

Clay-Fields Conference on Additive
Combinatorics, Number Theory, and
Harmonic Analysis

**On the exact structure of
multidimensional sets with small
doubling property**

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1. Direct and inverse problems of additive and combinatorial number theory

Additive number theory is the study of sums of sets and we can distinguish two main lines of research.

In a direct problem of additive number theory we start with a particular known set A and attempt to determine the structure and properties of the h -folds sumset hA . These are the classical direct problems in additive number theory: Waring's problem, Goldbach conjecture...

As a counterbalance to this direct approach, an inverse problem in additive number theory is a problem in which we study properties of a set A , if some characteristic of the h -fold sumset hA is given.

Sumsets can be defined in any Abelian group G , for example in

- \mathbb{Z}
the group of integers,
- $\mathbb{Z}/m\mathbb{Z}$
the group of congruence classes modulo m ,
- \mathbb{Z}^n
the group of integer lattice points,
- \mathbb{R}^d
the d -dimensional Euclidean space.

Freiman proposed an unifying “algorithm” for solving inverse additive problems:

- Step 1. Consider some (usually numerical) characteristic of the set under study.
- Step 2. Find an extremal value of this characteristic within the framework of the problem that we are studying.
- Step 3. Study the structure of the set when its characteristic is equal to its extremal value.
- Step 4. Study the structure of the set when its characteristic is near to its extremal value.
- Step 5.Continue, taking larger and larger neighborhoods for the characteristic.

Let us choose as characteristic the *cardinality of the sumset*:

$$2K = K + K,$$

or equivalently the “*measure of doubling*”:

$$\sigma = \frac{|K + K|}{|K|}.$$

We will examine in detail the **exact structure** of a finite set

$$K \subseteq G,$$

in the case of a torsion free Abelian group

$$G = \mathbb{Z}^n \quad \text{or} \quad G = \mathbb{R}^d,$$

assuming that the doubling constant is small.

REMARK: If σ is an *arbitrary* doubling constant, then *Freiman’s fundamental result* (1973) asserts that such a set is a large subset of a multidimensional arithmetic progression; see also Freiman (1987), Bilu (1993), Ruzsa (1994), Nathanson (1996), or Tao and Vu (2006).

2. Small doubling property on the plane \mathbb{Z}^2

Let us describe some results concerning the structure of *planar sets* with small sumset.

We begin with the following basic inequality:

Theorem 1 (Freiman 1973). *If $\mathcal{K} \subseteq \mathbb{Z}^2$ lies on exactly $s \geq 2$ parallel lines, then*

$$|\mathcal{K} + \mathcal{K}| \geq \left(4 - \frac{2}{s}\right)|\mathcal{K}| - 2s + 1 \geq 3k - 3. \quad (1)$$

Moreover, using Freiman's $3k - 4$ theorem we easily conclude that a planar set of lattice points $\mathcal{K} \subseteq \mathbb{Z}^2$ with

$$|\mathcal{K} + \mathcal{K}| < 3|\mathcal{K}| - 3$$

lies on a straight line and is contained in an arithmetic progression of no more than

$$v = |\mathcal{K} + \mathcal{K}| - |\mathcal{K}| + 1$$

terms. Step 2 is completely solved.

Therefore, a natural problem is to concentrate on the study of Steps 3 and 4.

We ask for the structure of a finite *planar* set of lattice points with small doubling $|\mathcal{K} + \mathcal{K}|$. As one can expect, this question is easier to answer when the cardinality $|\mathcal{K} + \mathcal{K}|$ is close to its minimal possible value $3|\mathcal{K}| - 3$, and becomes much more complicated if we choose bigger values for $|\mathcal{K} + \mathcal{K}|$. To be more specific, we may ask the following

Problem.

Find the exact structure of planar sets of lattice points under the small doubling hypothesis:

$$|\mathcal{K} + \mathcal{K}| < (4 - \frac{2}{s+1})|\mathcal{K}| - (2s + 1).$$

Let us examine the first case $s = 2$.

Though, the Freiman's $(2^n - \epsilon)$ theorem gives a first indication on the structure of \mathcal{K} , still this is not so precise as the following

Theorem 2 (Freiman 1973, S. 1998). *Let $\mathcal{K} \subseteq \mathbb{Z}^2$ be a finite of dimension $\dim \mathcal{K} = 2$.*

(i) $|\mathcal{K}| \geq 11$ and $|\mathcal{K} + \mathcal{K}| < \frac{10}{3}|\mathcal{K}| - 5$ then \mathcal{K} lies on two parallel lines.

