3. lecture. Nov. 8, 2007

Applications and examples.

Decomposable operators

X Banach space

$T \in B(X)$

T is called decomposable if for every covering of $\mathbb{C}$

$$\mathbb{C} = V \cup W$$

by two open sets V, W, there exists closed T-invariant subspaces Y, Z of X s.t.

$$X = Y + Z$$

and

$$\sigma(T_{|Y}) \subseteq V \quad , \quad \sigma(T_{|Z}) \subseteq W.$$

Theorem (Apostol, Foias $\approx$ 1968)

$T \in B(X)$ is decomposable if and only if it has a spectral capacity, i.e. there exists a function E from the set of closed subsets of $\mathbb{C}$ into the set of closed T-invariant subspaces of X, s.t.

(i) $E(\emptyset) = \{0\}$, $E(\mathbb{C}) = X$

(ii) $E(\bar{U}_1) + \ldots + E(\bar{U}_n) = X$
for every open covering of $\mathbb{C}$

(iii) $E(\bigcap_{n=1}^{\infty} F_n) = \bigcap_{n=1}^{\infty} E(F_n)$

(iv) $\sigma(T_{|E(F)}) \subseteq F$ (F closed set in $\mathbb{C}$

Thm. (Dykema + H, AMJ 2004)
Voiculescu's circular operator is decomposable. More generally every DT-operator is decomposable.

Main Theorem (joint work with Hanne Schultz 2006)

Let M be a II_1-factor and let $T \in M$. Then, for every Borelset B in $\mathbb{C}$, there is a unique closed T-invariant subspace $K = K(T, B)$ affiliated with M, such that w.r.t. the decomposition $H = K \oplus K^\perp$, T has the form

$$T = \begin{pmatrix} T_K & T_{12} \\ 0 & T_{22} \end{pmatrix}$$

where the Brown measures μ_{T_K} and $\mu_{T_{22}}$ are concentrated on B and $\mathbb{C} \setminus B$ respectively. Moreover

$$\tau_M(P_K) = \mu_T(B)$$

and K is T-hyperinvariant*.

Corollary

If μ_T is not a Dirac measure, then T has a non-trivial closed hyperinvariant subspace K ($\Rightarrow$ K is affiliated with $W^*(T) \subseteq M$).

Theorem (H + Schultz, 2006)

M II_1-factor.

If $T \in M$ is decomposable, then

$$F \to K(T, F), \quad (F \text{ closed})$$

is the spectral capacity for T

In particular

$$\circledast \quad \sigma(T_{|K(T,F)}) \subseteq F \quad (F \text{ closed})$$

Corollary

Every II_1-factor M has a non-decomposable operator.

Proof

The hyperfinite II_1-factor R embeds in M, and the operator

$$T = (0) \oplus \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \oplus \cdots$$

can be realized in R. Moreover

$$\sigma(T) = \overline{\mathbb{D}} \quad \text{and} \quad \mu_T = \delta_0$$

Hence $K(T, \{0\}) = H$, while $\sigma(T_{|H}) \not\subseteq \{0\}$

Theorem (H + Schultz, 2006)

M II_1-factor, $T \in M$. Then the limits

$$C := \text{s.o.-}\lim_{n\to\infty} \left((T^*)^n T^n\right)^{\frac{1}{2n}}$$

$$D := \text{s.o.-}\lim_{n\to\infty} \left(T^n (T^*)^n\right)^{\frac{1}{2n}}$$

both exists, and for all $r \geq 0$:

$$P_{K(T,\, \overline{B(0,r)})} = 1_{[0,r]}(C)$$

$$P_{K(T,\, \mathbb{C}\setminus B(0,r))} = 1_{[r,\infty)}(D)$$

Proof

$r \to P_{K(T,\, \overline{B(0,r)})}$ is increasing and s.o. continuous from right and

$$K(T, \overline{B(0,r)}) = H \quad \text{for} \quad r \geq \|T\|$$

Hence $\exists!\ C \in M_+$ $(\|C\| \leq \|T\|)$, s.t.

