Developments in Volatility Derivatives Pricing

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- Which dynamics are consistent with market prices?

Outline

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   - Problems with one-factor stochastic volatility models.
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    - Bergomi's variance curve model.
    - Buehler's consistent variance curve functionals.

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- We know from PCA of volatility surface time series that there are at least three important modes of fluctuation:
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- It makes sense to add at least one more factor.

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  - Variance curves are more realistic in the two-factor case. For example, they can have humps.

Historical attempts to add factors

- Dupire's unified theory of volatility (1996)
  - Local variances are driftless in the butterfly measure.
  - We can impose dynamics on local variances.
- Stochastic implied volatility (1998)
  - The implied volatility surface is allowed to move.
  - Under diffusion, complex no-arbitrage condition, impossible to work with in practice.
- Variance curve models (1993-2005)
  - Variances are tradable!
  - Simple no-arbitrage condition.

Dupire's unified theory of volatility

- The price of the calendar spread $\partial_T C(K,T)$ expressed in terms of the butterfly $\partial_{K,K} C(K,T)$ is a martingale under the measure $Q_{K,T}$ associated with the butterfly.

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- Local variance $v_L(K,T)$ is given by (twice) the current ratio of the calendar spread to the butterfly.
- We may impose any dynamics such that the above holds and local variance stays non-negative.
- For example, with one-factor lognormal dynamics, we may write:

$$v(S,t) = v_L(S,t)\,\frac{\exp\left\{-b^2/2\,t - b\,W_t\right\}}{\mathbb{E}\left[\exp\left\{-b^2/2\,t - b\,W_t\right\} \mid S_t = S\right]}$$

where it is understood that $v_L(\cdot)$ is computed at time $t=0$. Note that the denominator is hard to compute!

Stochastic implied volatility

- The evolution of implied volatilities is modeled directly as in $\sigma_{BS}(k, T, t) = G(\mathbf{z}; k, T - t)$ with $\mathbf{z} = \{z_1, z_2, ..., z_n\}$ for some factors z_i.

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  - SABR is a stochastic implied volatility model, albeit without mean reversion, but it's not arbitrage-free.
- *Stochastic implied volatility is a dead end!*

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  - and forward variance swaps are natural hedges for cliquets and other exotics.
- Thus, as originally suggested by Dupire in 1993, and then latterly by Duanmu, Bergomi, Buehler and others, we should impose dynamics on forward variance swaps.

Modeling forward variance

Denote the variance curve as of time t by $\hat{W}_t(T) = \mathbb{E}\left[\int_0^T v_s\, ds \middle| \mathcal{F}_t\right]$. The forward variance $\zeta_t(T) := \mathbb{E}\left[v_T \mid \mathcal{F}_t\right]$ is given by

$$\zeta_t(T) = \partial_T \hat{W}_t(T)$$

A natural way of satisfying the martingale constraint whilst ensuring positivity is to impose lognormal dynamics as in Dupire's (1993) example:

$$\frac{d\zeta_t(T)}{\zeta_t(T)} = \sigma(T-t)\, dW_t$$

for some volatility function $\sigma(\cdot)$.

Lorenzo Bergomi does this and extends the idea to n-factors.

Bergomi's model

In the 2-factor version of his model, we have

$$\frac{d\zeta_t(T)}{\zeta_t(T)} = \xi_1\, e^{-\kappa\,(T-t)}\, dW_t + \xi_2\, e^{-c\,(T-t)}\, dZ_t$$

This has the solution

$$\zeta_t(T) = \zeta_0(T)\, \exp\left\{\xi_1\, e^{-\kappa\,(T-t)}\, X_t + \xi_2\, e^{-c\,(T-t)}\, Y_t + \text{drift terms}\right\}$$

with

$$X_t = \int_0^t e^{-\kappa\,(t-s)}\, dW_s;\; Y_t = \int_0^t e^{-c\,(t-s)}\, dZ_s;$$

Thus, both X_t and Y_t are Ornstein-Ühlenbeck processes. In particular, they are easy to simulate. The Bergomi model is a market model: $\mathbb{E}\left[\zeta_t(T)\right] = \zeta_0(T)$ for any given initial forward variance curve $\zeta_0(T)$.

