Multiparameter Fourier Analysis

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Some early issues in harmonic anaysis

- representations of functions, decompositions into wave packets/discrete bases/atomic or simpler pieces.
- Convergence, i.e., the quantitative estimates which control certain limits.
- solutions of PDEs, with data in a variety of function spaces,
- complex function theory, algebras of operators, the Cauchy integral operator on graph domains
- geometric measure theory (covering lemmas, null sets and Hausdorff dimension, Kakeya sets)
- probability:martingale theory, Brownian motion (harmonic functions)
- number theory: additive and analytic
- representation of groups/group actions/Lie groups and symmetric spaces

Some current issues in harmonic anaysis

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Singular integrals and summing Fourier series

- $S_N f(x) = \sum_{-N}^{N} \hat{f}(n) e^{2\pi i n x}$, and $\hat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} dx$.
- If $\mathrm{P}^+(f)$ is the projection onto the positive frequencies, and $M_k(f) = f e^{2\pi i k x}$ then

$$S_N f(x) = M_N \mathrm{P}^+(M_{-N} f) - M_{-N} \mathrm{P}^+(M_N f).$$

- The function space L^2 plays an important role since $\|f\|_2^2 = \int |f|^2 dx = \sum |\hat{f}(n)|^2$. Thus, $\mathrm{P}^+ : L^2 \to L^2$.
- Enter the Hilbert transform: $H = i(\mathrm{P}^+ - \mathrm{P}^-)$.
- The continuous setting: If $\hat{f}(\xi) = \int f(x) e^{2\pi i k \xi} dx$, and $\widehat{\mathrm{P}(f)}(\xi) = \hat{f}(\xi) \chi_{\xi > 0}$.
- $Hf(x) = \text{p.v.} \int \frac{f(t)}{x-t} dt$

Other operators

- Hardy-Littlewood maximal function: $Mf(x) = \sup_Q \int_Q |f| \frac{dy}{|Q|}$, over cubes Q containing x.
- Riesz transforms R_j: convolution in $\mathbb{R}^n$ with $ix_j/|x|^{n+1}$. R_j is also a Fourier multiplier: $\widehat{R_j(f)}(\xi) = i\xi_j/|\xi|\hat{f}(\xi)$.
- General SIO's in $\mathbb{R}^n$: $T(f)(x) = \int f(y)k(x,y)dy$, where $|k(x,y)| \leq |x-y|^{-n}$, plus some regularity condition, such as, $|\nabla k(x,y)| \leq |x-y|^{-n-1}$. A Calderón-Zygmund (CZ) operator, such the Hilbert transform, is also bounded on L^2.

Other operators, continued.

- The Carleson operator: (the Hilbert transform + modulation)

$$\mathcal{C}(f(x)) = \int \frac{e^{iN(x)y}}{y} f(x-y)dy,$$

  with $N(x)$ bounded and measurable.
- $M : L^p \to L^p, 1 < p < \infty$, $H, R_j : L^p \to L^p$: Calderón-Zygmund theory: CZ "decomposition", etc.
- $\mathcal{C} : L^p \to L^p, 1 < p < \infty$, Carleson (1966), Hunt (1967),.C. Fefferman (1973), Lacey-Thiele (2001)

More connections with complex function theory

Let u be a harmonic function in the upper half space ($\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$). Then u is the Poisson integral of its boundary values, and so is determined by its data $f(x) = u(x, 0)$. If v is its conjugate function, so that $F = u + iv$ is analytic in $y > 0$, v is also harmonic and determined by its boundary values $v(x, 0) = g(x)$. The functions u and v are related by the C-R equations, but how are f and g related?

- Answer: $\mathrm{g} = \mathrm{H}(\mathrm{f})$.
- Limits of Poisson integrals: $u(x, t) = f * P_t(x)$, then u converges nontangentially to its boundary values, $f(x)$. (Here, the maximal function controls the Poisson integral in these nontangential regions.)

Harmonic analysis on Symmetric spaces

Let G be a connected non-compact semi-simple Lie group, with finite center, and let K denote a maximal compact subgroup. G/K is the symmetric space associated to G. There exists a left/right invariant Haar measure, and a maximal boundary such that (suitably) bounded functions on the boundary $B(G)$ have a Poisson extension to a harmonic function and conversely, certain harmonic functions have such a representation. (Furstenberg, Annals of Math. 1963).