(ii) If \mathcal{K} lies on two parallel lines and

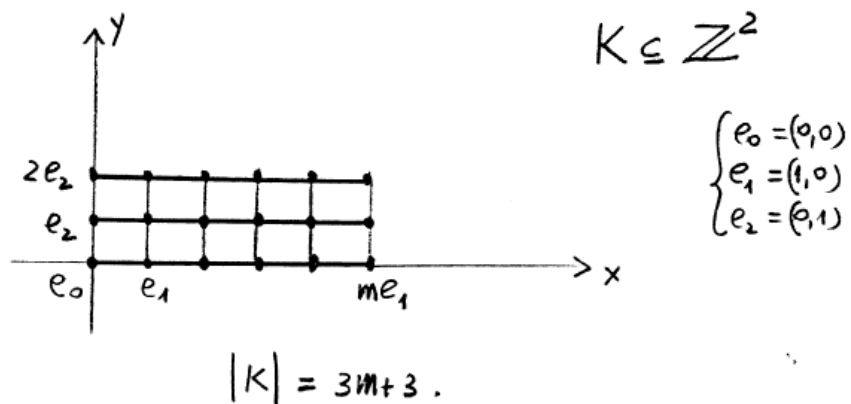
$$|\mathcal{K} + \mathcal{K}| < 4|\mathcal{K}| - 6$$

then \mathcal{K} is included in two parallel arithmetic progressions with the same common having together no more than $v = |2\mathcal{K}| - 2k + 3$ terms.

This means that the total number of holes satisfies

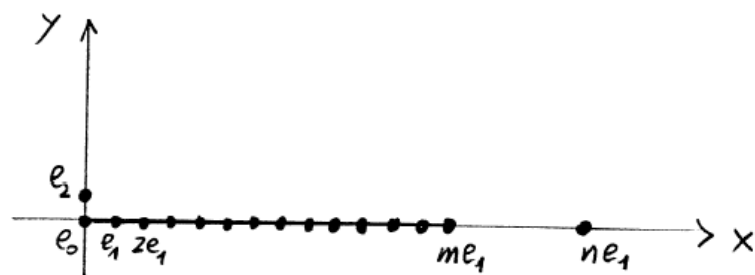
$$h \leq |2\mathcal{K}| - (3k - 3).$$

EXAMPLE 1



$$|2K| = 5(2m+1) = \frac{10}{3}|K| - 5.$$

EXAMPLE 2



$$|K| = m+3.$$

$$|2K| = n+2m+5.$$

$$v = n+2 = |2K| - 2|K| + 3.$$

The following theorem incorporates Freiman's previous result as a particular case:

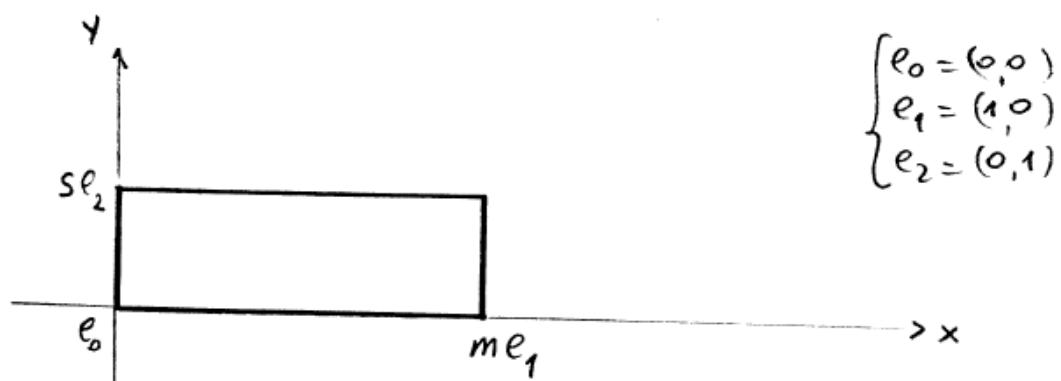
Theorem 3 (S. 1998). *Let \mathcal{K} be a finite set of \mathbb{Z}^2 and $s \geq 1$ be a natural number. If $|\mathcal{K}|$ is sufficiently large, i.e. $k \geq O(s^3)$, and*

$$|\mathcal{K} + \mathcal{K}| < \left(4 - \frac{2}{s+1}\right)|\mathcal{K}| - (2s+1) , \quad (2)$$

then there exist s parallel lines which cover the set \mathcal{K} .