$$1_{[0,r]}(C) = P_{K(T,\, \overline{B(0,r)})}$$

Claim : $((T^*)^n T^n)^{\frac{1}{2n}} \xrightarrow{s.o.} C \quad (n \to \infty)$

Recall from 2. lecture, that

$$K(T, \overline{B(0,r)}) = E(T,r)$$

$$K(T^*, \mathbb{C} \setminus B(0,r)) = F(T^*,r)$$

where

$$E(T,r) = \left\{ \xi \in H \mid \exists \xi_n \in H, \begin{array}{l} \|\xi_n - \xi\| \to 0 \\ \limsup \|T^n \xi_n\|^{1/n} \le r \end{array} \right\}$$

$$F(T^*,r) = \left\{ \eta \in H \mid \exists \eta_n \in H, \begin{array}{l} \|(T^*)^n \eta_n - \eta\| \to 0 \\ \limsup \|\eta_n\|^{1/n} \le \frac{1}{r} \end{array} \right.$$

$E(T,r) \perp F(T^*,s)$ for $s > r$

If $\mu_T(\partial B(0,r)) = 0$, then

$$E(T,r) = F(T^*,r)^\perp$$

and $r^0 \to P_{E(T,r^0)}$, $r^0 \to P_{F(T^*,r^0)}$ are

s.o. continuous at such $r > 0$.

Lemma 1

$A_n, A \in B(H)_+$, $\|A_n\| \leq K$, $\|A\| \leq K$.

If

$$1_{[0,r]}(A_n) \xrightarrow{s.o.} 1_{[0,r]}(A) \quad (n \to \infty)$$

for all, but countably many $r \in [0,K]$

then

$$A_n \xrightarrow{s.o.} A.$$

proof

Let $\delta > 0$ and choose stepfunction $s : [0,K] \to$ cont. from right such that

$$\|A_n - s(A_n)\| < \delta \qquad \|A - s(A)\| < \delta$$

Clearly $s(A_n) \xrightarrow{s.o.} s(A)$, ∎

lemma 2

$P_n, P \in B(H)$ projections. Then

TFAE:

(1) $P_n \xrightarrow{s.o.} P \quad (n \to \infty)$

(2) $\forall \xi \in P(H)$: $\operatorname{dist}(\xi, P_n(H)) \to 0 \quad (n \to$

$\forall \eta \in P(H)^\perp$: $\operatorname{dist}(\eta, P_n(H)^\perp) \to 0 \quad (n \to$

Proof of $((T^*)^n T^n)^{\frac{1}{2n}} \xrightarrow{s.o.} C$

Enough to show that

$$1_{[0,r]}\left(((T^*)^n T^n)^{\frac{1}{2n}}\right) \xrightarrow{s.o.} 1_{[0,r]}(C) = P_{E(T,r)}$$

for all but countably many $r>0$. Hence we may assume

$$\begin{cases} \mu_T(\partial B(0,r)) = 0 \\ r \text{ is not an eigenvalue of } ((T^*)^n T^n)^{\frac{1}{2n}} \\ \text{for any } n \in \mathbb{N}. \end{cases}$$

From the definition of $E(T,r)$ it follows (after some work), that for

$$\xi \in E(T,r'), \text{ where } r'<r$$

one has

$$(*) \quad d(\xi, 1_{[0,r]}(((T^*)^n T^n)^{1/n})(H)) \to 0$$

By s.o.-continuity of $r' \to P_{E(T,r')}$ at $r'=r$ we have

$$E(T,r) = \overline{\bigcup_{0<r'<r} E(T,r')}$$

Hence $(*)$ holds for all $\xi \in E(T,r)$

A similar argument based on the $F(T^*, r'')$-sets for $r'' > r$ shows that $\forall \xi \in F(T^*, r) = E(T, r)^\perp$:

$$d\left[\xi,\ 1_{[r,\infty)}\left(((T^*)^n T^n)^{\frac{1}{2n}}\right)(H)\right] \to 0$$

and since r is not an eigenvalue for $((T^*)^n T^n)^{\frac{1}{2n}}$, it is the same as

$$(**)\quad d\left(\xi,\ 1_{[0,r]}\left(((T^*)^n T^n)^{\frac{1}{2n}}\right)(H)^\perp\right) \to 0.$$