One may then ask the classical questions posed in harmonic function theory: Fatou theorems, analogs of nontangential convergence of Poisson integrals to the boundary. These questions hinge on qualitative estimates of appropriate maximal operators.

We return to some specific instances of these issues later.

Hardy spaces and *BMO*

- Associated to C-Z operators like the Riesz transforms are the natural "endpoint" spaces: H^1 and BMO, where $(H^1)^* = BMO$, and

$$f \in BMO \text{ iff } \exists C, \text{ for all cubes } Q, \int_Q |f - f_Q| \frac{dx}{|Q|} < C$$

  Here, $f_Q = \int_Q f(y) \frac{dy}{|Q|}$.
  Example: $f(x) = \log 1/|x|$.
- John-Nirenberg theorem: $f \in BMO$ iff $\exists \alpha, C$ such that $\sup_Q \int_Q e^{\alpha|f - f_Q|} dy < C$.
- C-Z operators map $H^1 \to L^1$ and $BMO \to BMO$, and we may define H^1 as the subset of L^1 s.t. all $R_j f \in L^1$.
- Interaction with transformations acting on R^n: translation invariance, commuting with (one parameter family of) dilations: $\vec{x} \to (\delta x_1, ..., \delta x_n)$

Commutators: Coifman-Rochberg-Weiss, C-L-M-S

Let T be C-Z operator associated to a standard kernel, and suppose $b \in BMO$ with norm $||b||_*$. Then, if b also denotes the operator of multiplication by b,

Theorem (C-R-W,1976)

The commutator $[T, b]f = bTf - T(bf)$ is a bounded map of $L^p(\mathbb{R}^n)$ to itself, $1 < p < \infty$. Conversely, if all the commutators $[R_j, b]$ are bounded, $1 \leq j \leq n$, then $b \in BMO$.

- By duality, a multilinear operator: $T(f, g) = fR_jg + gR_jf$ maps $L^2 \times L^2$ into H^1. More: weak factorization of H^1. Upper bound in $\mathbb{R}^1$: $fHg + gHf = P^+fP^+g - P^-fP^-g$ is easily seen to belong to H^1.
- Importance of BMO: it captures the cancellation conditions required of a SIO in order to be C-Z. David-Journé $T(1)$ theorem: T is bounded on L^2 iff $T(1) \in BMO$, and weak-boundedness.

Dyadic decompositions: Haar series, a simple martingale

- The dyadic grid, $\mathcal{D}$, in $[0,1]$: intervals $[k2^{-n},(k+1)2^{-n}]$, $n \geq 0$, $k = 0, 1, ..., 2^n - 1$.
- Definition of dyadic or martingale BMO: Let $h_{\mathrm{J}}(x)$ denote the Haar function associated to $\mathrm{J} \in \mathcal{D}$,

$$h_{\mathrm{J}}(x) = |\mathrm{J}|^{-1/2}, x \in \mathrm{J}_l, h_{\mathrm{J}}(x) = -|\mathrm{J}|^{-1/2}, x \in \mathrm{J}_r.$$

- The Haar functions form a basis for L^2 :$f \in L^2$: $\|f\|_2^2 = \sum_{J\in\mathcal{D}}(f, h_{\mathrm{J}})^2$
- $f \in BMO$ iff there exists a C such that for all $\mathrm{J} \in \mathcal{D}$,

$$\sum_{\mathrm{I}\subset\mathrm{J}}(f, h_{\mathrm{J}})^2 \leq C|J|.$$

- Same property for arbitrary open sets instead of dyadic intervals J.

Singular integrals arising in pointwise convergence

- Partial sums:

$$S_{N,M}f(x,y) = \sum_{|n|<N} \sum_{|m|<M} \hat{f}(n,m)e^{2\pi i(nx+my)}$$

- Double Hilbert transform H_1H_2, $k(x_1,x_2) = 1/(x_1x_2)$
- By iteration, map $L^p \to L^p, 1 < p < \infty$.
- Generalize this tensor product structure (R.Fefferman-Stein, Journé), to a class of operators invariant under $\vec{x} \to (\delta_1 x_1, \delta_2 x_2, ..., \delta_n x_n)$
- Pointwise convergence fails (C. Fefferman), so the Carleson operator (with independent phases $e^{iN(x_1,x_2)y_1}$ and $e^{M(x_1,x_2)y_2}$) is unbounded, but there are positive results for other families of phases, leading to bounded Carleson operators, and hence applications to convergence of multiple Fourier series (E. Prestini). The story is far from complete.

