Recovering credit portfolio loss dynamics from CDO tranches : solution of an inverse problem via intensity control

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Outline

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- Reconstruction of the loss intensity by relative entropy minimization under constraints
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- Nonlinear representation as expectation
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Portfolio credit derivatives

Contracts whose payoffs depend on the losses due to defaults in some underlying reference portfolio (of loans, bonds or credit default swaps).

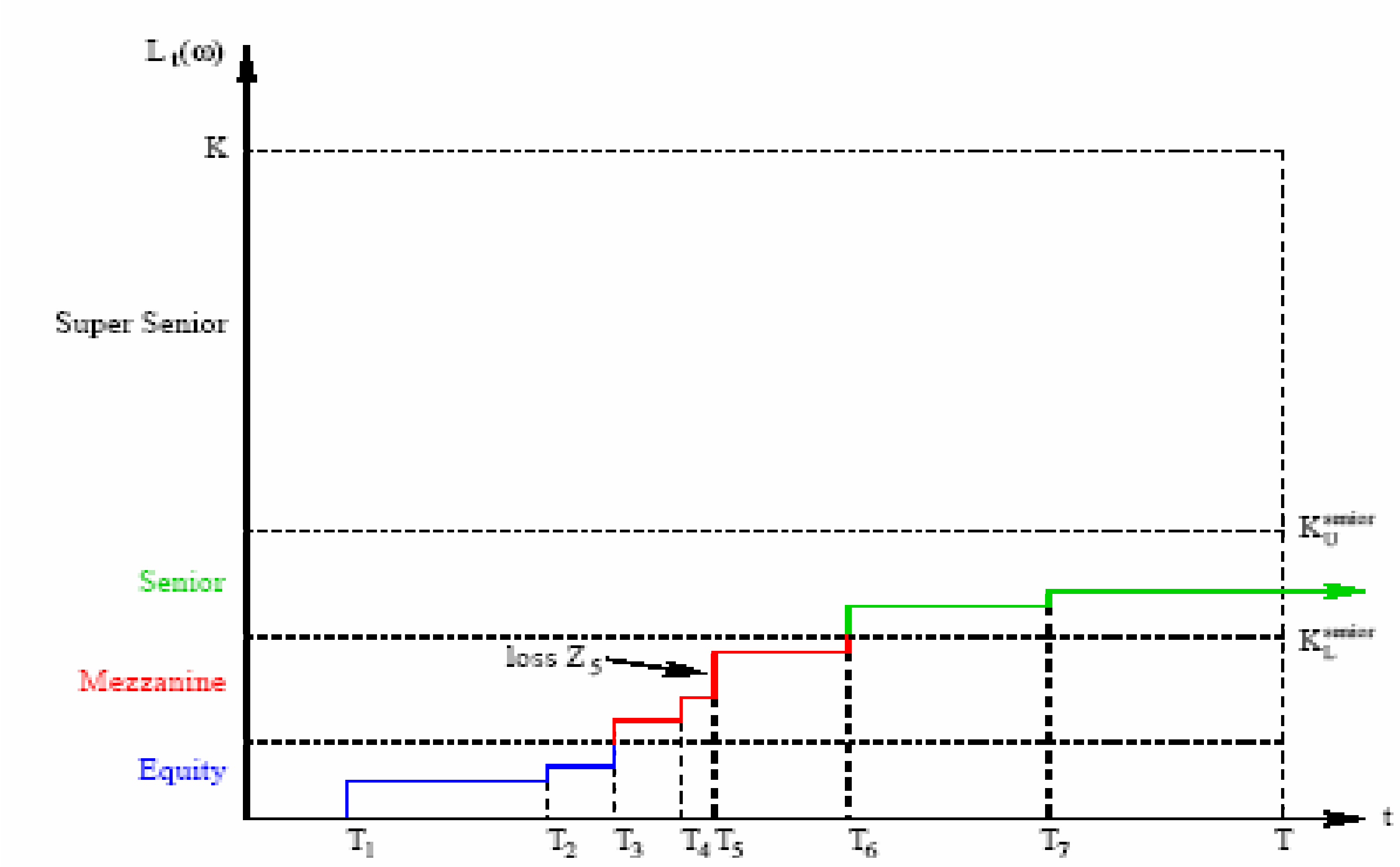
Most common example: Collateralized Debt Obligations (CDOs).

Commonly used approach to pricing of portfolio credit derivatives: value = discounted expectation of cash flows computed under a *pricing measure* ("risk neutral probability") $\mathbb{Q}$:

$$V_t = \sum_{t_j > t} E^{\mathbb{Q}}[B(t, t_j) H_j(L(t_j))] \tag{1}$$

where t_j are cash flow dates, $L(t_j)$ is the loss due to default in the reference portfolio, $B(t, t_j)$ is the discount factor and $H_j(L(t_j))$ is the random, default dependent cash flow paid at t_j.

Sample path of the loss process



ITRAXX CDO tranches

Maturity	Low	High	Bid\ Upfront	Ask\ Upfront
5Y	0%	3%	11.75%	12.00%
	3%	6%	53.75	55.25
	6%	9%	14.00	15.50
	9%	12%	5.75	6.75
	12%	22%	2.13	2.88
	22%	100%	0.80	1.30
7Y	0%	3%	26.88%	27.13%
	3%	6%	130	132
	6%	9%	36.75	38.25
	9%	12%	16.50	18.00
	12%	22%	5.50	6.50
	22%	100%	2.40	2.90

Maturity	Low	High	Bid\ Upfront	Ask\ Upfront
10Y	0%	3%	41.88%	42.13%
	3%	6%	348	353
	6%	9%	93	95
	9%	12%	40	42
	12%	22%	13.25	14.25
	22%	100%	4.35	4.85

Table 1: ITRAXX tranche spreads, in bp. For the equity tranche the periodic spread is 500bp and figures represent upfront payments.

Ingredients

- Nominals $N_i, i = 1..n$, Total nominal $N = \sum N_i$
- Default dates $\tau_i, i = 1..n$
- Risk neutral probability of default $F_i(t) = \mathbb{Q}(\tau_i \leq t)$
- Survival function $S_i(t) = 1 - F_i(t)$
- Recovery rate R_i
- Risk-free discount factor $B(t, T)$
- Portfolio loss (as percentage of total nominalO:
  $L_t = \frac{1}{N} \sum_{i=1}^{n} N_i(1 - R_i) 1_{\tau_i \leq t}$
- Tranche loss: $L_{a,b}(t) = (L(t) - a)^+ - (L(t) - b)^+$

Cash flow structure of a CDO tranche

Default leg: tranche loss due to defaults between t_{j-1} and t_j

$$\begin{aligned}
\text{Cash flow at } t_j \qquad & N[L_{a,b}(t_j) - L_{a,b}(t_{j-1})] \\
\text{Value at } t=0 \qquad & N\sum_{j=1}^{J} B(0,t_j)E^{\mathbb{Q}}[L_{a,b}(t_j) - L_{a,b}(t_{j-1})\ ] \qquad (2) \\
= \qquad & N\sum_{j=1}^{J} B(0,t_j)E^{\mathbb{Q}}[(L(t_j)-a)^+ - (L(t_j)-b)^+ \\
& \quad -(L(t_{j-1})-a)^+ \ + (L(t_{j-1})-b)^+]
\end{aligned}$$

Similar to pricing of a portfolio of calls on $L(t)$.