Variance curve models

- The idea (similar to the stochastic implied volatility idea) is to obtain a factor model for forward variance swaps. That is,

$$\zeta_t(T) = G(\mathbf{z}; T - t)$$

  with $\mathbf{z} = \{z_1, z_2, ..., z_n\}$ for some factors z_j and some *variance curve functional* $G(\cdot)$.
- Specifically, we want $\mathbf{z}$ to be a diffusion so that

$$d\mathbf{z}_t = \mu(\mathbf{z}_t)\,dt + \sum_j^d \sigma^j(\mathbf{z}_t)\,dW_t^j \tag{1}$$

  - Note that both μ and σ are $n-$dimensional vectors.

Buehler's consistency condition

Theorem

The variance curve functional $G(\mathbf{z}_t, \tau)$ is consistent with the dynamics (1) if and only if

$$\begin{aligned}\partial_\tau G(\mathbf{z};\tau) &= \sum_{i=1}^{n} \mu_i(\mathbf{z})\,\partial_{z_i} G(\mathbf{z};\tau) \\ &\quad + \frac{1}{2}\sum_{i,k=1}^{n}\left(\sum_{j=1}^{d} \sigma_i^j(\mathbf{z})\,\sigma_k^j(\mathbf{z})\right)\,\partial_{z_i,z_k} G(\mathbf{z};\tau)\end{aligned}$$

To get the idea, apply Itô's Lemma to $\zeta_t(T) = G(z, T-t)$ with $dz = \mu\, dt + \sigma\, dW$ to obtain

$$\mathbb{E}\left[d\zeta_t(T)\right] = 0 = \left\{-\partial_\tau G(z,\tau) + \mu\,\partial_z G(z,\tau) + \frac{1}{2}\sigma^2\,\partial_{z,z} G(z,\tau)\right\} dt$$

Example: The Heston model

- In the Heston model, $G(v,\tau) = v + (v - \bar{v})\, e^{-\kappa\,\tau}$.
  - This variance curve functional is obviously consistent with Heston dynamics with time-independent parameters κ, ρ and η.
- Imposing the consistency condition, Buehler shows that the mean reversion rate κ cannot be time-dependent.
- By imposing a similar martingale condition on forward entropy swaps, Buehler further shows that the product $\rho\,\eta$ of correlation and volatility of volatility cannot be time-dependent.

Buehler's affine variance curve functional

- Consider the following variance curve functional:

$$G(\mathbf{z};\tau) = z_3 + (z_1 - z_3)\, e^{-\kappa\,\tau} + (z_2 - z_3)\,\frac{\kappa}{\kappa - c}\,\left(e^{-c\,\tau} - e^{-\kappa\,\tau}\right)$$

  - This looks like the Svensson parametrization of the yield curve.
  - The short end of the curve is given by z_1 and the long end by z_3.
  - The middle level z_2 adds flexibility permitting for example a hump in the curve.

Consistent dynamics

- Buehler's affine variance curve functional is consistent with double mean reverting dynamics of the form:

$$\begin{aligned}
\frac{dS}{S} &= \sqrt{v}\,dW \\
dv &= -\kappa\,(v - v')\,dt + \eta_1\,v^{\alpha}\,dZ_1 \\
dv' &= -c\,(v' - z_3)\,dt + \eta_2\,v'^{\beta}\,dZ_2
\end{aligned}$$

  for any choice of α, $\beta \in [1/2, 1]$.
  - We will call the case $\alpha = \beta = 1/2$ *Double Heston*,
  - the case $\alpha = \beta = 1$ *Double Lognormal*,
  - and the general case *Double CEV*.
- All such models involve a short term variance level v that reverts to a moving level v' at rate κ. v' reverts to the long-term level z_3 at the slower rate $c < \kappa$.