Hence by lemma 2 we get from (*) and (**):

$$1_{[0,r]}\left(((T^*)^n T^n)^{\frac{1}{2n}}\right) \xrightarrow{s.o.} P_{E(T,r)} = 1_{[0,r]}(C)$$

for all such r, and hence

$$\left((T^*)^n T^n\right)^{\frac{1}{2n}} \xrightarrow{s.o.} C$$

The second statement in the Theorem follows from the same proof applied to T^* instead of T. ∎

Example 1

Voiculescu's circular operator Y:

Let X_1, X_2 be a free semicircular system in a II_1-factor M, i.e. $X_i = X_i^*$, X_1, X_2 are free w.r.t. $tr = tr_M$, and

$$tr(X_i^p) = \frac{1}{2\pi}\int_{-2}^{2} t^p \sqrt{4-t^2}\,dt \quad \begin{cases} p \in \mathbb{N} \\ i = 1,2 \end{cases}$$

Then

$$Y = \frac{1}{\sqrt{2}}(X_1 + i X_2)$$

is Voiculescu's circular operator.
Note that

$$VN(Y) \cong L(F_2) \text{ is a } \mathrm{II}_1\text{-factor}$$

and the pair $(Y, VN(Y))$ is unique up to $*$-isomorphism.

Theorem (H + Larsen, JFA 2000)

1) $\sigma(Y) = \bar{D}$

2) μ_Y = the uniform distribution on $\bar{D}$
(i.e. $d\mu_Y = \frac{1}{\pi} 1_{\bar{D}}(x,y)\,dx\,dy$)

Remark

It follows from Voiculescu's random matrix model that the circular operator Y is the limit in $*$-distribution of the series of random matrices

$$Y_n = \begin{pmatrix} y_{11}^{(n)} & \cdots & y_{1n}^{(n)} \\ \vdots & & \vdots \\ y_{n1}^{(n)} & \cdots & y_{nn}^{(n)} \end{pmatrix}$$

where for each $n \in \mathbb{N}$, $(y_{ij}^{(n)})_{i,j=1,\dots n}$ is a set of n^2 independent identical distributed complex Gaussian random variables with density

$$\frac{n}{\pi} \exp(-n|z|^2), \quad z \in \mathbb{C}$$

By results of GINIBRE and DYSON for the GO'es, it is known that the eigenvalue distribution the Y_n's converges almost surely to the uniform distribution on the unit disk $\overline{\mathbb{D}}$ (i.e to μ_Y) in the weak*-topology on $\mathrm{Prob}(\mathbb{C})$.

Theorem 1 (Dykema + H, GAFA 2001)

Let $Y = \frac{1}{\sqrt{2}}(X_1 + iX_2)$ be Voiculescu's circular operator and put $M = W^*(Y) \simeq L(F_2)$. Then for each $r \in (0,1)$ there is a unique closed T-invariant subspace K_r affiliated with M such that when

$$Y = \begin{pmatrix} Y_{11} & Y_{12} \\ 0 & Y_{22} \end{pmatrix}$$

according to the decomposition $H = K_r \oplus K_r^\perp$ then

$$\sigma(Y_{11}) = \{z \in \mathbb{C} \mid |z| \leq r\}$$

$$\sigma(Y_{22}) = \{z \in \mathbb{C} \mid r \leq |z| \leq 1\}$$

Moreover $\tau_M(P_K) = r^2$.

Finally

$$K_r = \{\xi \in H \mid \limsup_{n \to \infty} \| Y^n \xi \|^{1/n} \leq r\}$$

In particular K_r is T-hyperinvariant.