Requires knowledge of the risk neutral distribution of total portfolio loss $L(t)$

Premium leg: pays fixed spread S(a,b) at dates t_j on remaining principal

Cash flow at t_j $\quad S(a,b)N(t_j - t_{j-1})[(b - L(t_j))^+ - (a - L(t_j))^+]$

Value at $t = 0$ $\quad S(a,b)N \sum_{j=1}^{J} B(0,t_j)(t_j - t_{j-1})$

$$E^{\mathbb{Q}}[(b - L(t_j))^+ - (a - L(t_j))^+]$$

Computation of $E^{\mathbb{Q}}[(L(t_j) - K)^+]$ requires knowledge of the (risk neutral) distribution of total loss $L(t_j)$ which depends on dependence among defaults

Fair spread of a CDO tranche swap with attachment point a and detachment b initiated at $t = 0$:

$$S_0(a,b) = \frac{\sum_{j=1}^{J} B(0,t_j) E^{\mathbb{Q}}[L_{a,b}(t_j) - L_{a,b}(t_{j-1})\ ]}{\sum_{j=1}^{J} B(0,t_j)(t_j - t_{j-1}) E^{\mathbb{Q}}[(b - L(t_j))^+ - (a - L(t_j))^+]}$$

Computation of CDO spread involves $E^{\mathbb{Q}}[(L(t_j) - K)^+]$ which requires knowledge of the (risk neutral) distribution of total loss $L(t_j)$: involves assumptions on dependence among defaults ("default correlation")

Mark to market value of the value of a protection seller on the tranche: premium leg- default leg

$$
\begin{aligned}
MTM(t) = NS_0(a,b) \sum_{t_j > t} B(t,t_j)\delta_j E^{\mathbb{Q}}[(b - L(t_j))^+ - (a - L(t_j))^+ | \mathcal{F}_t] \\
-N \sum_{t_j > t} B(t,t_j) E^{\mathbb{Q}}[L_{a,b}(t_j) - L_{a,b}(t_{j-1}) | \mathcal{F}_t] \\
= N(b-a) \sum_{t_j > t} B(t,t_j)[S_0(a,b)\delta_j(1 - P_{a,b}(t,t_j)) -
\end{aligned}
$$

$$
= [S_0(a,b) - S_t(a,b)]\ N \sum_{t_j > t} B(t,t_j)\delta_j E^{\mathbb{Q}}[(b - L(t_j))^+ - (a - L(t_j))^+ | \mathcal{F}_t]
$$

where $\delta_j = t_j - t_{j-1}$.

Case of the equity tranche $[0, K]$

Default leg: tranche loss due to defaults between t_{j-1} and t_j

Cash flow at t_j $\qquad N[\min(L(t_j), K) - \min(L(t_{j-1}), K)]$

Value at $t = 0$ $\qquad N \sum_{j=1}^{J} B(0, t_j) E^{\mathbb{Q}}[\min(L(t_j), K) - \min(L(t_{j-1}), K)]$

Premium leg: upfront fee $U(K)\%$ of the nominal of the tranche+ fixed spread f (usually 500 bp) at dates t_j on remaining principal

Cash flow at t_j	$f(t_j - t_{j-1})(K - L(t_j))^+$
Value at $t = 0$	$Nf \sum_{j=1}^{J} B(0, t_j)(t_j - t_{j-1})E^{\mathbb{Q}}[(K - L(t_j))^+]$
	$+NKU(K)$

Upfront fee for equity tranche with detachment point K:

$$KU(K) = \sum_{j=1}^{J} B(0,t_j) E^{\mathbb{Q}}[\min(L(t_j),K) - \min(L(t_{j-1},K)\ ]$$
$$-f \sum_{j=1}^{J} B(0,t_j)(t_j - t_{j-1}) E^{\mathbb{Q}}[(K - L(t_j))^+]$$

Computation requires knowledge of the (conditional) distribution $P_x(t,t_j) = \mathbb{Q}(L(t_j) \leq x | \mathcal{F}_t)$ of total loss $L(t_j)$ which depends on dependence among defaults

Bottom-up approach in credit portfolio modeling

Idea:

- calibrate implied default probabilities for portfolio components to credit default swap term structures
- add extra ingredient (copula, dependence structure) to obtain joint distribution $F(t_1, .., t_n)$ of default times (n-dimensional probability distribution)
- Use numerical procedure to compute the risk–neutral distribution of portfolio loss L_t from F: recursion methods, FFT, quadrature, Monte Carlo,...
- Imply correlation parameters from tranche spreads

Issues:

- High dimensional models: $n \simeq 100 - -500$.
- Need to separate joint distribution into copula + marginals and parameterize them separately otherwise calibration to CDS and CDO tranches cannot be separated $\rightarrow$ high-dimensional nonlinear optimization problem
- scarcity of data $\rightarrow$ crude parametrization of joint distribution/copula $\rightarrow$ restrictions on default dependence structure.

Disadvantages of default time copula models

Copula models

- are unable to reproduce implied correlations for quoted CDO tranches in a simple manner.
- are static: no dynamics for spreads, no spread volatility, no way to update prices as time goes on.
- do not tell us how to compute conditional default probabilities, forward tranche prices,...

Prototype of dynamic credit portfolio model: Duffie & Garleanu (2005)

Default in each of $i = 1..N \sim 100$ names driven by a random intensity process $\lambda^i(t)$ modeled as an affine jump-diffusion

$$\lambda^i(t) = \sqrt{\rho}\lambda^0(t) + \sqrt{1-\rho}\lambda_i(t) \tag{3}$$

$$d\lambda_i(t) = (a + b\lambda_i)dt + c\sqrt{\lambda_i(t)}dW_t^i + dJ_i(t) \tag{4}$$

Parameter ρ is difficult to calibrate: it cannot be calibrated separately from parameters describing dynamics of N individual spreads.

As a result, getting market-consistent prices is a challenge.

Recall the expression for a CDO tranche spread

$$S_t(a,b) = \frac{\sum_{j=1}^{m} B(0,t_j) E^{\mathbb{Q}}[L_{a,b}(t_j) - L_{a,b}(t_{j-1})\ |\mathcal{F}_t]}{\sum_{j=1}^{m} B(0,t_j)\delta_j E^{\mathbb{Q}}[(b - L(t_j))^+ - (a - L(t_j))^+ |\mathcal{F}_t]}$$

Key observation: only involves the (conditional) distribution of total portfolio loss L_t:

$$p_{t,T}(x) = \mathbb{Q}(\ L_T \leq x|\mathcal{F}_t) \tag{5}$$

Top-down representation of the portfolio loss

Loss process is a (pure jump) process with increasing sample paths, whose jump times T_j are the default events and whose jump sizes L_j are default losses:

$$L_t = \frac{1}{N}\sum_{i=1}^{n} N_i(1-R_i)1_{\tau_i \le t} = \sum_{j=1}^{N_t} L_j \tag{7}$$

where $N_t = \sum_{i=1}^{n} 1_{\tau_i \le t}$ is the number of defaults in portfolio before t and L_j is loss at j-th default event.

Idea: model the occurrence of jumps via the *aggregate default rate* λ_t defined as probability per unit time of the next default conditional on current market information

$$\mathbb{Q}[N_{t+\Delta t} = N_t + 1 | \mathcal{F}_t] = \lambda_t \Delta t + o(\Delta t)$$

Market convention: $L_j = (1-R)/N$ is constant.

The top-down approach

Idea: view credit derivatives as options on the portfolio loss L_t model risk–neutral/ market–implied dynamics of L_t.

1. Model the (spot) loss process L_t: compound Poisson process (insurance literature), conditional Poisson process (Brigo & Pallavicini 05), self-exciting point process (Giesecke & Goldberg), Cox process (Longstaff & Rajan), conditional Markov chain (Ehlers & Schonbucher 07).

2. Build a model for the term–structure of conditional default probabilities (Schonbucher 05, Andersen et al 05):

$$p_{t,T}(x) = \mathbb{Q}(\ L_T \leq x | \mathcal{F}_t) \tag{6}$$

Wide variety of specifications for portfolio loss process L_t: which one to pick? how to choose its parameters (loss intensity) consistently with market observations of CDO spreads?