Check of consistency condition

- Because $G(\cdot)$ is affine in z_1 and z_2, we have that

$$\partial_{z_i,z_j} G\left(\{z_1, z_2\}; \tau\right) = 0 \qquad i, j \in \{1, 2\}.$$

- Then the consistency condition reduces to

$$\begin{aligned} \partial_\tau G(\{z_1, z_2\}; \tau) &= \sum_{i=1}^{2} \mu_i(\{z_1, z_2\})\, \partial_{z_i} G(\{z_1, z_2\}; \tau) \\ &= -\kappa\,(z_1 - z_2)\,\partial_{z_1} G - c\,(z_2 - z_3)\,\partial_{z_2} G \end{aligned}$$

  - It is easy to verify that this holds for our affine functional.
- In fact, the consistency condition looks this simple for affine variance curve functionals with any number of factors!

Dufresne's trick for computing moments

Dufresne (2001) shows how to compute any desired moment of the state variables in the Heston model through repeated application of Itô's Lemma.

For example, suppose we want to compute the second moment of integrated variance $W_T := \int_0^T v_t \, dt$. We first note that

$$dW_t = v_t \, dt$$

Then,

$$d(W_t)^2 = 2 \, W_t \, v_t \, dt$$

so

$$\mathbb{E}\left[(W_t)^2\right] = 2 \int_0^t \mathbb{E}\left[W_s \, v_s\right] \, ds$$

We may repeat this procedure to compute $\mathbb{E}\left[W_t\, v_t\right]$. Specifically, applying Itô's Lemma,

$$d(W_t\, v_t) = W_t\, dv_t + v_t\, dW_t = W_t\,\left[-\kappa\,(v_t - \bar{v})\,dt + \eta\,\sqrt{v_t}\,dZ\right] + v_t^2\,dt$$

Thus

$$\mathbb{E}\left[W_t\, v_t\right] = \kappa\,\bar{v}\,\int_0^t e^{-\kappa\,(t-s)}\,\mathbb{E}\left[W_s\right]\,ds + \int_0^t e^{-\kappa\,(t-s)}\,\mathbb{E}\left[v_s^2\right]\,ds$$

We can apply Itô's Lemma once more to find $\mathbb{E}\left[v_t^2\right]$ and integrate to get our result.

- This trick will also work for
  - the Double Heston model (so long as $\mathbb{E}[dZ_1\, dZ_2] = 0$)
  - for the Double Lognormal model (even if $\mathbb{E}[dZ_1\, dZ_2] \neq 0$).

A digression: formulations of lognormal stochastic volatility

There are at least two obvious ways of writing down a lognormal stochastic volatility model:

$$dv = -\kappa\,(v - \bar{v})\,dt + \xi\,v\,dZ \tag{2}$$

and

$$d(\log v) = -\kappa\,(\log v - \theta)\,dt + \xi\,dZ \tag{3}$$

(2) allows for easy computation of moments, including moments of integrated variance, using Dufresne's trick. On the other hand, with the Ornstein-Ühlenbeck formulation (3), $\log v$ is normally distributed with easy expressions for the mean and variance, so exact big-step Monte Carlo becomes possible.

Double Lognormal vs Bergomi

- Recall that the Bergomi model has dynamics (with $\tau = T - t$)

$$\frac{d\zeta_t(T)}{\zeta_t(T)} = \xi_1\, e^{-\kappa\,\tau}\, dZ_1 + \xi_2\, e^{-c\,\tau}\, dZ_2$$