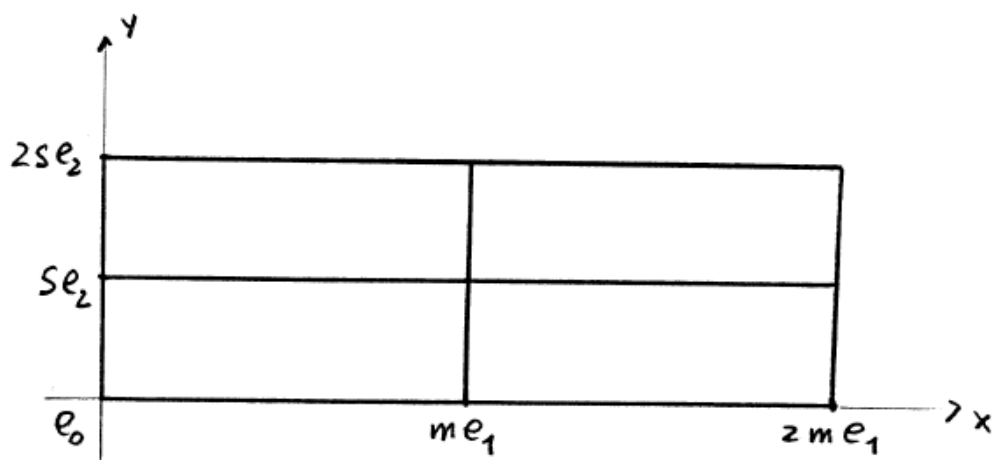
This is a best possible result, because it cannot be improved by increasing the upper bound for $|\mathcal{K} + \mathcal{K}|$, or by reducing the number of lines that cover \mathcal{K} :

EXAMPLE 3



$$K = ([0, m] \times [0, s]) \cap \mathbb{Z}^2.$$

$$|K| = (m+1)(s+1).$$



$$2K = ([0, 2m] \times [0, 2s]) \cap \mathbb{Z}^2.$$

$$|2K| = (2m+1)(2s+1) = \left(4 - \frac{2}{s+1}\right) |K| - 2s - 1.$$

The theorem is effective and recently Serra and Grynkiewicz obtained an explicit value for the constant $k_0(s) = 2s^2 + s + 1$. They also succeeded to extend the result for sums of different sets $A + B$:

Theorem 4 (Grynkiewicz and Serra 2007).
Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^2$ be finite subsets and $s \geq 1$ be a natural number.

(i) *If $\left| |\mathcal{A}| - |\mathcal{B}| \right| \leq s + 1, |\mathcal{A}| + |\mathcal{B}| \geq 4s^2 + 2s + 1$ and*

$$|\mathcal{A} + \mathcal{B}| < \left(2 - \frac{1}{s+1}\right)(|\mathcal{A}| + |\mathcal{B}|) - (2s + 1)$$

then there exist $2s$ (not necessarily distinct) parallel lines which cover the sets \mathcal{A} and \mathcal{B} .

(ii) *If $|\mathcal{A}| > |\mathcal{B}| + s, |\mathcal{B}| \geq 2s^2 + \frac{s}{2}$ and*

$$|\mathcal{A} + \mathcal{B}| < |\mathcal{A}| + \left(3 - \frac{2}{s+1}\right)|\mathcal{B}| - (s + 1)$$

then there exist $2s$ (not necessarily distinct) parallel lines which cover the sets \mathcal{A} and \mathcal{B} .

The next natural question is to consider a finite set \mathcal{K} of lattice points on a plane having the *small doubling property*

$$|2\mathcal{K}| < (4 - \frac{2}{s+1})|\mathcal{K}| - (2s+1)$$

and ask for a reasonable estimate for the number of lattice points of a "minimal" parallelogram that covers the set \mathcal{K} .

More precisely, if \mathcal{L} is a lattice generated by \mathcal{K} , we are interested in precise upper bounds for the number of points of \mathcal{L} that lie in the convex hull of \mathcal{K} . Our main result asserts that \mathcal{K} is located inside a parallelogram that lies on a few lines which are well filled:

Theorem 5 (S. 2007). *Let $s \geq 19$ be an integer and let \mathcal{K} be a finite subset of \mathbb{Z}^2 that lies on exactly s parallel lines. If*

$$|2\mathcal{K}| < \left(4 - \frac{2}{s+1}\right)|\mathcal{K}| - (2s+1),$$

then there is a lattice $\mathcal{L} \subseteq \mathbb{Z}^2$ and a parallelogram \mathcal{P} such that

$$\mathcal{K} \subseteq (\mathcal{P} \cap \mathcal{L}) + v$$

and

$$|\mathcal{P} \cap \mathcal{L}| \leq 24(|\mathcal{K} + \mathcal{K}| - 2|\mathcal{K}| + 1),$$

for some $v \in \mathbb{Z}^2$.

Conjecture. We believe that for a best possible result, the constant factor 24 of Theorem 5 should be replaced by $\frac{1}{2}(1 + \frac{1}{s-1})$, i.e.

$$|\mathcal{P} \cap \mathcal{L}| \leq \frac{s}{2(s-1)}(|\mathcal{K} + \mathcal{K}| - 2|\mathcal{K}| + 2s - 1).$$

So far inequality this estimate has been proved only for $s = 2$ (see Theorem 2) and $s = 3$ (S. 1999).