Modeling ingredients:

- Intensity λ_t of (next) default event:

$$\lambda_t(\omega) = \lim_{\Delta t \to 0} \frac{\mathbb{Q}(N(t+\Delta t) = N(t-) + 1 | \mathcal{F}_t)}{\Delta t}$$

  – Poisson: $\lambda_t = f(t)$

  – "Doubly stochastic": default intensity driven by other "market factors", not by default itself

$$d\lambda_t = \mu(t, \lambda_t)dt + \sigma(\lambda_t)dW_t$$

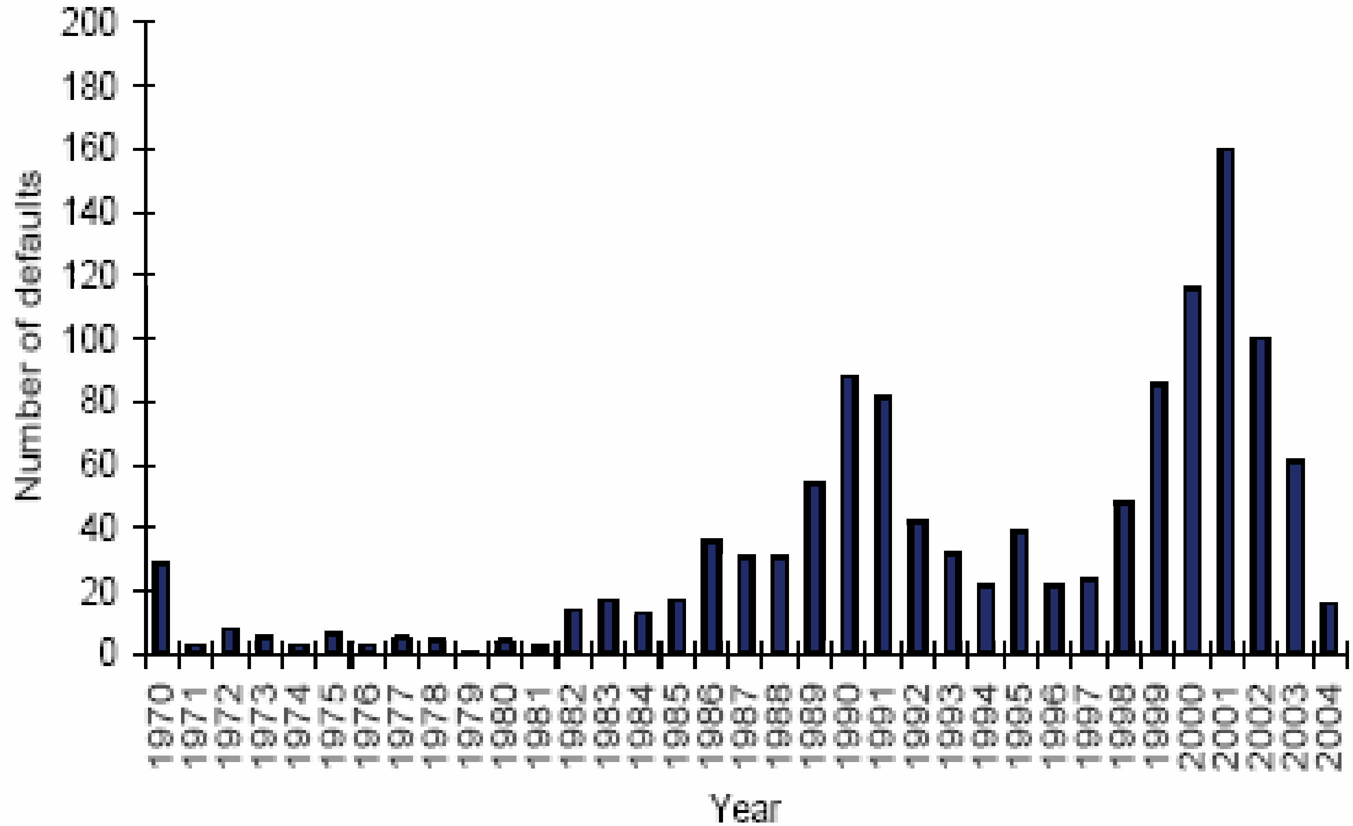
  – Inhomogeneous Markov process: $\lambda_t = f(t, N_t) = a_{N_t}(t)$ where $a_n(t)$ are transition rates from n to $n+1$

  – Dependence on history of defaults/ losses:

$$\lambda_t = g(t_j, L_j, j = 1..N_t - 1)$$

- (Distribution of) Loss given default L_j.

Clustering of defaults



Information content of credit portfolio derivatives

Market observations consist of fair spreads for (index) CDO tranches. These can be represented in terms of expected tranche notionals

$$C(t_j, K_i) = C_i = E^{\mathbb{Q}}[(K_i - L_{t_j})^+] \tag{8}$$

Common procedure is to "strip" CDO spreads to get expected tranche notionals $C(t_j, K_i)$ and then calibrate these using a model.

Problem: we need $C(t_j, K_i)$ for all payment dates t_j: many more than data observed! Ill-posed linear problem $\rightarrow$ parametrization of $C(.,.)$ / interpolation usually used

Here we will avoid this step altogether and use a nonparametric approach

Information content of credit portfolio derivatives

Proposition 1. *Consider any non-explosive jump process $(L_t)_{t\in[0,T^*]}$ with a intensity process $(\lambda_t(\omega))_{t\in[0,T^*]}$ and IID jumps with distribution F. Define $(\tilde{L}_t)_{t\in[0,T^*]}$ as the Markovian jump process with jump size distribution F and intensity*

$$\lambda_{\text{eff}}(t,l) = E^{\mathbb{Q}}[\lambda_t | L_{t-} = l, \mathcal{F}_0] \tag{9}$$

Then, for any $t \in [0,T^*]$, L_t and $\tilde{L}_t$ have the same distribution conditional on $\mathcal{F}_0$. In particular, the flow of marginal distributions of $(L_t)_{t\in[0,T^*]}$ only depends on the intensity $(\lambda_t)_{t\in[0,T^*]}$ through its conditional expectation $\lambda_{\text{eff}}(.,.)$.

Analogy with *local volatility*.

Proof. Consider any bounded measurable function $f(.)$. Using the pathwise decomposition of L_T into the sum of its jumps we can write

$$f(L_T) \quad = \quad f(L_0) + \sum_{0<s\leq T} (f(L_{s-} + \Delta L_s) - f(L_{s-})\ ) \tag{10}$$

so

$$\begin{aligned} E[f(L_T)|\mathcal{F}_0] \quad &= \quad f(L_0) + E[\sum_{0<s\leq T} (f(L_{s-} + \Delta L_s) - f(L_{s-})\ )|\mathcal{F}_0] \\ &= \quad f(L_0) + \int_0^T dt \quad E[(f(L_{t-} + \Delta L_t) - f(L_{t-})\ )\lambda_t|\mathcal{F}_0] \end{aligned} \tag{11}$$

Denote

$$\mathcal{G}_t = \sigma(\mathcal{F}_0 \vee L_{t-})$$

the information set obtained by adding the knowledge of L_{t-} to the current information set $\mathcal{F}_0$. Define the *local intensity* function

$$\lambda_{\text{eff}}(t, l) = E^{\mathbb{Q}}[\lambda_t|\mathcal{F}_0, L_{t-} = l]. \tag{12}$$

Noting that $\mathcal{F}_0 \subset \mathcal{G}_t$ we have

$$E[(f(L_{t-} + \Delta L_t) - f(L_{t-})\ )\lambda_t | \mathcal{F}_0]$$

$$= E[E[(f(L_{t-} + \Delta L_t) - f(L_{t-})\ )\lambda_t | \mathcal{G}_t] | \mathcal{F}_0]$$

$$= E[\int_0^1 F(dy)\ (f(L_{t-} + y) - f(L_{t-})\ E[\lambda_0 | \mathcal{G}_t] | \mathcal{F}_0]$$

$$= E[\lambda_{\text{eff}}(t, L_{t-}) \int F(dy)\ (f(L_{t-} + y) - f(L_{t-})\ | \mathcal{F}_0] \qquad \text{so}$$

$$E[f(L_T) | \mathcal{F}_0] = f(L_t) +$$

$$E[\int_0^T dt\ \lambda_{\text{eff}}(t, L_{t-}) \int F(dy)\ (f(L_{t-} + y) - f(L_{t-})\ | \mathcal{F}_0]$$

The above equality shows that $E[f(L_T)|\mathcal{F}_0] = E[f(\tilde{L}_T)|\mathcal{F}_0]$ where $(\tilde{L}_t)_{0 \leq t \leq T}$ is the Markovian loss process with intensity $\gamma_t = \lambda_{\text{eff}}(t, \tilde{L}_{t-})$ and jump size distribution F hence $\tilde{L}_t =^d L_t$. $\square$

Corollary 1 (Information content of non-path dependent portfolio credit derivatives). *The value* $E^{\mathbb{Q}}[f(L_T)|\mathcal{F}_0]$ *at* $t = 0$ *of any derivative whose payoff depends on the aggregate loss* L_T *of the portfolio at on a fixed grid of dates, only depends on the default intensity* $(\lambda_t)_{t\in[0,T^*]}$ *through its risk-neutral conditional expectation with respect to the current loss level:*

$$\lambda_{\text{eff}}(t,l) = E^{\mathbb{Q}}[\lambda_t | L_{t-} = l, \mathcal{F}_0] \tag{13}$$

In particular, CDO tranche spreads and mark-to-market value of CDO tranches only depends on the transition rate $(\lambda_t)_{t\in[0,T^*]}$ *through the effective default intensity* $\lambda_{\text{eff}}(.,.)$.