Example. (The quasi-nilpotent DT-operator

Theorem (Dykema + H., ~~JFA~~ 2004)

Let for each $n \in \mathbb{N}$, T_n denote the strictly upper triangular random matrix

$$T_n = \begin{pmatrix} 0 & t_{12}^{(n)} & \cdots & t_{1n}^{(n)} \\ & 0 & \ddots & \vdots \\ & & \ddots & t_{n-1,n}^{(n)} \\ & 0 & & 0 \end{pmatrix}$$

where $(t_{ij}^{(n)})_{i<j}$ are $\frac{n(n-1)}{2}$ independent identical distributed complex random variables each with density

$$\frac{n}{\pi} e^{-n|z|^2}, \quad z \in \mathbb{C}.$$

Then there exists an operator T in a II_1-factor M, such that $T_n \to T$ for $n \to \infty$ in *-distribution, i.e.

$$tr_M(p(T, T^*)) = \lim_{n\to\infty} \mathbb{E}\, tr_n(p(T_n, T_n^*))$$

for every polynomial p in 2 non-commuting variables. Here

$$tr_n = \frac{1}{n} \mathrm{Tr}$$

is the normalized trace on $M_n(\mathbb{C})$

The pair $(T, W^*(T))$ is unique up to $*$-isomorphism, and we call T the quasi-nilpotent DT-operator.

Properties of T (Dykema + H, AJM 2004)

1) $VN(T)$ is a factor

2) $\sigma(T) = \{0\}$, $\mu_T = \delta_0$

3) $\ker(T) = \{0\}$,

4) $\|T\| = \sqrt{e}$

5) $tr((T^*T)^n) = \dfrac{n^n}{(n+1)!}$, $n = 0, 1, 2, \ldots$

6) $tr((T^*T)^n) = \displaystyle\int_0^e t^n \varphi(t)\, dt$

where $\varphi : (0, e) \to (0, \infty)$ is given by

$$\varphi\left(\frac{\sin v}{v} e^{v \cot v}\right) = \frac{1}{\pi} \sin v \, e^{-v \cot v}, \quad 0 < v < \pi$$

Question

Does T have a non-trivial invariant

Piotr Śniady proved in 2003 the following moment formula for the quasi-nilpotent DT-operator

$$\mathrm{tr}[((T^*)^k T^k)^n] = \frac{n^{nk}}{(nk+1)!} \qquad n,k \in \mathbb{N}$$

Based on Śniady's moment formula and its proof, Dykema and I proved:

Theorem 3 (Dykema + H, IFA 2004)

a) $W^*(T) \cong L(F_2)$

b) T has a one-parameter family of non-trivial hyperinvariant closed subspaces.

Remark

Theorem 3(b) does not follow by the general result of [H + Schultz], because

$$\sigma(T) = \{0\} \Rightarrow \mu_T = \text{Dirac measure in } 0.$$

About the proof of Thm. 3

The quasinilpotent DT-operator T can be realized in $L(F_2)$ in the following way:

$$L(F_2) \simeq W^*(D, X)$$

where D and X are selfadjoint and free (in the sense of Voiculescu) and their distribution measures μ_D and μ_X have no atoms. In particular we can take

$$\mu_D = 1_{[0,1]}(x)\,dx$$

$$\mu_X = \frac{1}{2\pi}\sqrt{4-x^2}\,1_{[-2,2]}(x)\,dx$$

(standard semicircular)

Put

$$T_N = \sum_{1 \le i < j \le 2^N} P_i^{(N)} X P_j^{(N)}$$

where $P_i^{(N)} = 1_{\left[\frac{i-1}{2^N}, \frac{i}{2^N}\right]}(D)$. V2.