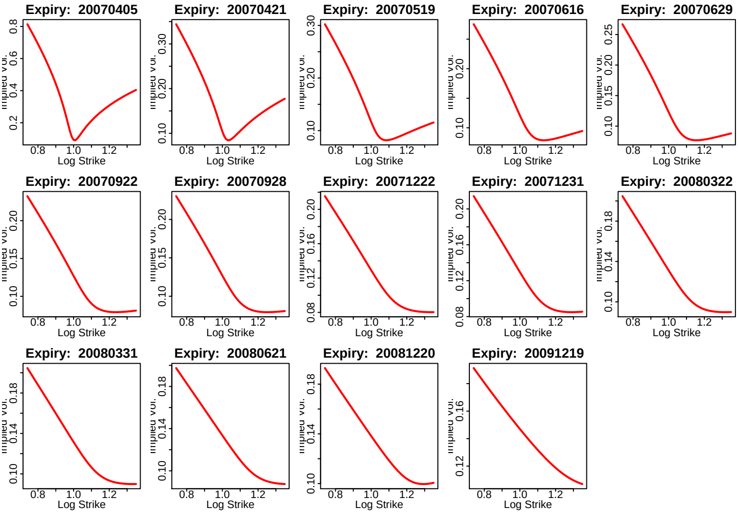
- Now in the Double Lognormal model

$$\begin{aligned} d\zeta_t(T) &= dG(v, v'; \tau) \\ &= \xi_1\, v\, e^{-\kappa\,\tau}\, dZ_1 + \xi_2\, v'\, \frac{\kappa}{\kappa - c}\, \left(e^{-c\,\tau} - e^{-\kappa\,\tau}\right)\, dZ_2 \end{aligned}$$

  - We see that the two sets of dynamics are very similar.
- Bergomi's model is a market model and Buehler's affine model is a factor model.
  - However any variance curve model may be made to fit the initial variance curve by writing

$$\zeta_t(T) = \frac{\zeta_0(T)}{G(\mathbf{z}_0, T)}\, G(\mathbf{z}_t, T)$$

SPX option implied volatilities as of 03-Apr-2007



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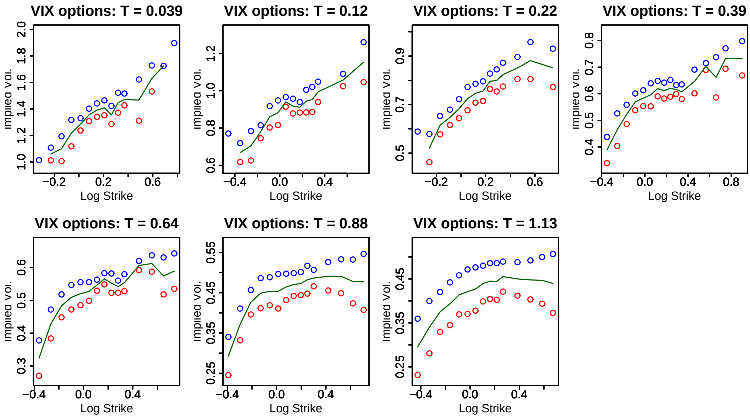


- We note that skews are steeply positive and that implied volatilities decline with time to expiry.

How to price options on VIX

A VIX option expiring at time T with strike K_{VIX} is valued at time t as

$$\mathbb{E}_t\left[\left(\sqrt{\mathbb{E}_T\left[\int_T^{T+\Delta} v_s\, ds\right]} - K_{VIX}\right)^+\right]$$

where Δ is around one month (we take $\Delta = 1/12$).
In the affine models under consideration, the inner expectation is linear in v_T, v'_T and z_3 so that

$$VIX_T{}^2 = \mathbb{E}_T\left[\int_T^{T+\Delta} v_s\, ds\right] = a_1\, v_T + a_2\, v'_T + a_3\, z_3$$

with some coefficients a_1,a_2 and a_3 that depend only on Δ.

A simple lognormal model

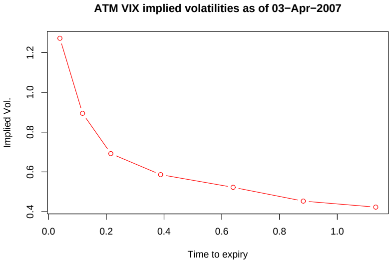
As in Friz-Gatheral, assume (wrongly of course) that VIX is lognormally distributed: $\log VIX \sim N(\mu, s^2)$. Then VIX^2 is also lognormal with $\log VIX^2 \sim N(2\,\mu, 4\,s^2)$. Then

$$
\begin{aligned}
\mathbb{E}_t\left[{VIX_T}^2\right] &= \mathbb{E}_t\left[a_1\,v_T + a_2\,v_T' + a_3\,z_3\right] = \exp\left\{2\,\mu + 2\,s^2\right\} \\
\mathbb{E}_t\left[{VIX_T}^4\right] &= \mathbb{E}_t\left[\left(a_1\,v_T + a_2\,v_T' + a_3\,z_3\right)^2\right] = \exp\left\{4\,\mu + 8\,s^2\right\}
\end{aligned}
$$