Forward equation for expected tranche loss In the markovian case where portfolio loss intensity only depends on time/loss, the expected tranche loss $C(T,K) = E^{\mathbb{Q}^\lambda}[(K - L_T)^+]$ solves a Dupire-type forward equation (Cont & Savescu 2006)

$$\begin{aligned} \frac{\partial C(T,K)}{\partial T} &= -\lambda^*(T, K - \delta K) C(T,K) \\ &-(\lambda^*(T, K - 2\delta K) - 2\lambda^*(T, K - \delta K)) C(T, K - \delta K) \\ -\sum_{i=1}^{k-2} (\lambda^*(T,(i-1)\delta K) &- 2\lambda^*(T, i\delta K) + \lambda^*(T,(i+1)\delta K)) C(T,K) \end{aligned}$$

Problem 1 (Calibration problem). *Given a set of observed CDO tranche spreads* $(S_0(K_i, K_{i+1}, T_k), i = 1..I-1, k = 1..m)$ *for a reference portfolio, construct a (risk–neutral) default rate/ loss intensity* $\lambda = (\lambda_t)_{t\in[0,T]}$ *such that the spreads computed under the model* $\mathbb{Q}^\lambda$ *match the market observations*

$$S_0(K_i, K_{i+1}, T_k) = \frac{\sum_{t_j \le T_k} B(0,t_j) E^{\mathbb{Q}^\lambda}[L_{K_i,K_{i+1}}(t_j) - L_{K_i,K_{i+1}}(t_{j-1})\ ]}{\sum_{t_j \le T_k} B(0,t_j)(t_j - t_{j-1}) E^{\mathbb{Q}^\lambda}[(K_{i+1} - L(t_j))^+ - (K_i - L(t_j))^+]}$$

Calibration by Relative entropy minimization under constraints

One period case: Buchen & Kelly, Avellaneda 1998

Diffusion models: Avellaneda Friedman Holmes Samperi 1997

Monte Carlo setting: Avellaneda et al 2001

Lévy processes: Cont & Tankov 2004, 2006)

Given market prices $C(K_i)$ of tranche payoffs and a prior guess λ^0 for the loss intensity process, the reconstruction of the default intensity process $(\lambda_t)_{t\in[0,T^*]}$ can be formalized as

$$\inf_{\mathbb{Q}^\lambda \in \Lambda} E^{\mathbb{Q}_0}\left[\frac{d\mathbb{Q}^\lambda}{d\mathbb{Q}_0} \ln \frac{d\mathbb{Q}^\lambda}{d\mathbb{Q}_0}\right] \tag{14}$$

under the constraint that the model $\mathbb{Q}^\lambda$ prices correctly the observed CDO tranches, where $\mathbb{Q}^\lambda$ is the law of the point process with intensity process λ and $\mathbb{Q}_0$ is the law of the point process with intensity λ^0.

Problem 2 (Calibration via relative entropy minimization)**.** *Given a prior loss process with law $\mathbb{Q}_0$, find a default intensity $(\lambda_t)_{t\in[0,T^*]}$ which minimizes*

$$\inf_{\mathbb{Q}^\lambda\in\Lambda} E^{\mathbb{Q}_0}[\frac{d\mathbb{Q}^\lambda}{d\mathbb{Q}_0}\ln\frac{d\mathbb{Q}^\lambda}{d\mathbb{Q}_0}] \quad \text{under} \quad E^{\mathbb{Q}^\lambda}[H_{i,k}]=0 \tag{15}$$

$$H_{ik} = S_0(K_i, K_{i+1}, T_k)\sum_{t_j\le T_k} B(0,t_j)(t_j-t_{j-1})[(K_{i+1}-L(t_j))^+ - (K_i - L(t_j))^+]$$
$$-\sum_{t_j\le T_k} B(0,t_j)[(K_{i+1}-L(t_j))^+ - (K_i-L(t_j))^+ - (K_{i+1}-L(t_{j-1}))^+ + (K_i-L(t_{j-1}))^+))\ ] \tag{16}$$

and $\mathbb{Q}^\lambda$ denotes the law of the point process with intensity $(\lambda_t)_{t\in[0,T^*]}$ and $\mathbb{Q}_0$ is the law of the point process with intensity λ^0.

Using the previous result we can restrict Λ to *Markovian intensities* $\lambda(t, L_t)$.

Computation of entropy

Equivalent change of measure for point processes (Jacod 1980, Bremaud 1981)

Proposition 2. *Let N_t be a Poisson process with intensity γ_0 on $(\Omega, \mathcal{F}_t, \mathbb{Q}_0)$. Let $\lambda = (\lambda_t)_{t\in[0,T]}$ be an $\mathcal{F}_t$-predictable process such that*

$$\int_0^t \lambda_s ds < \infty \quad \mathbb{Q}_0 - a.s. \tag{17}$$

Define the probability measure $\mathbb{Q}^\lambda$ on $\mathcal{F}_T$ by

$$\frac{d\mathbb{Q}^\lambda}{d\mathbb{Q}_0} = Z_T \quad \text{where} \quad Z_t = \left(\prod_{\tau_j \le t} \frac{\lambda_{\tau_j}}{\gamma_0}\right) \exp\left\{\int_0^t (\gamma_0 - \lambda_s)\, ds\right\}$$

Then N_t is a point process with $\mathcal{F}_t$ intensity $(\lambda_t)_{t\in[0,T]}$ under $\mathbb{Q}^\lambda$.

Proposition 3 (Computation of relative entropy)**.** *Denote by*

- $\mathbb{Q}_0$ *the law on* $[0, T]$ *of a (standard unit intensity) Poisson process and*
- $\mathbb{Q}^\lambda$ *the law on* $[0, T]$ *of the point process with intensity* $(\lambda_t)_{t\in[0,T]}$ *verifying hypothesis (17).*

The relative entropy of $\mathbb{Q}^\lambda$ *with respect to* $\mathbb{Q}_0$ *is given by:*

$$E^{\mathbb{Q}_0}[\frac{d\mathbb{Q}^\lambda}{d\mathbb{Q}_0}\ln\frac{d\mathbb{Q}^\lambda}{d\mathbb{Q}_0}] = E^{\mathbb{Q}^\lambda}[\int_0^T \lambda_t \ln \lambda_t dt + T - \int_0^T \lambda_t dt] \tag{18}$$

Duality

Define the Lagrangian

$$\mathcal{L}(\lambda, \mu) = E^{\mathbb{Q}^\lambda}[\int_0^T \lambda_s \ln \lambda_s ds + T - \int_0^T \lambda_s ds - \sum_{i=1}^{I} \sum_{k=1}^{m} \mu_{i,k} H_{ik}]$$