Then $(T_N)_{N=1}^{\infty}$ is norm-convergent in $W^*(D,X) \cong L(F_2)$, and

$$T = \lim_{N\to\infty} T_N$$

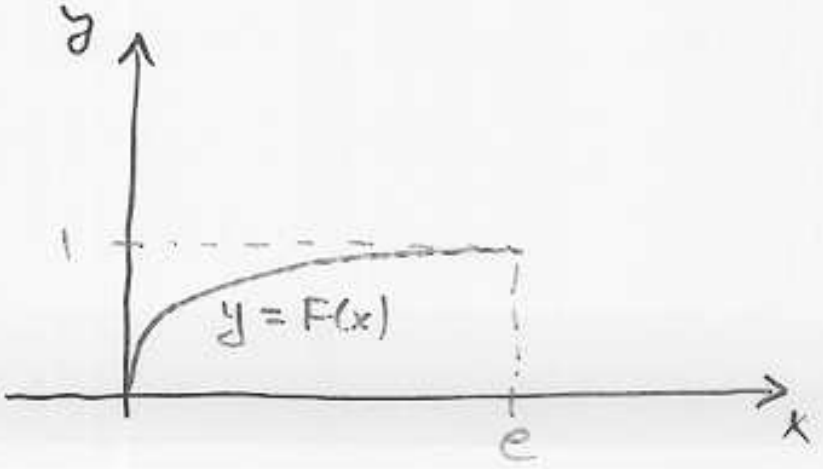
is "a copy" of the quasinilpotent DT-operator (i.e. has the same $*$-moments as that). Moreover

$$X = T + T^*$$

Recall that $\sigma(T^*T) = [0,e]$ and that μ_{T^*T} has a density $\varphi(t)$ w.r.t. Lebesgue measure on $[0,e]$.

Put

$$F(x) = \int_0^x \varphi(t)\,dt, \qquad 0 \le x \le e$$



Theorem 4 (Dykema + H, JFA 2008)

Put

$$S_k = k\left((T^*)^k T^k\right)^{\frac{1}{k}} \qquad \mu_{S_k} \simeq \mu_{T^*T}$$

Then $\sigma(S_k) = [0, e]$. Moreover

$$\lim_{k\to\infty} \| F(S_k) - D \|_2 = 0$$

In particular $D \in W^*(T)$.

Proof of Theorem 4 $\Rightarrow$ Theorem 3

a) $T \in W^*(D, X)$

$D \in W^*(T)$ and $X = T + T^* \in W^*(T)$

Hence

$$W^*(T) = W^*(D, X) \simeq L(F_2)$$

b) It is easy to check that

$$1_{[0,t]}(D)(H), \quad 0 < t < 1$$

form a one-parameter family of closed T-invariant subspaces. Moreover these subspaces are affiliated with $W^*(T)$, because $D \in W^*(T)$.

Finally

$$H_t = 1_{[0,t]}(D)(H) \quad , \quad 0 < t < 1$$

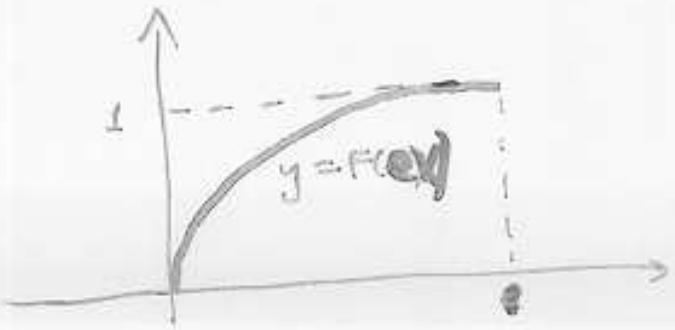
are T-hyperinvariant because it can be proved, that

$$H_t = \{\xi \in H : \lim_{k\to\infty} \sqrt[k]{e}\, \|T^k \xi\|^{1/k} \le t\}$$

Remark

If one instead consider the spaces

$$H_t' = \{\xi \in H : \exists\, \xi_n \in H \text{ s.t. } \lim_{n\to\infty} \|\xi_n - \xi\| = 0 \text{ and } \lim_{k\to\infty} \sqrt[k]{e}\, \|T^k \xi_k\|^{1/k} \le t\}$$

Then it turns out, that

$$H_t' = H_{F(et)} \supsetneq H_t .$$