...
- ... typically no explicit expressions for $C_t(K,T)$ ...
- ... hence **no explicit joint dynamics** for $S$ and $C$ ...
- ... and good **calibration can be difficult.**

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- ... hence **no explicit joint dynamics** for $S$ and $C$ ...
- ... and good **calibration can be difficult.**
- This is **no solution** to our problem **!**
- So what now ???

# Market models

- **Basic idea:** specify dynamics (SDEs) for **all tradable assets**.
- Joint model for $S$ and all $C(K,T)$ with $K \in \mathcal{K}$ and $T \in \mathcal{T}$.
- **Advantages:**
  - we know **joint dynamics** of $S$ and $C$ by construction.
  - **calibration is automatic** since market prices of options at time 0 are input as initial conditions.
- But: must observe **arbitrage restrictions:**
- How to handle these constraints ?

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  - No **static** arbitrage $\longrightarrow$ restrictions on **state space of processes**.
  - No **dynamic** arbitrage $\longrightarrow$ restrictions on **SDE coefficients** (**drift restrictions** à la HJM).
- How to handle these constraints ?

# Ideas and questions

# The simplest example

- Consider one stock $S$ and **one** call $C(K, T)$. **Restrictions** are
  - **static:** $(S_t - K)^+ \leq C_t \leq S_t$ and $C_T = (S_T - K)^+$.
  - **dynamic:** $S$ and $C$ both martingales under some $Q \approx P$.
- How to write down **explicit SDE** for $(S, C)$ satisfying this **???**

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  - **dynamic:** $S$ and $C$ both martingales under some $Q \approx P$.
- How to write down **explicit SDE** for $(S, C)$ satisfying this **???**
- Way out ($\longrightarrow$ Lyons 1997, Babbar 2001):
  - **reparametrize:** Instead of $C_t$, use **implied volatility** $\hat{\sigma}_t$ via $C_t = c_{BS}\big(S_t, K, (T-t)\hat{\sigma}_t^2\big)$.
  - more precisely: work with $V_t := (T-t)\hat{\sigma}_t^2$.
  - **static** arbitrage constraint is equivalent to $0 \leq V_t < \infty$ and $V_T = 0$; so **state space** is nice.
  - **dynamic** arbitrage constraint reduces to **drift restriction** for $V$ in SDE model for $(S, V)$.
  - SDE is still tricky (nonlinear), but feasible; explicit examples.

# Multiple maturities

- Now consider one stock $S$ and **many** calls $C(K,T)$ with one fixed strike $K$ and **maturities** $T \in \mathcal{T}$.
- Use **new parametrization:**
  - **forward implied volatilities** defined by

$$X_t(T) := \frac{\partial}{\partial T}\big((T-t)\hat{\sigma}_t^2(K,T)\big) = \frac{\partial}{\partial T} V_t(T).$$

  - **static** arbitrage constraints are equivalent to
    **(i)** $V_t(T) \geq 0$ and $V_T(T) = 0$ as before.
    **(ii)** $T \mapsto V_t(T)$ is increasing, i.e., $0 \leq X_t(T) < \infty$.
    So: **state space** is nice.
  - **dynamic** arbitrage constraints reduce to **drift restrictions** for all $X(T)$ in SDE model for $\big(S, X(T)\big)_{T\in\mathcal{T}}$.
- $\longrightarrow$ Schönbucher 1999, Brace et al. 2001, Ledoit et al. 2002

# Problems with multiple maturities I

- Structure of model: start with

$$\begin{aligned} dS_t &= S_t \mu_t \, dt + S_t \sigma_t \, dW_t, \\ dX_t(T) &= \alpha_t(T) \, dt + v_t(T) \, dW_t. \end{aligned}$$

- **Dynamic arbitrage constraints:** $(S_t)$ and all

$$C_t(T) = c_{BS}\left( S_t, K, \int_t^T X_t(s) \, ds \right)$$

must be (local) martingales under some $Q \approx P$.

- **Drift restrictions:**

$$\begin{aligned} \mu_t &= -\sigma_t b_t, \\ \sigma_t &= f\big(X_t(t), S_t, v_t(\cdot)\big), \\ \alpha_t(T) &= g\big(X_t(T), S_t, v_t(\cdot), X_t(\cdot)\big). \end{aligned}$$

# Problems with multiple maturities II

- Structure of model with **drift restrictions:**

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dS_t &= S_t f\big(X_t(t), S_t, v_t(\cdot)\big)(dW_t - b_t\,dt), \\
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- $f$ and $g$ are **nonlinear**; so even if $v$ is Lipschitz and of linear growth, $dt$-coefficients and $\sigma$ are not !
- **Existence problem for (infinite, nonlinear) SDE system !**
- No results in the literature (except classical HJM, with severe conditions: bounded and Lipschitz).