Using convex duality arguments, the primal problem:

$$\inf_{\mathbb{Q}^\lambda \in \Lambda} E^{\mathbb{Q}_0}[\frac{d\mathbb{Q}^\lambda}{d\mathbb{Q}_0} \ln \frac{d\mathbb{Q}^\lambda}{d\mathbb{Q}_0}] \quad \text{under} \quad E^{\mathbb{Q}^\lambda}[H_{ik}] = 0 \tag{19}$$

is equivalent to the dual problem

$$\sup_{\mu \in \mathbb{R}^{m.I}} \inf_{\lambda \in \Lambda} E^{\mathbb{Q}^\lambda}[\int_0^T \lambda_s \ln \lambda_s ds + T - \int_0^T \lambda_s ds - \sum_{i=1}^{I} \sum_{k=1}^{m} \mu_{i,k} H_{ik}] \tag{20}$$

Intensity control problem

An *intensity control* problem is an optimization problem with a criterion of the type

$$E^{\mathbb{Q}^\lambda}[\int_0^T \varphi(t,\lambda_t,L_t)dt + \sum_{j=1}^{J}\Phi_j(t_j,L_{t_j})],$$

where $\varphi(t,\lambda_t,N_t)$ is a *running cost* and $\Phi_j(t_j,L_{t_j})$ represents the terminal cost. Here

$$\varphi(t,\lambda,L) = \lambda\ln\lambda + 1 - \lambda \qquad \text{and} \quad \Phi_j(t_j,L_{t_j}) = \sum_{i=1}^{I} M_{ij}(K_i - L_{t_j})^+$$

$$\text{where} \quad M_{ij} = B(0,t_{j+1})\sum_{T_k\geq t_{j+1}}(\mu_{ik}-\mu_{i-1,k}) +$$

$$B(0,t_j)\sum_{T_k\geq t_j}[\mu_{ik}(1-\Delta S(K_i,K_{i+1},T_k)) - \mu_{i-1,k}(1-\Delta S(K_{i-1},K_i,T_k))]$$

Single horizon case

$$E^{\mathbb{Q}^\lambda}[\int_0^T (\lambda_t \ln \lambda_t + 1 - \lambda_t)dt + \Phi(T, L_T)],$$

Solution by dynamic programming: introduce the value function

$$V(t,k) = E^{\mathbb{Q}^\lambda}[\int_0^T \varphi(t, \lambda_t, L_t)dt + \Phi(T, L_T)|N_t = k]$$

The value function can be characterized in terms of a Hamilton Jacobi equation (Bismut 1975, Bremaud 1982).

Proposition 4. *(Hamilton-Jacobi equations) Suppose there exists a bounded function* $V : [0, T^*] \times N \to V(t, n)$ *differentiable in* t*, such that*

$$\frac{\partial V}{\partial t}(t,k) + \inf_{\lambda \in]0, infty[} \{\lambda[V(t,k+1) - V(t,k)] + \lambda \ln \lambda - \lambda + 1\} = 0 \quad (21)$$

$$\text{for} \quad t \in [0,T] \quad \text{and} \quad V(T,k) = \Phi(T, k\delta) \quad (22)$$

and suppose there exists for each $n \in N^+$ *an* $\mathcal{F}_t$*-predictable mapping* $t \to u^*(t, N_t)$ *such that for each* $n \in N^+,\ t \in [t_0, T]$

$$\lambda^*(t,k) = \underset{\lambda \in]0,\infty[}{argmin} \{\lambda[V(t,k+1) - V(t,k)] + \lambda \ln \lambda - \lambda + 1\} \quad (23)$$

Then $\lambda_t^* = \lambda^*(t, N_t)$ *is an optimal control. Moreover* $V(t_0, N_{t_0}) = \inf_{\lambda \in \Lambda_t} E^{\mathbb{Q}_\lambda}[\int_{t_0}^T C_s(\lambda) ds + \Phi_T(\lambda) | \mathcal{F}_{t_0}]$.

In our problem, in the case of a single maturity, the dual problem is an intensity control problem with running cost

$$(\ln \lambda(t, N_t) - 1)\lambda(t, N_t) + 1$$

and terminal cost is of the type $\Phi_j(L) = \sum M_{ij}(K_i - L)^+$.

The Hamilton Jacobi equations are given by

$$\frac{\partial V}{\partial t}(t,n) + \inf_{\lambda \in \Lambda} \{\lambda[V(t,n+1) - V(t,n)] + (\ln \lambda(t,n) - 1)\lambda(t,n) + 1) = 0$$

which is a system of $n = 125$ coupled nonlinear ODEs.

The maximum in the nonlinear term can be explicitly computed:

$$\lambda^*(t,n) = e^{-[V(t,n+1)-V(t,n)]} \tag{24}$$

$$\frac{\partial V}{\partial t}(t,n) + 1 - e^{-[V(t,n+1)-V(t,n)]} = 0 \tag{25}$$

$$V(T,k) = \Phi(T,k) \tag{26}$$

Proposition 5 (Value function)**.** *Consider any terminal condition Φ such that $\Phi(x) = 0$ for $x > n\delta$. Then the solution of (26)-27 is given by*

$$V(t,k,\mu) = T - t - \ln \sum_{j=0}^{n-k} \frac{(T-t)^j}{j!} e^{-\Phi(T,(j+k)\delta)} \tag{27}$$

The key is to note that if we consider the exponential change of variable $u(t,k) = e^{-V(t,k)}$ then u solves a *linear* equation

$$\frac{\partial u(t,k)}{\partial t} + u(t,k+1) - u(t,k) = 0 \qquad \text{with} \quad u(T,k) = \exp(-\Phi(T,k\delta))$$

which is recognized as the backward Kolmogorov equation associated with the Poisson process (i.e. the prior process, with law $\mathbb{Q}_0$). The solution is thus given by the Feynman-Kac formula

$$u(t,k;\mu) = E^{\mathbb{Q}_0}[e^{-\Phi(T,\delta N_T)}|N_t = k] = E^{\mathbb{Q}_0}[e^{-\Phi(T,k\delta+\delta N_{T-t})}]$$

using the Markov property and the independence of increments of the Poisson process. The expectation is easily computed using the Poisson distribution:

$$u(t,k;\mu) = \sum_{j=0}^{n-k} e^{-(T-t)} \frac{(T-t)^j}{j!} e^{-\Phi(T,(k+j)\delta)} \tag{28}$$

which leads to (28).

Case of several maturities

Recursive algorithm via dynamic programming principle

1. Start from the last payment date $j = J$ and set $F_J(k) = \Phi_J(t_J, \delta k)$.
2. Solve the Hamilton–Jacobi equations (26) on $]t_{j-1}, t_j]$ backwards starting from the terminal condition

$$V(t_j, k) = F_j(k) \tag{29}$$

   which can be explicitly solved to yield $V(t, k; \mu)$ on $t \in]t_{j-1}, t_j]$ using (28).
3. Set $F_{j-1}(k) = V(t_{j-1}, k) + \Phi_{j-1}(t_{j-1}, k\delta)$
4. Go to step 2 and repeat.

Discontinuities may appear in value function at junction dates.

Reconstruction algorithm

1. Solve the dynamic programming equations (26)–(27) $\mu \in \mathbb{R}^I$ to compute $V(0,0,\mu)$.
2. Optimize $V(0,0,\mu)$ over $\mu \in \mathbb{R}^{I\times J}$ using a gradient–based method:

$$\inf_{\mu\in\mathbb{R}^I} V(0,0,\mu) = V(0,0,\mu^*) = V^*(0,0)$$

3. Compute the calibrated default intensity (optimal control) as follows:

$$\lambda^*(t,k) = e^{V(t,k)-V(t,k+1)} \tag{30}$$

4. Compute the term structure of loss probabilities by solving the Fokker-Planck equations.
5. The calibrated default intensity $\lambda^*(.,.)$ can then be used to compute CDO spreads for different tranches, forward tranches

etc. in a straightforward manner: first we compute the expected tranche loss $C(T,K)$ by solving the forward equation:

$$\frac{\partial C(T,K)}{\partial T} = -\lambda^*(T, K-\delta K)C(T,K)$$

$$-(\lambda^*(T, K-2\delta K) - 2\lambda^*(T, K-\delta K))C(T, K-\delta K)$$

$$-\sum_{i=1}^{k-2} (\lambda^*(T,(i-1)\delta K) - 2\lambda^*(T, i\delta K) + \lambda^*(T,(i+1)\delta K))C(T,K)$$

In particular the calibrated default intensity can be used to "fill the gaps" in the base correlation surface in an arbitrage-free manner, by first computing the expected tranche loss for all strikes and then computing the base correlation for that strike.