3. Planar sets with no three collinear points on a line

Let $\mathcal{A} \subseteq \mathbb{Z}^2$ be a finite set, not containing any three collinear points. Freiman asked in 1966 for a lower bound for $|\mathcal{A} + \mathcal{A}|$. As a first step in the investigation of this problem we showed that $\frac{|\mathcal{A} \pm \mathcal{A}|}{|\mathcal{A}|}$ is unbounded, as $\lim |\mathcal{A}| = \infty$:

Theorem 6 (S. 2002). *Let $\mathcal{A} \subseteq \mathbb{Z}^2$ be a finite set of n lattice points. If \mathcal{A} does not contain any three collinear points, then there is a positive absolute constant $\delta > 0$ such that*

$$|\mathcal{A} \pm \mathcal{A}| \gg n(\log n)^\delta. \quad (3)$$

The constant δ can be easily computed: for instance, any positive δ smaller than 0.125 will do.

There is an intimate connection between two seemingly unrelated problems:

- (i) non-averaging sets of integers of order t and
- (ii) planar sets with no three points on a line.

Definition. A finite set of integers $\mathcal{B} \subseteq \mathbb{Z}$ is called a non-averaging set of order t , if for every $1 \leq m, n \leq t$ the equation

$$mX_1 + nX_2 = (m + n)X_3,$$

have no nontrivial solutions with $X_i \in \mathcal{B}$.

Let

$$s_t(n)$$

be the maximal cardinality of a *non-averaging set of order t* included in the interval $[1, n]$.

It is clear that a non-averaging set of order 1 is simply an integer set containing no arithmetic progressions. Bourgain's bound (1999) for Roth's theorem gives:

$$s_t(n) \leq s_1(n) = r_3(n) \ll \frac{n}{(\log n)^{\frac{1}{2}}} (\log \log n)^{\frac{1}{2}}.$$

Remark. We also obtained a *more exact* inequality, valid for sets $\mathcal{A} \subseteq \mathbb{Z}^2$ containing no k -terms arithmetic progressions: for every integer $t \geq 1$ we have

$$|\mathcal{A} \pm \mathcal{A}| \geq \frac{1}{2} |\mathcal{A}| \left(\frac{n}{s_t(n)} \right)^{\frac{1}{4t}}. \quad (4)$$

We formulate the following:

Problem S. *Suppose that $t \geq 1$ is a fixed, positive, but rather large integer. Is it true that $s_t(n) \ll \frac{n}{(\log n)^{4t}}$, or at least $s_t(n) \ll \frac{n}{(\log n)^c}$, for a positive absolute constant $c \geq \frac{1}{2}$?*

Note that Freiman's question asks for a non trivial lower estimate of $|\mathcal{A} + \mathcal{A}|$ for a set $\mathcal{A} \subseteq \mathbb{Z}^2$ containing no three collinear points and in Problem S we want to estimate the density of a sequence of natural numbers \mathcal{B} , assuming that t linear equations does not hold for \mathcal{B} . Inequality (4) shows that any upper bound for $s_t(n)$, better than the trivial one $r_3(n)$ will lead to a corresponding sharpening of (3) and (4).

As regards lower bounds, we have:

Theorem 7 (S. 2002).

(i) *For every $t \geq 1$, there is a positive constant c_t such that for every n one has*

$$s_t(n) \geq n \exp(-c_t \sqrt{\log n}).$$

(ii) *There is no $\epsilon_0 > 0$ such that the inequality*

$$|\mathcal{A} + \mathcal{A}| \gg |\mathcal{A}|^{1+\epsilon_0}$$

holds for every finite set $\mathcal{A} \subseteq \mathbb{Z}^2$ containing no three collinear points.

The proof uses Freiman's fundamental concept of isomorphism, Behrend's method (1946) and a result of Ruzsa about sets of integers containing no non-trivial three term arithmetic progressions.

A recent improvement of the lower bound (3), was obtained by T. Sanders (2006):

$$|\mathcal{A} + \mathcal{A}| \gg_{\epsilon} |\mathcal{A}| (\log |\mathcal{A}|)^{\frac{1}{3}-\epsilon}.$$

4. The simplest inverse problem for sums of sets in several dimensions

It is a well known fact that $|A+B| \geq |A|+|B|-1$ for every two finite sets A and B of \mathbb{Z}^d , equality being attained when A and B are arithmetic progressions with the same difference.

It is possible to obtain a much better estimate. The first result connecting geometry and additive properties is

Theorem 8 (Freiman 1973). *For every finite set $\mathcal{A} \subseteq \mathbb{Z}^d$ of affine dimension $\dim \mathcal{A} = d$, one has*

$$|\mathcal{A} + \mathcal{A}| \geq (d+1)|\mathcal{A}| - \frac{1}{2}d(d+1). \quad (5)$$

This lower bound is tight, i.e. Step 2 is solved.