- $\mathbb{E}_t\left[a_1\,v_T + a_2\,v_T' + a_3\,z_3\right]$ is easy to evaluate; the result does not depend on whether we choose Heston or lognormal dynamics.
- $\mathbb{E}_t\left[\left(a_1\,v_T + a_2\,v_T' + a_3\,z_3\right)^2\right]$ may be computed using the Dufresne trick; in this case, the result does depend on our choice of dynamics.

Calibration to VIX options

- We now have explicit expressions for μ and s under both Heston and lognormal dynamics.
- Moreover, with our lognormal assumption, the volatility smile of VIX options will be flat at the level $s/\sqrt{t}$.
- We proceed by fitting the model parameters jointly to the term structure of VIX forwards and ATM VIX implied volatilities.



Calibration to VIX options

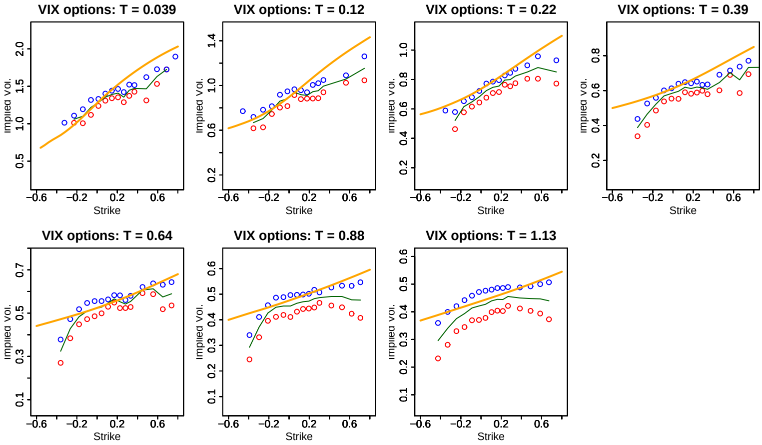
- Of course, because VIX is not lognormally distributed, this calibration doesn't work very well.
  - It gets us close enough to be able to fit with manual tweaking of parameters
- In the next slide, we see the result of tweaking.

Fit of Double Lognormal model to VIX options

From Monte Carlo simulation with parameters

$$z_1 = 0.0137;\ z_2 = 0.0208;\ z_3 = 0.0421;\ \kappa = 12;\ \xi_1 = 7;\ c = 0.34;\ \xi_2 = 0.94;$$

we get the following fits (orange lines):