Empirical results: ITRAXX

Maturity	Low	High	Bid\ Upfront	Ask\ Upfront
5Y	0%	3%	11.75%	12.00%
	3%	6%	53.75	55.25
	6%	9%	14.00	15.50
	9%	12%	5.75	6.75
	12%	22%	2.13	2.88
	22%	100%	0.80	1.30
7Y	0%	3%	26.88%	27.13%
	3%	6%	130	132
	6%	9%	36.75	38.25
	9%	12%	16.50	18.00
	12%	22%	5.50	6.50
	22%	100%	2.40	2.90

Maturity	Low	High	Bid\ Upfront	Ask\ Upfront
10Y	0%	3%	41.88%	42.13%
	3%	6%	348	353
	6%	9%	93	95
	9%	12%	40	42
	12%	22%	13.25	14.25
	22%	100%	4.35	4.85

Table 2: ITRAXX tranche spreads, in bp. For the equity tranche the periodic spread is 500bp and figures represent upfront payments.

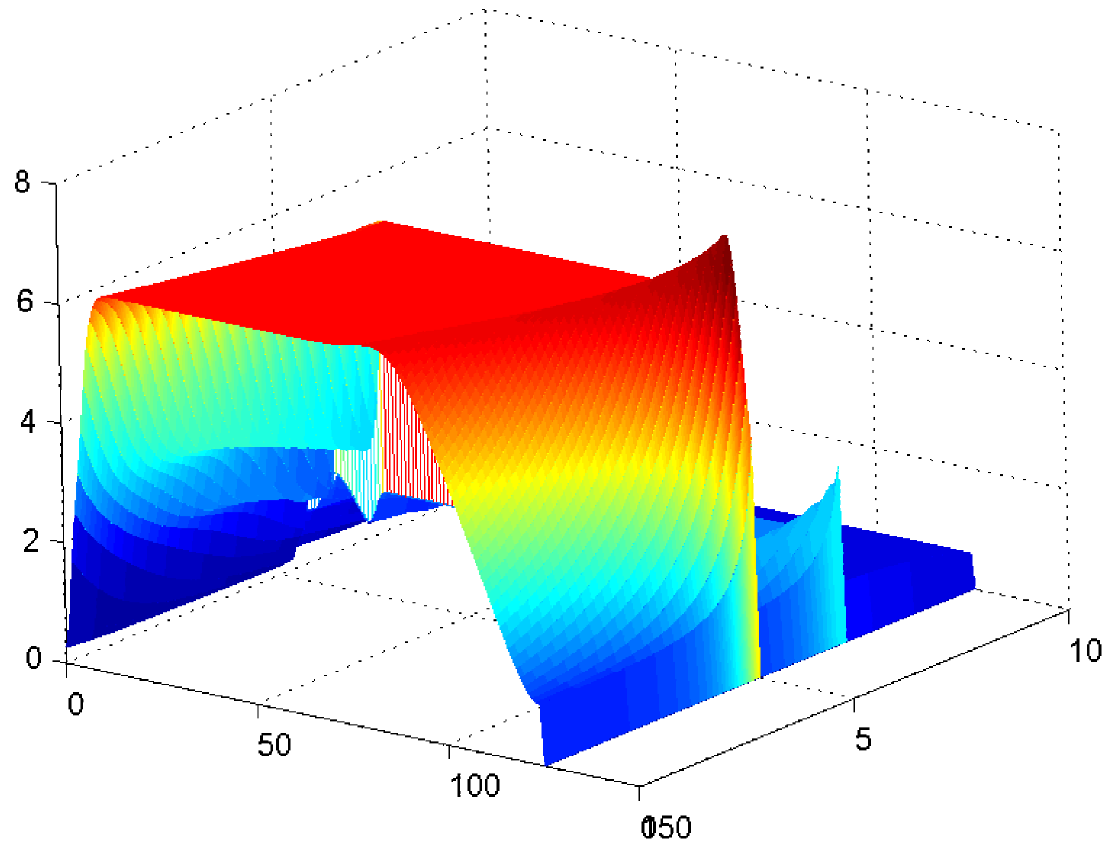


Figure 1: Calibrated intensity function $\lambda(t, L)$: ITRAXX Europe Series 6, March 15 2007.

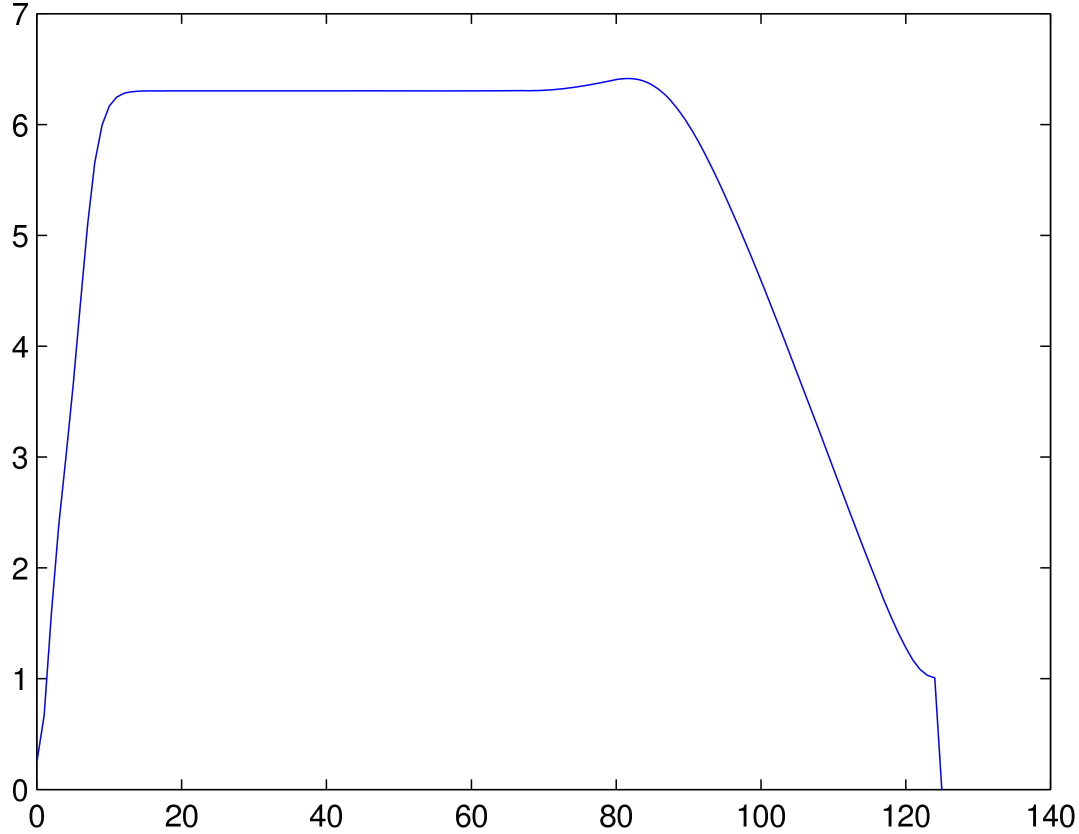


Figure 2: Dependence of default intensity on number of defaults for $t = 1 year$: ITRAXX Europe Series 6, March 15 2007..