Let us investigate now Step 3. What is the *exact structure* of multi-dimensional sets having the *smallest cardinality* of the sumset?

The following result is an analogue of the well known Vosper's theorem (1956), $\mathbb{Z}/p\mathbb{Z}$ being here replaced by the d -dimensional space \mathbb{R}^d .

Theorem 9 (S. 1998). *Let $\mathcal{A} \subseteq \mathbb{R}^n$ be a finite set such that $\dim \mathcal{A} \geq d$ and*

$$|\mathcal{A} + \mathcal{A}| = (d + 1)|\mathcal{A}| - \frac{1}{2}d(d + 1).$$

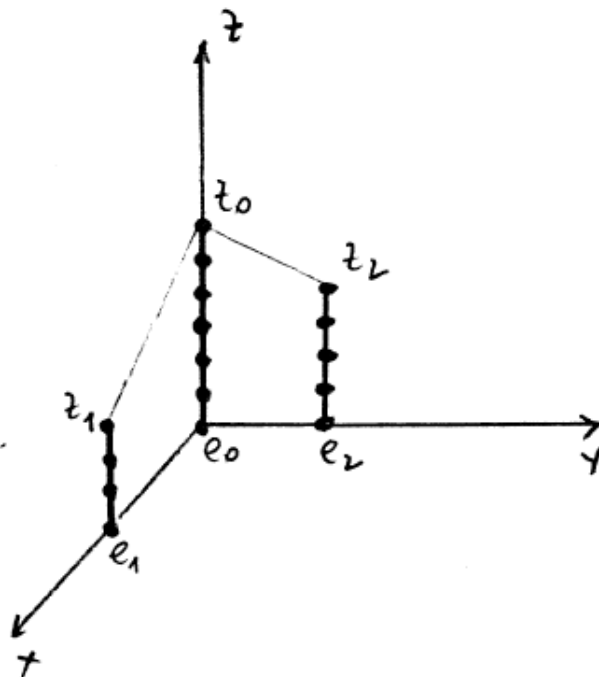
If $|\mathcal{A}| \neq d + 4$, then \mathcal{A} is a d -dimensional set and \mathcal{A} consists of d parallel arithmetic progressions with the same common difference.

Moreover, if $|\mathcal{A}| = d + 4$, then

$$\mathcal{A} = \{v_0, v_1, \dots, v_d\} \cup \{2v_1, v_1 + v_2, 2v_2\},$$

where v_i are the vertices of a d -dimensional simplex.