# Multiple strikes: even more problems

- Next consider one stock $S$ and **many** calls $C(K, T)$ with one fixed maturity $T$ and **strikes** $K \in \mathcal{K}$.
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  - **static:** $K \mapsto C_t(K,T)$ is convex and satisfies

$$-1 \le \frac{\partial}{\partial K} C_t(K,T) \le 0.$$

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  - using (classical or forward) implied volatilities does not help either.
- Before even thinking about SDEs: **How to choose parametrization ??**

# Results

# Infinite SDE systems

- **Key mathematical tool:** consider SDE system

$$dX_t(\theta) = A_t\big(\theta, X_\cdot(\cdot)\big)\, dt + B_t\big(\theta, X_\cdot(\cdot)\big)\, dW_t$$

with $\theta \in \Theta$ $\big($usually $[0, T^*]$ or $[0, \infty)\big)$ and $0 \le t \le T_0$.

- $\longrightarrow$ **J. Wissel 2006:**
  - **Existence and uniqueness** result for **strong solution** under only **local Lipschitz-type conditions** on $A, B$.
  - Includes sufficient conditions on growth for non-explosion.
  - Key idea: work on product space $\Theta \times \Omega$.
- **Important:** global Lipschitz condition is too strong for the required applications.

# Multiple maturities

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- **Not direct** from general SDE results, because
  - only $dW$-coefficient $v$ can be chosen here.
  - in addition, must have $X \geq 0$.
- Classes of explicit examples for such models, for **first time in literature.**
- $\longrightarrow$ **S/Wissel 2006**

Explicitly:

$$\begin{aligned}\alpha_t(T) \;=\; & -\frac{1}{2}\left(\big(\mathcal{R}_t(T)\big)^2 - \frac{1}{\mathcal{Z}_t(T)} - \frac{1}{4}\right) v_t(T)\cdot\int_t^T v_t(s)\,ds \\ & + \frac{1}{2}\left(\big(\mathcal{R}_t(T)\big)^2 - \frac{1}{2}\frac{1}{\mathcal{Z}_t(T)}\right)\frac{X_t(T)}{\mathcal{Z}_t(T)}\left|\int_t^T v_t(s)\,ds\right|^2 \\ & + \left(\mathcal{R}_t(T) - \frac{1}{2}\right)\sigma_t v_t^1(T) \\ & - \mathcal{R}_t(T)\frac{X_t(T)}{\mathcal{Z}_t(T)}\sigma_t\int_t^T v_t^1(s)\,ds - b_t\cdot v_t(T)\end{aligned}$$

with

$$Y_t(t) := \log S_t, \quad \mathcal{R}_t(T) := \frac{Y_t(t) - \log K}{\mathcal{Z}_t(T)}, \quad \mathcal{Z}_t(T) := \int_t^T X_t(s)\,ds.$$

## Multiple strikes: new parametrization

- Recall key difficulty: **how to parametrize ?**
- Call option prices **admissible** if for each $t$, $K \mapsto C_t(K)$
  - is $C^2$,
  - is strictly convex,
  - satisfies $-1 < C_t'(K) < 0$ for all $K$,
  - satisfies $\lim_{K \to \infty} C_t(K) = 0$.
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- (This is slight strengthening of **static arbitrage constraints.**)
- **New concept: local implied volatilities**

$$X_t(K) := \frac{1}{\sqrt{T-t}\,KC_t''(K)}\varphi\Big(\Phi^{-1}\big(-C_t'(K)\big)\Big)$$

and, for fixed $K_0$, **price level**

$$Y_t := \sqrt{T-t}\,\Phi^{-1}\big(-C_t'(K_0)\big).$$

- **Theorem:** There is a **bijection** between **admissible option price models** and all pairs $(X, Y)$ of **positive local implied volatility curves** $X$ and **real-valued price levels** $Y$.
- In other words:
  - State space of $(X, Y)$ is nice ...
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  - interpretation for the $X_t(T)$ as "local implied volatilities".
  - explicit formulae relating the classical and the above new local implied volatilities.
  - recovers standard volatility in Black-Scholes setting.