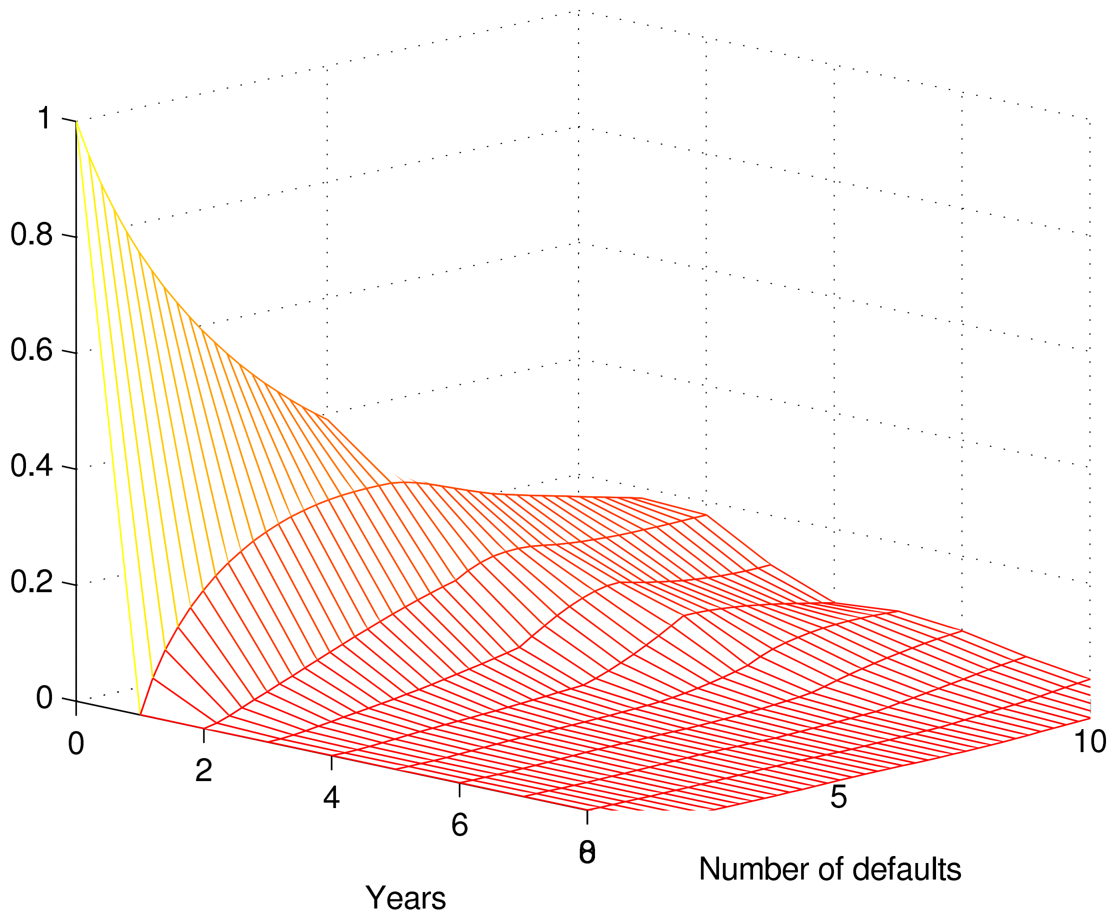


Figure 3: Term structure of loss distributions computed from calibrated default intensity: ITRAXX Europe Series 6, March 15 2007..

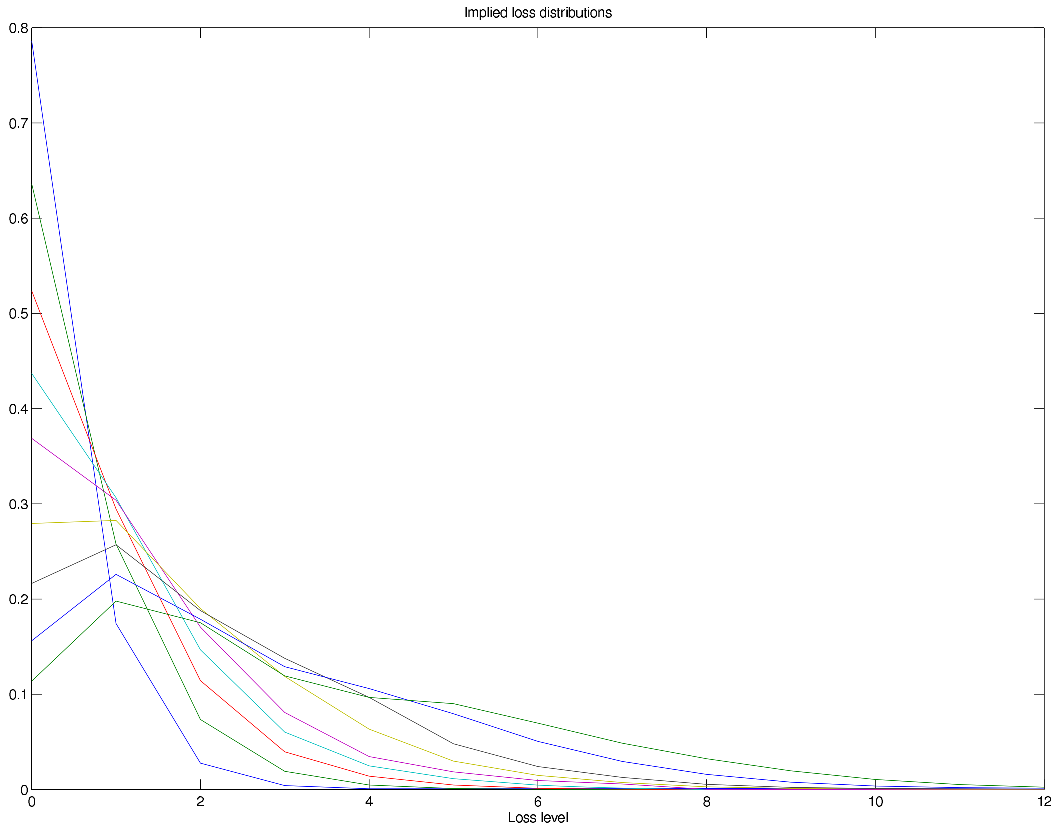


Figure 4: Implied loss distributions at various maturities: ITRAXX Europe Series 6, March 15 2007.

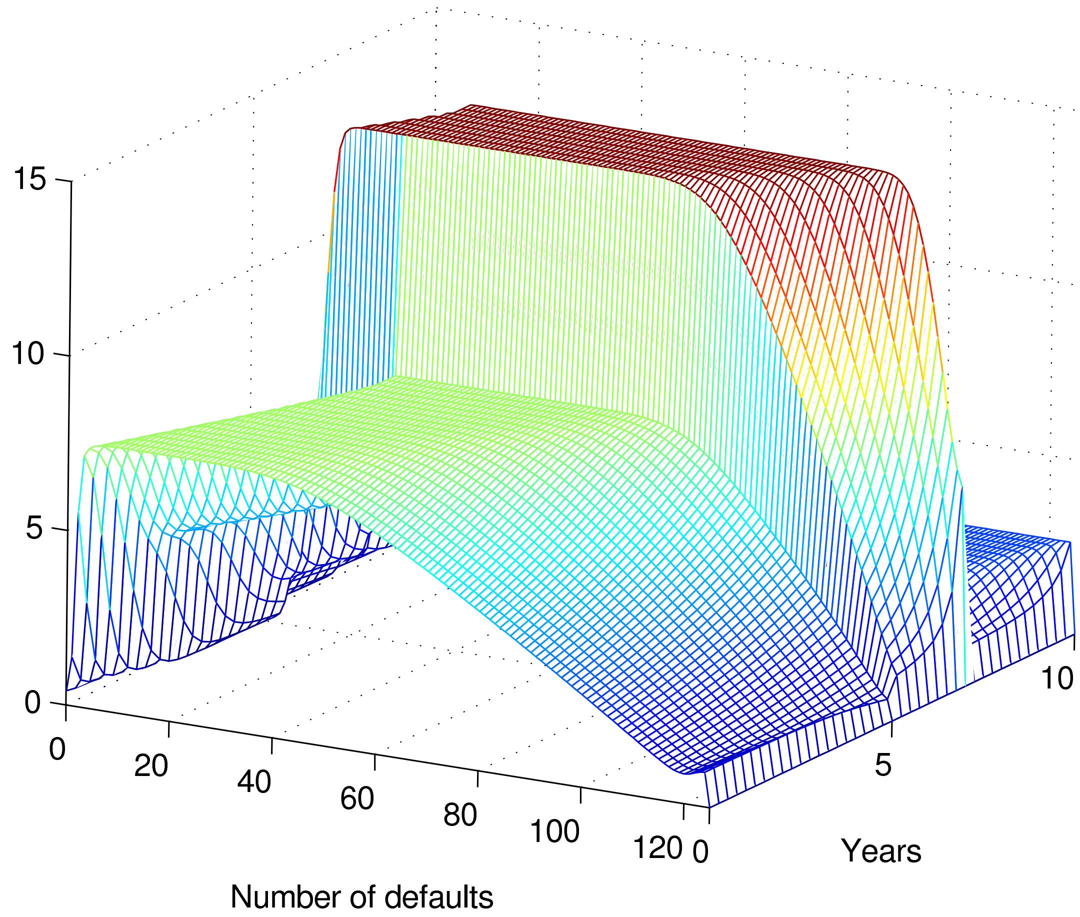


Figure 5: Calibrated intensity function $\lambda(t, L)$: ITRAXX September 26, 2005