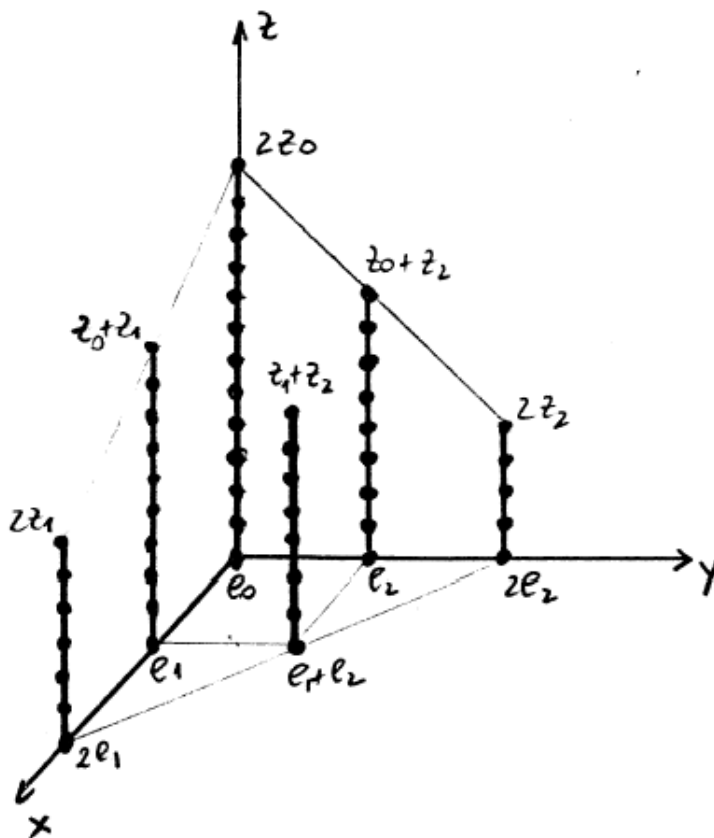
EXAMPLE 4



$$K = K_0 \cup K_1 \cup K_2 \in \mathbb{Z}^3$$

$$|K_i| = z_i + 1 \quad (i=0,1,2)$$

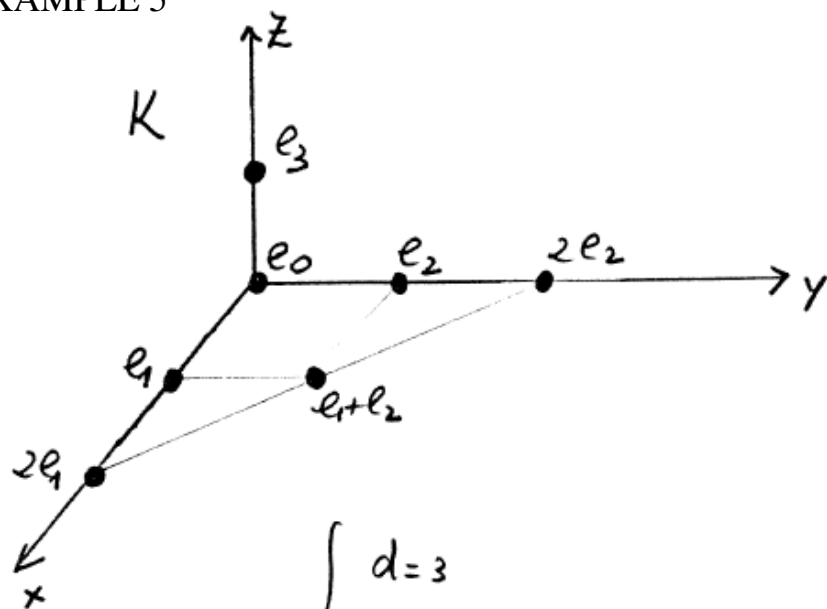
$$|K| = z_0 + z_1 + z_2 + 3$$



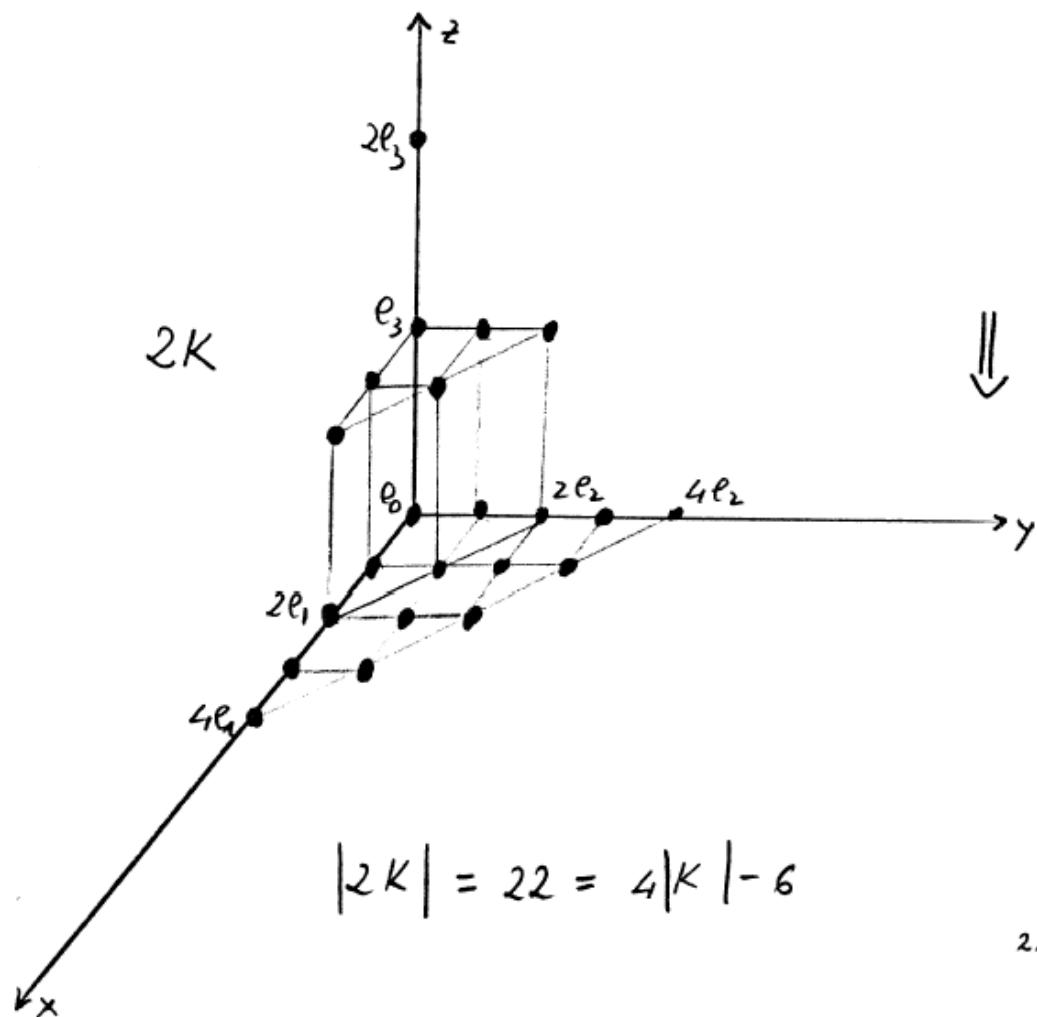
$$|2K| = 4z_0 + 4z_1 + 4z_2 + 6$$

$$|2K| = 4|K| - 6$$

EXAMPLE 5



$$\begin{cases} d=3 \\ K \subseteq \mathbb{Z}^3, \quad |K|=7 \end{cases}$$



Further developments:

Ruzsa (1994): If $|A| \geq |B|$ and $\dim(A+B) = d$, then

$$|A+B| \geq |A| + d|B| - \frac{d(d+1)}{2}.$$

Gardner and Gronchi (2001): If $|A| \geq |B|$ and $\dim(B) = d$, then

$$\begin{aligned} |A+B| &\geq \\ &\geq |A| + (d-1)|B| + \sqrt[d]{(|A|-d)^{d-1}(|B|-d)} - \frac{d(d-1)}{2} \end{aligned}$$

Green and Tao (2006)

Suppose that $A \subseteq \mathbb{R}^m$ is a finite set which contains a parallelepiped $P = \{0, 1\}^d \subseteq \mathbb{Z}^d \subseteq \mathbb{R}^m$.

Then

$$|A+A| \geq 2^{d/2}|A|.$$

5. Exact Structure Results for Multidimensional Inverse Additive Problems

A natural question is to generalize Theorem 3 to the multidimensional case $d = \dim(\mathcal{K}) \geq 3$:

Assume that the doubling coefficient of the sum set $2\mathcal{K}$ is not much exceeding the minimal one, i.e.

$$d + 1 \leq \sigma = \frac{|2\mathcal{K}|}{|\mathcal{K}|} < \rho_d.$$