Connections with complex function theory

- On the polydisc, $\mathbb{T}^n$, consider the real parts of functions analytic in each variable separately. These are harmonic functions, in each variable separately.
- Multiple Poisson extensions, convolutions with products of poisson kernels.
- Product spaces: Hardy spaces, *BMO*, etc.
- Early interest in harmonic function theory on the polydisc was connected to its role as a symmetric space, and the resulting new notions of nontangential convergence. This required the analysis of appropriate associated maximal operators (product maximal functions), which we will return to later.

- What is product BMO? (in the sense: $H_1 H_2 : L^\infty \to BMO$)
- The natural conjecture (replace cubes by rectangles, sides independent, in the def'n of one parameter BMO) fails. (L. Carleson)
- BMO_{prod} theory was developed by S.-Y. A. Chang-R.Fefferman: duality, Carleson measures, C-Z decomposition, weighted inequalities
- Definition of dyadic or martingale BMO_{prod}: Let $h_{\mathrm{J}}(x)$ denote the Haar function associated to $\mathrm{J} \in \mathcal{D}$,

$$h_{\mathrm{J}}(x) = |\mathrm{J}|^{-1/2}, x \in \mathrm{J}_l, h_{\mathrm{J}}(x) = -|\mathrm{J}|^{-1/2}, x \in \mathrm{J}_r$$

and for $R = \mathrm{I} \times \mathrm{J} \in \mathcal{D} \times \mathcal{D}$, set $h_{\mathrm{R}}(x, y) = h_{\mathrm{I}}(x) h_{\mathrm{J}}(y)$.

Endpoint spaces

A function f belongs to dyadic BMO if there exists a constant such for that all open sets Ω,

$$\sum_{\mathrm{R}\subset\Omega}(f, h_{\mathrm{R}})^2 \lesssim |\Omega|.$$

Note that in $\mathbb{R}^1$,

$$\sum_{\mathrm{I}\subset\mathrm{J}}(f, h_{\mathrm{I}})^2 = \int_{\mathrm{J}} |f - f_{\mathrm{J}}|^2 \frac{dx}{|\mathrm{J}|}$$

- Continuous BMO: $h_{\mathrm{R}} \longleftrightarrow \omega_{\mathrm{R}}$
- Product SIOs: $BMO \to BMO$, $T(1)$ theorem, duality with product H^1,...
- Open sets bad for maximality arguments, stopping times...
- Relationship between rectangles and open sets (a reduction to rectangle estimates): Journé's Lemma (1986), and its variants (Pipher 1987, Ferguson-Lacey 2002, Cabrillo-Lacey-Molter-Pipher 2006,etc).

Journé's Lemma

- If $\Omega \subset \mathbb{R}^2$, let $\mathcal{M}_2(\Omega)$ denote the collection of dyadic rectangles R maximal in the x_2-direction.
- For $\mathrm{R} = \mathrm{I} \times \mathrm{J} \in \mathcal{M}_2(\Omega)$, say that $emb(\mathrm{R}) = 2^k$, if $k = \inf\{j : |2^j\mathrm{I} \times \mathrm{J} \cap \Omega| > \frac{1}{2}|2^j\mathrm{I} \times \mathrm{J}|\}$
- This is one of several possible notions of "embeddedness" of the rectangle R in the open set Ω.
- For $k \in \mathbb{N}$, let $\mathcal{F}_k \subset \mathcal{M}_2(\Omega)$ denote the collection of all $\mathrm{R} \in \mathcal{M}_2(\Omega)$ with $emb(\mathrm{R}) = 2^k$.

Lemma

For all $\Omega \subset \mathbb{R}^2$,

1. $\sum_{\mathrm{R} \in \mathcal{F}_k} |R| \lesssim k|\Omega|$,
2. $\bigcup\{2^k\mathrm{I} \times \mathrm{J} : \mathrm{I} \times \mathrm{J} \in \mathcal{F}_k\} \subset \tilde{\Omega}$ *where* $|\tilde{\Omega}| \leq C|\Omega|$

The proof of Journé's Lemma

Why? Useful in context of estimates relative to $\mathrm{R} \subset \Omega$ which have small but exponential decay as a function of distance to Ω. Example: boundedness of product singular integrals on Hardy spaces.