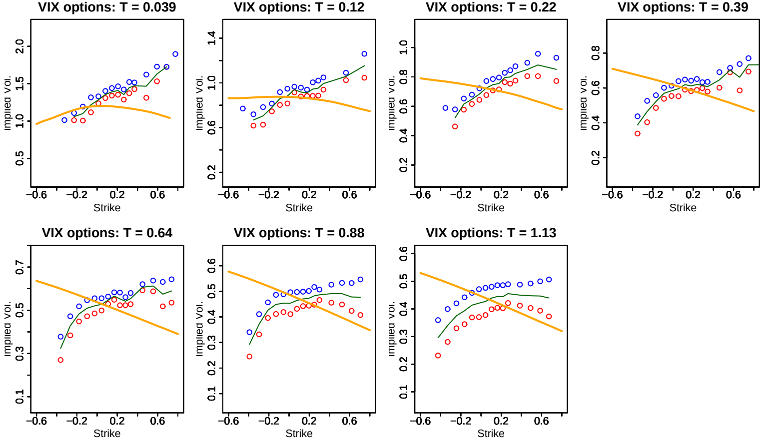


Fit of Double Heston model to VIX options

From Monte Carlo simulation with parameters

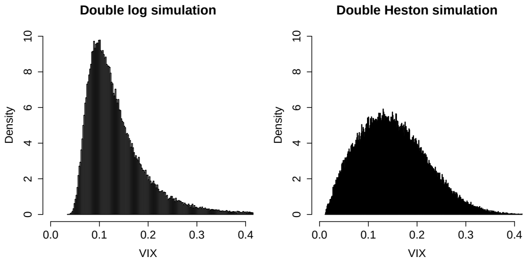
$$z_1 = 0.0137;\ z_2 = 0.0208;\ z_3 = 0.0421;\ \kappa = 12;\ \eta_1 = 0.7;\ c = 0.34;\ \eta_2 = 0.14;$$

we get the following fits (orange lines):



In terms of densities of VIX

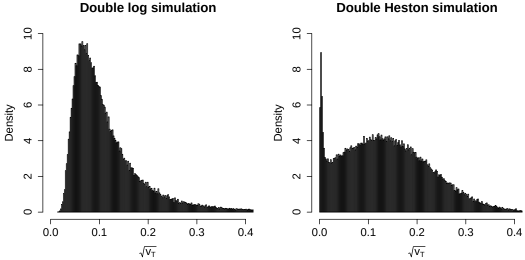
- When we draw the densities of VIX for the last expiration ($T = 1.13$) under each of the two modeling assumptions, we see what's happening:



- In the (double) Heston model, v_t spends too much time in the neighborhood of $v = 0$ and too little time at high volatilities.

In terms of densities of $\sqrt{v_T}$

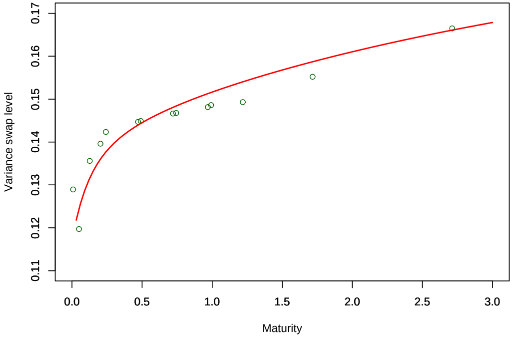
- We see this really clearly when we plot the densities of $\sqrt{v_T}$ (again with $T = 1.13$):



- The distribution of v_T in the Heston model is completely unrealistic. What do you think is the probability of instantaneous volatility being less than 2%?

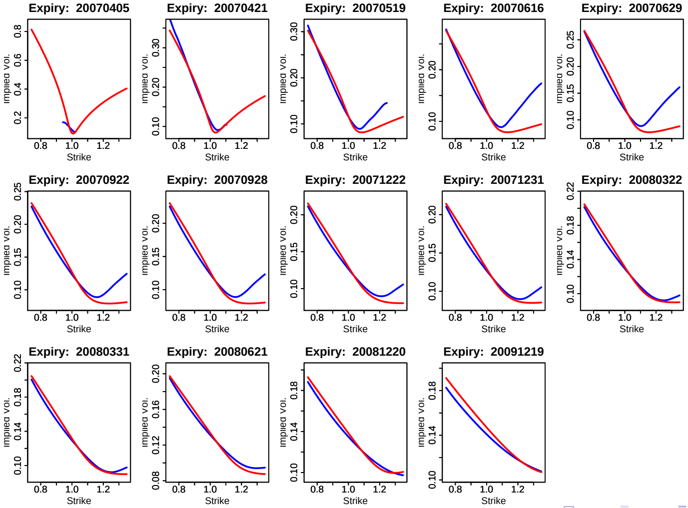
Fit to SPX variance swaps

Variance swap fits are independent of the specific dynamics. Then as before with
$z_1 = 0.0137$; $z_2 = 0.0208$; $z_3 = 0.0421$; $\kappa = 12$; $c = 0.34$, we obtain the following fit (green points are market prices):