What can be said about the *exact structure* of \mathcal{K} ? The expected result is: if

$$\rho_d = d + 1 + \frac{1}{3},$$

then the set K is contained in d "short" arithmetical progressions.

The problem was first solved for the first open case $d = 3$:

Theorem 10 (S. 2005). *Let \mathcal{K} be a finite subset of \mathbb{Z}^3 of affine dimension $\dim \mathcal{K} = 3$.*

(i) *If $|\mathcal{K}| > 12^3$ and*

$$|\mathcal{K} + \mathcal{K}| < \frac{13}{3}|\mathcal{K}| - \frac{25}{3}$$

then \mathcal{K} lies on three parallel lines.

(ii) *If \mathcal{K} lies on three parallel lines and*

$$|\mathcal{K} + \mathcal{K}| < 5|\mathcal{K}| - 10,$$

then \mathcal{K} is contained in three arithmetic progressions with the same common difference, having together no more than

$$v = |\mathcal{K} + \mathcal{K}| - 3|\mathcal{K}| + 6$$

terms.

The structure of \mathcal{K} can be also be described for sets of dimension $d \geq 3$:

Theorem 11 (S. 2008). *Let $\mathcal{K} \subseteq \mathbb{Z}^d$ be a finite set of dimension $d \geq 2$.*

(i) *If $k > 3 \cdot 4^d$ and*

$$|\mathcal{K} + \mathcal{K}| < (d + \frac{4}{3})|\mathcal{K}| - c_d,$$

where $c_d = \frac{1}{6}(3d^2 + 5d + 8)$, then \mathcal{K} lies on d parallel lines.