Proof.

- For $\mathrm{I} \in \mathcal{D}$, set
$$\mathcal{E}_{\mathrm{I}} = \bigcup \{\mathrm{J} : \mathrm{I} \times \mathrm{J} \subset \Omega\}.$$
- We have
$$|\Omega| = \sum_{\mathrm{I}} |\mathrm{I}||\mathcal{E}_{\mathrm{I}} \setminus \mathcal{E}_{2\mathrm{I}}|$$
- and
$$\sum_{\mathrm{R} \in \mathcal{F}_k} |\mathrm{R}| \leq 2 \sum_{\mathrm{I}} |\mathrm{I}||\mathcal{E}_{\mathrm{I}} \setminus \mathcal{E}_{2^k\mathrm{I}}| \leq 2k \sum_{\mathrm{I}} |\mathrm{I}||\mathcal{E}_{\mathrm{I}} \setminus \mathcal{E}_{2\mathrm{I}}|.$$
- using the "stopping time" def'n of k :
$k =$ the smallest integer such that $|2^k\mathrm{I} \times \mathrm{J} \cap \Omega| > \frac{1}{2}|2^k\mathrm{I} \times \mathrm{J}|\}$.

SIO's in $\mathbb{R}^n \times \mathbb{R}^m$:

- $Tf(x_1, x_2) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} k(x_1, y_1, x_2, y_2) f(y_1, y_2) dy_1 dy_2$
- For each $x_1, y_1 \in \mathbb{R}^n$, let $K^{(1)}(x_1, y_1)$ denote the operator on $\mathbb{R}^m$ associated to the kernel $k^{(1)}(x_1, y_1)(x_2, y_2) = k(x_1, y_1, x_2, y_2)$, and similarly define $K^{(2)}$.
- Then T is a C-Z operator on $\mathbb{R}^n \times \mathbb{R}^m$ if
  - $T : L^2 \to L^2$
  - $\int_{|x_1 - y_1| > \gamma |x_1 - x_1'|} \|K^{(1)}(x_1, y_1) - K^{(1)}(x_1, x_1')\|_{CZ} dy_1 \leq C\gamma^{-\delta}$, for all γ and some δ.
  - Likewise for $K^{(2)}$.
- In Journé's Revista 85 paper: extended earlier work on convolution structure operators to general setting, using new geometric constructions to induct on dimension, and proved a $T(1)$ theorem.

Commutators: one basic example of a bilinear operator

What is the analog in the multiparameter situation of the characterization of BMO_{prod} in terms of boundedness of commutators like $[H, b]$?

Formulated in Ferguson-Sadosky (2000), and proven in in Ferguson-Lacey (2002):

$$\|[[H_1, b], H_2]\|_{2\to 2} \sim \|b\|_{BMO}$$

The upper bound for the operator is the same as

$$fH_1H_2g + H_1fH_2g + H_2fH_1g + gH_1H_2f \in H^1$$

and a proof follows by expressing the iterated Hilbert transform in terms of projections, and using the characterization of H^1 by singular integrals.

The F-S work established a lower bound in terms of rectangle BMO which turned out to be important in the definitive F-L proof.

Later (2006) Lacey-Terwilliger found an (ingenious) inductive scheme defining new BMO spaces, to extend the result to arbitrary iterates of Hilbert transforms.

Commutators in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3} \times \ldots$

Theorem (Lacey-Petermichl-Pipher-Wick 2007)

Let $R_{j_k}^{(k)}$ denote a Riesz transform acting on $\mathbb{R}^{n_k}$. The iterated commutators

$$[\ldots[R_{j_1}^{(1)}, b], R_{j_2}^{(2)}], \ldots]R_{j_d}^{(d)}]$$

are bounded on $L^2(\mathbb{R}^{n_1}) \times \ldots L^2(\mathbb{R}^{n_d})$ with norm $\lesssim \|b\|_{BMO}$. Conversely, if all possible iterated Riesz transform commutators are bounded, then $b \in BMO_{prod}$.