Fit of double lognormal model to SPX options

From Monte Carlo simulation with the same parameters as before plus $\rho_1 = -0.66$, $\rho_2 = -0.60$, we get the following fits (blue lines):



Parameter stability

- Suppose we keep all the parameters unchanged from our 03-Apr-2007 fit. Can we fit the VIX option smiles from a later date?
- Recall the parameters:
  - Lognormal parameters:

$$\kappa = 12;\ \xi_1 = 7;\ c = 0.34;\ \xi_2 = 0.94;$$

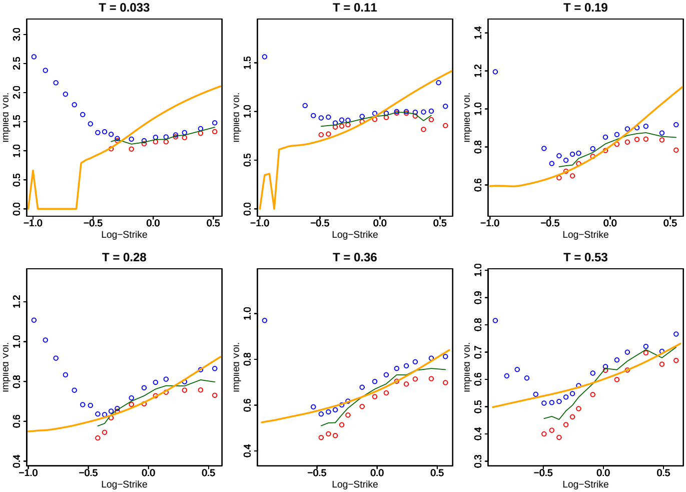
  - Heston parameters:

$$\kappa = 12;\ \eta_1 = 0.7;\ c = 0.34;\ \eta_2 = 0.14;$$

- Specifically, consider 09-Nov-2007 when volatilities were much higher than April.

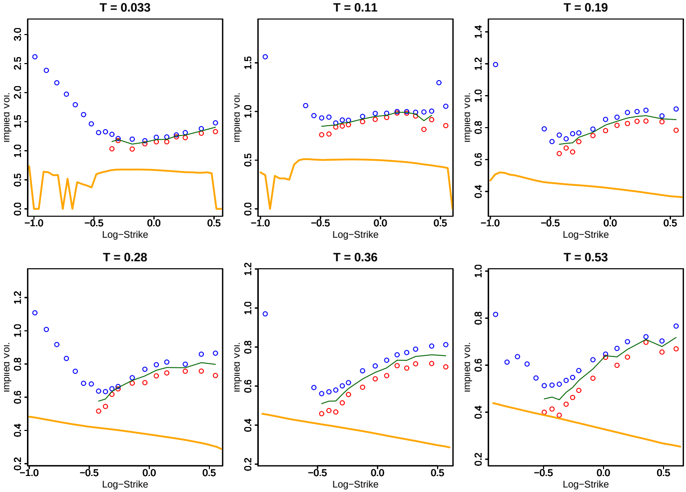
Double Lognormal fit to VIX options as of 09-Nov-2007

We get the following fits (orange lines):



Double Heston fit to VIX options as of 09-Nov-2007

We get the following fits (orange lines):



Observations

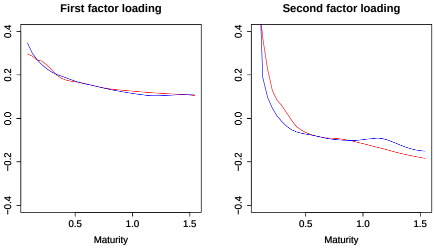
- The double lognormal model clearly fits the market better than double Heston.
  - Not only does lognormal fit better on a given day, but parameters are more stable over time.
- We can fit both short and long expirations with the same parameters in contrast to single-factor stochastic volatility models.
- The fitted lognormal parameters are such that the fast timescale ($\log 2/12 \approx 3$ weeks) is shorter than the expiration of most VIX options; the slow timescale ($log(2)/.34 \approx 2$ years) is longer than the expiration of the longest-dated VIX option.
  - We therefore have time-scale separation and can apply the methods of (for example) Fouque, Papanicolaou, Sircar and Solna.