(ii) *If \mathcal{K} lies on d parallel lines and*

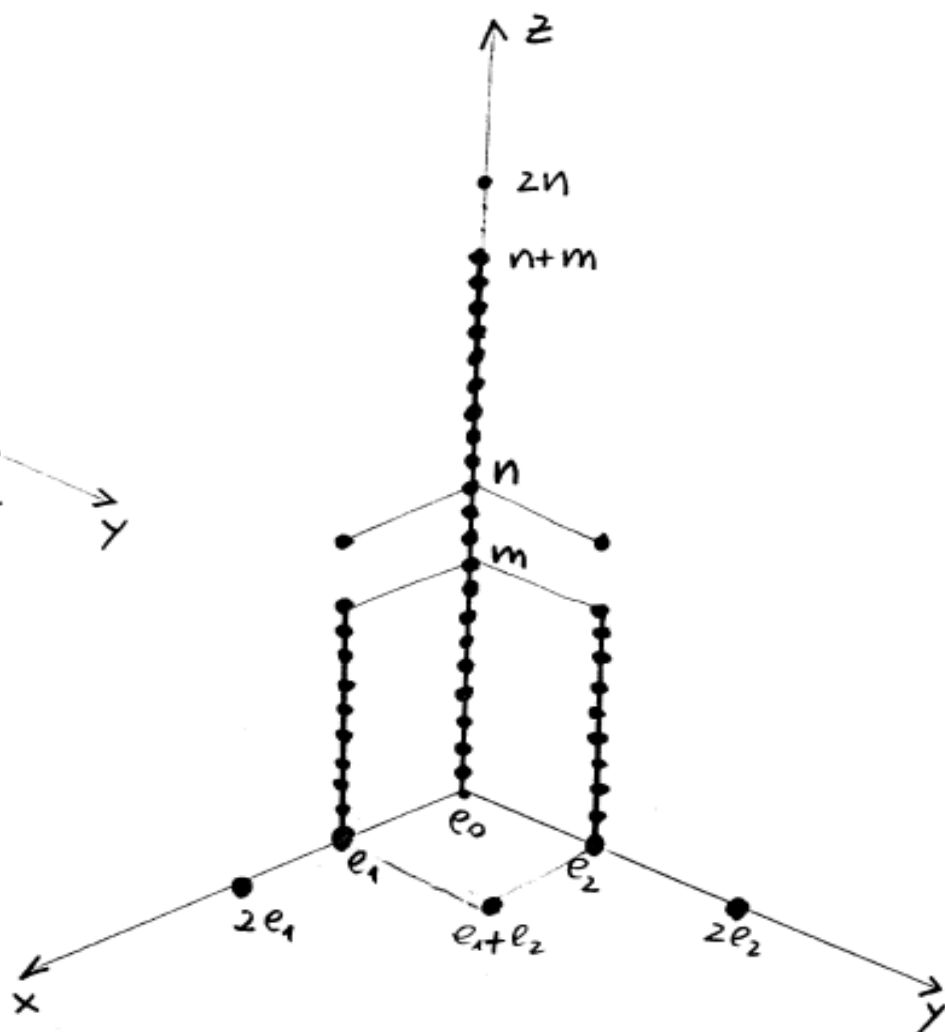
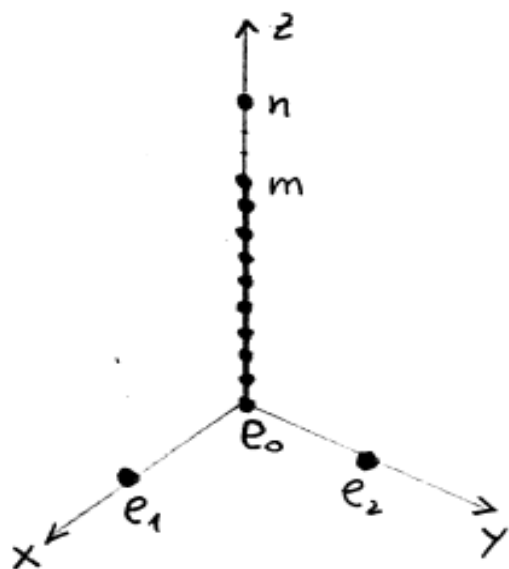
$$|\mathcal{K} + \mathcal{K}| < (d + 2)|\mathcal{K}| - \frac{1}{2}(d + 1)(d + 2),$$

then \mathcal{K} is contained in d parallel arithmetic progressions with the same common difference, having together no more than

$$v = |\mathcal{K} + \mathcal{K}| - d|\mathcal{K}| + \frac{1}{2}d(d + 1) \quad \text{terms.}$$

These results are best possible and cannot be sharpened by reducing the quantity v or by increasing the upper bounds for $|\mathcal{K} + \mathcal{K}|$:

EXAMPLE 6



$$d=3$$

$$K = \{e_0, e_3, 2e_3, \dots, me_3\} \cup \{ne_3\} \cup \{e_1, e_2\} \subseteq \mathbb{Z}^3$$

$$\begin{cases} |K| = m+4, & m+1 \leq n \leq 2m. \\ |2K| = 3m+n+9. \\ v = n+3 = |2K| - 3|K| + 6. \end{cases}$$

We found that a similar inequality can be formulated for d -dimensional sets that have a doubling coefficient less than

$$\rho_d = d + 2 - \frac{2}{s - d + 3}$$

(where $s \geq d$ is a positive integer). In this case we prove that \mathcal{K} lies on no more than s parallel lines.

These results can be used to make Freiman's Main Theorem more precise.

In a joint work with Freiman (2008) we study the *exact structure* of d -dimensional sets satisfying the small doubling property

$$|2K| < (d + 2 - \epsilon)|K|.$$

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