- The upper bound extends to iterates of other SIOs.
- The proof uses (a version of) the multilinear paraproduct estimates.
- Div-curl: If, for $x, y \in \mathbb{R}^n$, $\vec{E}(x,y) \in L^p$ and $\vec{B}(x,y) \in L^{p'}$ satisfy $\text{div}_x \vec{E} = 0 = \text{div}_y \vec{E}$ and $\text{curl}_x \vec{B} = 0 = \text{curl}_y \vec{B}$ then $\vec{E} \cdot \vec{B} \in H^1_{prod}$

Commutators continued

Missing:

- Convergence in weak H^1. Jones-Journé Theorem: If $\|f_n\|_{H^1} \lesssim 1$, and $f_n \to f$ a.e., then $f \in H^1$ and $\int f_n\varphi \to \int f\varphi$, for all $\varphi \in VMO$. "Not abstract". Proof depends on the construction (Coifman-Rochberg) of a family of BMO functions of the form $\log(Mh)$ where $0 < h \in L^1$ and M is the (Hardy-Littlewood) maximal function.
- Generalized commutators which involve product singular integrals, as opposed to iterates of singular integrals?

Operators in $\mathbb{R}^n$ associated to k-parameter dilation groups

Singular integrals, maximal functions associated with k-parameter families (SIO's on surfaces,)

Role of maximal operators in the singular integral theory, and their boundedness properties.

- Hardy Littlewood maximal operator $\leftrightarrow$ cubes, one parameter dilations, endpoint estimate: weak L^1: Besicovitch covering lemma.

$$|\{Mf > \lambda\}| \lesssim \int \frac{f}{\lambda}.$$

- Strong maximal operator in $\mathbb{R}^n \leftrightarrow$ rectangles with n independent sides, n-parameter dilations, endpoint estimate: $L(\mathrm{Log}\ L)^{n-1}$. Jessen-Marcinkiewicz-Zygmund, Cordoba-R.Fefferman covering lemma.

$$|\{Mf > \lambda\}| \lesssim \int \frac{f}{\lambda}(\log \frac{f}{\lambda} + 1)^{n-1}.$$

Covering lemmas and k-parameter maximal functions

Sharp covering lemmas follow from:

Theorem

(R.Fefferman-Pipher) Let R_i be a collection of measurable sets in $\mathbb{R}^n$. Take $\beta \in (0,1]$ (for example, $\beta = 1/(n-1)$). Select a subcollection $\widetilde{R}_i$ according to the rule

$$\int_{R_m} \exp(\sum_{i<m} \chi_{\widetilde{R}_i}(x))^{\beta} dx \leq c|R_m|.$$

Then, one has the improvement

$$\int (\sum_i \chi_{\widetilde{R}_i}(x))^{1-\beta} \exp(\sum_i \chi_{\widetilde{R}_i}(x))^{\beta} dx \leq C \sum_m |\widetilde{R}_m|.$$

When the R_i are rectangles in $\mathbf{R}^n$, further assumptions on the ordering will also give $|\cup R_i| \leq C|\cup \widetilde{R}_m|$.

k-parameter dilation groups in $\mathbb{R}^n$

Example (Stein):

$$\mathbb{R}^3 \longleftrightarrow \begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix}$$

and $\Gamma = \{Y \in \mathbb{R}^3 : Y \text{ is pos. def. }\}$,
$T_\Gamma = \{X + iY : Y \in \Gamma\}$ is the Siegel upper half space.
For $f \in L^1(\mathbb{R}^3)$, set $u(X + iY) = P_Y * f$ where $P_Y(X) = \frac{C(\det Y)^{3/2}}{|\det(X+iY)|^3}$.
The maximal operator which controls a certain restricted convergence of u to its boundary values is associated to the 2 parameter dilation family: $(x_1, x_2, x_3) \to (\delta_1 x_1, \delta_2 x_2, \delta_1 \delta_2 x_3)$.

- Two independent variables: like M in $\mathbb{R}^2$ (Cordoba).
- Zygmund's conjecture.