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  - recovers standard volatility in Black-Scholes setting.
- So: **good solution to parametrization problem with multiple strikes !**

# Multiple strikes: structure of models

- **Dynamic arbitrage constraints:**

$$S_t = C_t(0) = \int_0^\infty \Phi\left(\frac{Y_t - \int_{K_0}^k \frac{1}{hX_t(h)}\,dh}{\sqrt{T-t}}\right) dk$$

and all call prices

$$C_t(K) = \int_K^\infty \Phi\left(\frac{Y_t - \int_{K_0}^k \frac{1}{hX_t(h)}\,dh}{\sqrt{T-t}}\right) dk$$

must be (local) martingales under some $Q \approx P$.

- **Drift restrictions** on SDEs for $X$ and $Y$ **?**

- Model for local implied volatilities $X$ and price level $Y$:

$$\begin{aligned} dX_t(K) &= X_t(K)u_t(K)\,dt + X_t(K)v_t(K)\,dW_t, \\ dY_t &= \beta_t\,dt + \gamma_t\,dW_t. \end{aligned}$$

- **Drift restrictions** from **dynamic arbitrage constraints:**

$$\begin{aligned} \beta_t &= -\gamma_t \cdot b_t + \frac{1}{2}\frac{Y_t}{T-t}\left(|\gamma_t|^2 - 1\right), \\ u_t(K) &= -v_t(K)\cdot b_t + \frac{1}{T-t}\Bigg(\frac{1}{2}\left(1 - |\gamma_t + \mathcal{I}_t^v(K)|^2\right) \\ &\qquad + \left(Y_t + \mathcal{I}_t^1(K)\right)\left(\gamma_t + \mathcal{I}_t^v(K)\right)\cdot v_t(K)\Bigg) + |v_t(K)|^2 \end{aligned}$$

with

$$\mathcal{I}_t^1(K) := \int_{K_0}^{K} \frac{1}{hX_t(h)}\,dh, \qquad \mathcal{I}_t^v(K) := \int_{K_0}^{K} \frac{v_t(h)}{hX_t(h)}\,dh.$$

# Multiple strikes: existence of models

- **Theorem:** Sufficient conditions on $v$ (**volatility structure** of local implied volatilities) for **existence and uniqueness** of solution.
- Again, **not direct** from general SDE results, because
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  - only $dW$-coefficient $v$ can be chosen here.
  - in addition, must have $X > 0$.
- Up to now, **no result on existence** of such models in the literature.
- First **tractable parametrization** to tackle this problem at all !
- In addition, explicit class of **examples** for models.
- $\longrightarrow$ **S/Wissel 2007**

# Towards the end

# Open problems (many . . . )

- Model construction and parametrization for **full option surface** (all maturities $T$ and all strikes $K$): **??**
- Practical implementation ?
- Numerical solution ?
- Analogous results for finite family of options ?
  $\longrightarrow$ **new recent results** by **Johannes Wissel**
- Recalibration ?
- Markov property ?
- Specific applications ?
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- Specific applications ?
- . . .
- [*Motto: You name it, we do not know it (yet) . . .*]

# Some related work I

- **Dupire:** **local volatility** model:
  - can also fit any initial term structure of option prices ...
  - ... but seems not rich enough for recalibration over time.
  - no explicit formulas, only PDEs for $C_t(K,T)$ with $t>0$ ...
  - ... and hence no joint dynamics for $S$ and $C$.
- **Bühler:** market models for **variance swaps:**
  - only maturity parameter $T$; no strike structure.
  - special payoff function (log) yields easy infinite SDE system.
  - some more explicit results.
- **Durrleman:** links between **spot and implied volatilities:**
  - classical martingale modelling, no market models.
  - results for at-the-money options and shortly before maturity.
  - asymptotic results; but no dynamics for $S$ and $C$.

# Some related work II

- **Alexander/Nogueira:** **stochastic local volatility** model:
  - extension of Dupire to more stochastic factors ...
  - ... but no existence results for models.
- **Derman/Kani, Carmona/Nadtochiy:** **full option surface:**
  - parametrization and drift restrictions.
  - use “local volatilities”.
  - but no existence result for specified volatility structure.
- **Jacod/Protter:** **fixed payoff function, all maturities:**
  - no strike structure.
  - martingale approach, hence no explicit joint dynamics.
  - “abstract existence of models”.

# References


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# A reminder

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## The end (for the time being . . . )

**Thank you for your attention !**

**http://www.math.ethz.ch/∼mschweiz**

**http://www.math.ethz.ch/∼wissel**