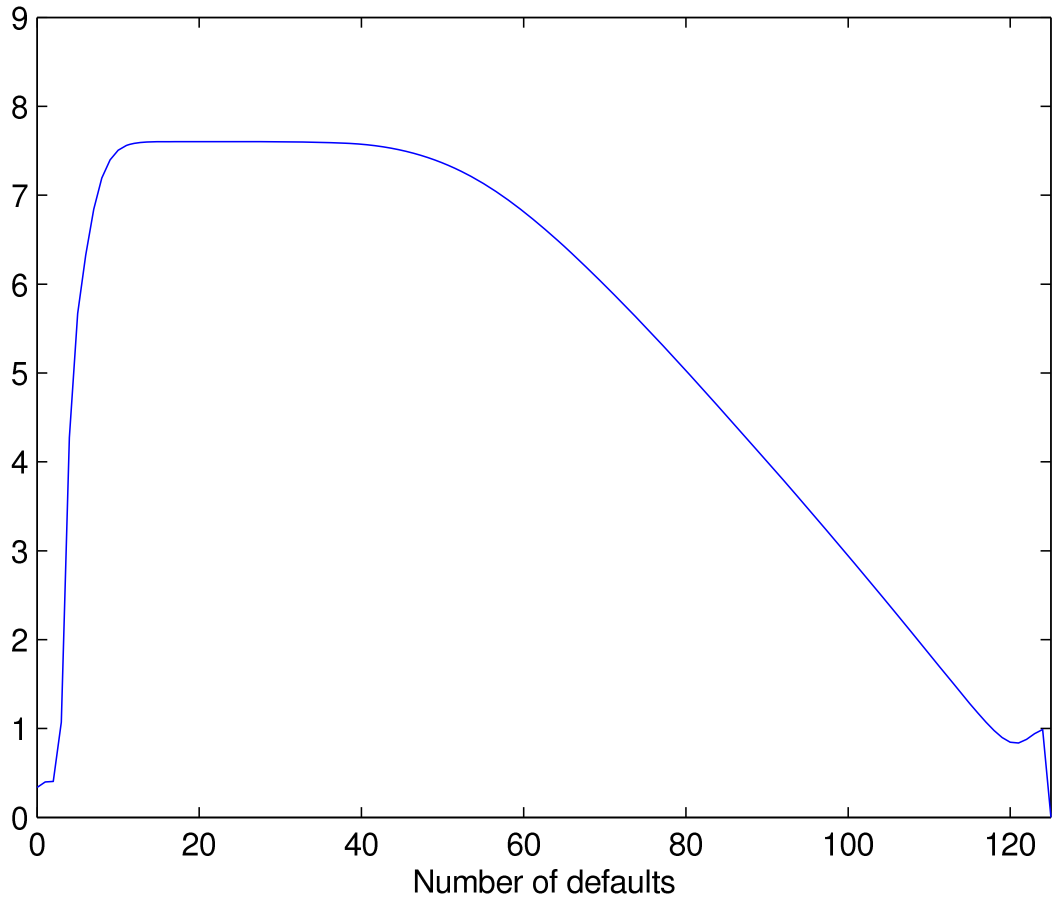


Figure 6: Dependence of default intensity on number of defaults for $t = 1year$: ITRAXX September 26, 2005.

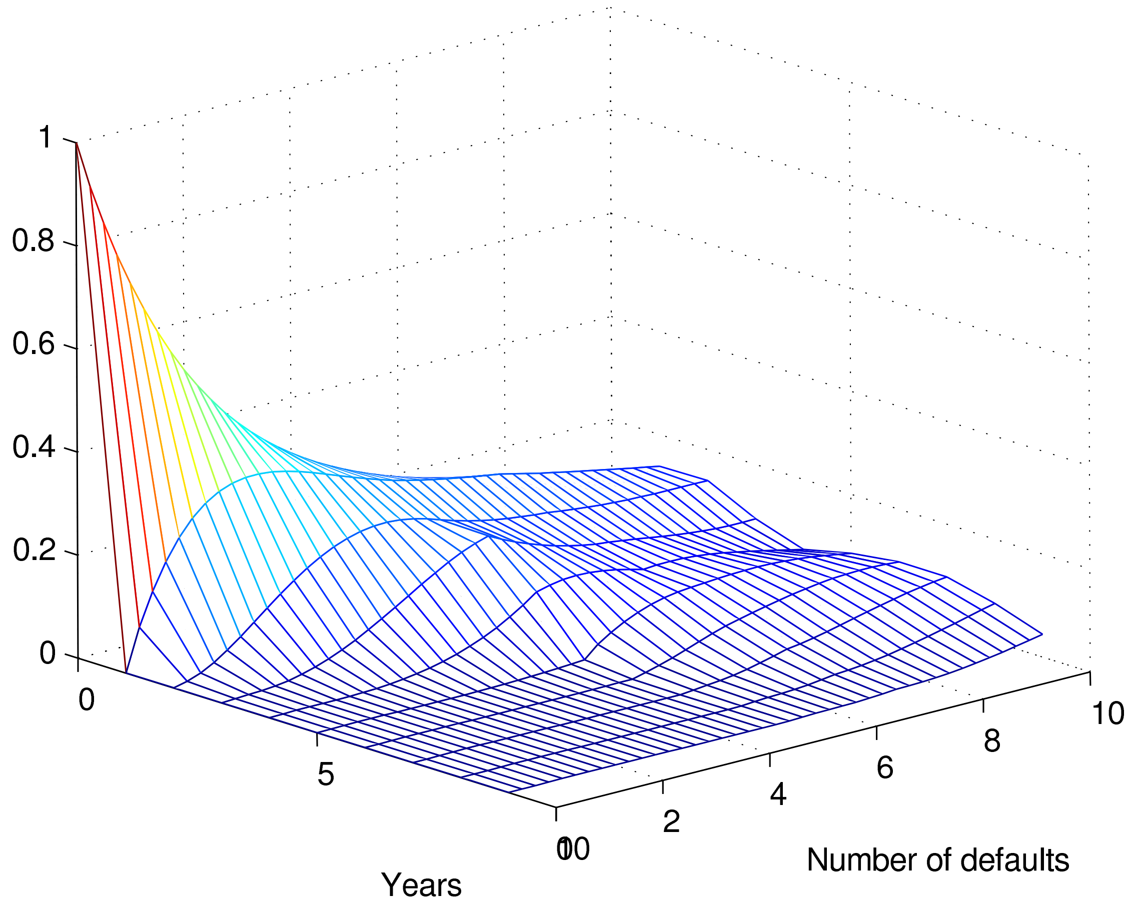


Figure 7: Term structure of loss distributions computed from calibrated default intensity: ITRAXX September 26, 2005.

Conclusion

- Stochastic control method for solving a model calibration problem.
- Rigorous methodology for calibrating a top-down CDO pricing model to market data.
- Stable calibration algorithm based on intensity control method.
- No black box optimization.
- Nonparametric: no arbitrary functional form for the default intensity.
- No need to interpolate CDO data in maturity or strike!
- Involves unconstrained convex minimization in dimension $\simeq 20$: few seconds on laptop!
- Results point to default contagion effects in the riskneutral loss process.