Some related and current developments:

- Flag singular integrals: Singular integrals in the work of Muller-Ricci-Stein and Nagel-Ricci-Stein arise in their study of Heisenberg groups and multiplier operators. The multiparameter structure of the flag singular integrals on $\mathbb{R}^n \times \mathbb{R}^m$ in N-R-S is a special case of product singular integrals. For example, $k(x, y) = (x(x + iy))^{-1}$ is a flag kernel on $\mathbb{R}^2$. Recent work (preprint) of Han-Lu on the Hardy space theory, reprising and extending the approach of R.Fefferman-Stein for convolution singular integrals in the product theory.
- Discrepancy theory-Brownian sheet-Small Ball conjecture. Recent work of Lacey, Bilyk-Lacey, and Bilyk-Lacey-Vagharshakyan. United by the tools and methodology of multiparameter Fourier analysis are questions related to number theory (the irregularity of distributions of points in the unit cube), probability (estimates for maximal functions associated with Gaussian processes) and analysis (L^∞ lower bounds for sums of multiple Haar series). This a big subject: for further information see the references at www.math.gatech.edu/ lacey.

Bilinear operators

These operators will be the generalization of tensor products of bilinear operators, in the same sense that the SIO's generalized products. However, here the L^p theory is interesting, as tensor products of bounded bilinear operators *need not be bounded*. (Example: Krikeles, citing Y. Meyer (1986), Muscalu-Pipher-Tao-Thiele (2004))

- The biparameter bilinear operators are

$$T(f,g)(x,y) = \int_{\vec{\eta},\vec{\xi}} \hat{f}(\eta_1,\eta_2)\hat{g}(\xi_1,\xi_2)e^{2\pi i(x\cdot\eta+y\cdot\xi)}m(\vec{\eta},\vec{\xi})d\eta d\xi$$

- A Coifman-Meyer paraproduct will have a symbol m which (like the product of two such symbols in $\mathbb{R}^2$) satisfies:

$$|D^{\alpha_1}_{\xi_1}D^{\alpha_2}_{\xi_2}D^{\beta_1}_{\eta_1}D^{\beta_2}_{\eta_2}m| \lesssim \frac{1}{|(\xi_1,\eta_1)|^{\alpha_1+\beta_1}}\frac{1}{|(\xi_2,\eta_2)|^{\alpha_2+\beta_2}}$$

Bilinear paraproducts

Theorem (M-P-T-Th)

If T is a bilinear operator with a symbol satisfying the product Coifman-Meyer condition, then $T : L^p \times L^q \to L^r$ for $1/p + 1/q = 1/r, r > 0 p, q > 1.$

Note: Special case $L^2 \times L^\infty \to L^2$ (Journé)

Corollary

If $D_1^\alpha D_2^\beta$ denotes the operator with symbol $|\xi_1|^\alpha |\xi_2|^\beta|$, then a multilinear version of the Christ-Weinstein, Kato-Ponce inequality holds:

$$\begin{aligned} \|D_1^\alpha D_2^\beta (fg)\|_r &\lesssim \|D_1^\alpha D_2^\beta (f)\|_p \|g\|_q + \|D_1^\alpha (f)\|_p \|D_2^\beta (g)\|_q \\ &+ \|D_1^\alpha (g)\|_q \|D_2^\beta (f)\|_p + \|f\|_p \|D_1^\alpha D_2^\beta (g)\|_q \end{aligned}$$

An unbounded tensor product

Recall bilinear Hilbert transform:

$$
\begin{aligned}
B(f,g)(x) &= \int_{\mathbb{R}^1} f(x-t)g(x+t)\frac{dt}{t} \\
&= \int\int \hat{f}(\eta)\hat{g}(\xi)e^{2\pi i(\eta+\xi)x}\mathrm{sgn}(\xi-\eta)\,d\eta\,d\xi
\end{aligned}
$$

It has the biparameter counterpart B_{prod}:

$$
\int_{\mathbb{R}^4} \hat{f}(\eta_1,\xi_1)\hat{g}(\eta_2,\xi_2)e^{2\pi i(x,y)\cdot((\eta_1,\xi_1)+(\eta_2,\xi_2))}\mathrm{sgn}(\eta_1-\eta_2)\mathrm{sgn}(\xi_1-\xi_2)\,d\vec{\eta}\,d\vec{\xi}
$$

But B_{prod} does not satisfy *any* L^p estimates. Take $f(x,y) = g(x,y) = e^{ixy}\mathbf{1}_{[-N,N]}(x)\mathbf{1}_{[-N,N]}(y)$. C. Fefferman's counterexample to rectangular convergence of multiple Fourier series!