Implied vs Historical

- Just as option traders like to compare implied volatility with historical volatility, we would like to compare the risk-neutral parameters that we got by fitting the Double Lognormal model to the VIX and SPX options markets with the historical behavior of the variance curve.
- First, we check to see (in the time series data) how many factors are required to model the variance curve.

PCA on historical variance swap data

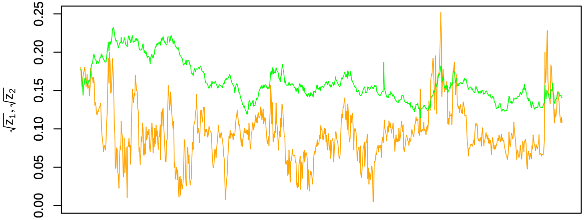
- We proxy variance swaps by the log-strip for each expiration.
- Spline-interpolate to get standardized variance curves.
- Perform PCA on first differences to obtain the following two factors:



- The blue line is conventional PCA and the red line is robust PCA.

Extracting time series for z_1 and z_2

- In our affine model, given estimates of κ, c and z_3, we may estimate z_1 and z_2 using linear regression.
- From two years of SPX option data with parameters $\kappa = 12$, $c = 0.34$ and $z_3 = 0.0421$, we obtain the following time series for $\sqrt{z_1}$ (orange) and $\sqrt{z_2}$ (green):

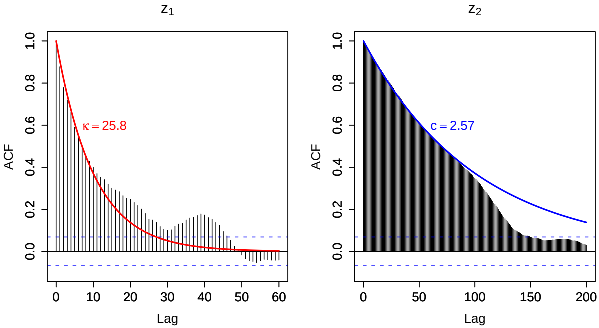


Statistics of z_1 and z_2

- Let's naïvely compute the standard deviations of log-differences of z_1 and z_2. We obtain

Factor	Historical vol.	Implied vol. (from VIX)
z_1	8.6	7.0
z_2	0.84	0.94

- The two factors have the following autocorrelation plots



Observations

- Historical and implied volatilities are similar
    - in contrast to single-factor stochastic volatility models.
- Historical decay rates are greater than implied
    - price of risk effect just as in single-factor stochastic volatility models.

Summary I

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- It makes sense to model tradables such as variance swaps rather than non-tradables such as implied volatilities.
- The affine variance curve functional introduced by Buehler is particularly attractive
  - It is consistent with many dynamics of interest.
- Dufresne's trick of recursively applying Itô's Lemma allows us to compute moments for both Heston and lognormal dynamics.
- Although Double Heston is more analytically tractable, Double Lognormal agrees much better with the market.
  - Whilst the rough levels of VIX option implied volatilities are determined by SPX option prices, VIX option skews are seen to be very sensitive to dynamical assumptions.

Summary II

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Summary II

- By adding a second volatility factor, we have achieved the following:
  - The term structure of SPX skew seems right even for short expirations with no need for jumps.
  - We are able to fit VIX options with time-homogeneous parameters.
  - Historical and risk-neutral estimates of the volatilities of the factors are similar
    - Recall that implied and historical vol. of vol. are very different in single-factor volatility models.

Current and future research

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- Investigate alternative dynamics:
  - More general CEV models with $\alpha, \beta \neq 1/2$ or 1
  - Add jumps in volatility and stock price.
- Add more tradable factors.

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