# Plünnecke's inequality for different summands 

Máté Matolcsi<br>(joint with K. Gyarmati and I. Z. Ruzsa)

Alfréd Rényi Institute of Mathematics, Budapest
matomate@renyi.hu
www.math.bme.hu/~matolcsi

Toronto, Fields Institute, 2008

## Overview

(1) Superadditivity and submultiplicativity of sumsets

- Superadditivity and related inequalitites
- Submultiplicativity and related inequalities
- Restricted sumsets, generalization of submultiplicativity

2) Plünnecke's inequality revisited

- Overview of Plünnecke-type inequalities
- Plünnecke's inequality for different summands
- Application: submultiplicativity for restricted sumsets

3 Follow-up's and open problems

## Superadditivity of sumsets I.

Let $A_{1}, A_{2}, \ldots A_{n}$ be finite sets of integers. How does the cardinality of the $n$-fold sumset $S=A_{1}+A_{2}+\cdots+A_{n}$ compare to the cardinalities of the $n-1$-fold sums $S_{i}=A_{1}+\cdots+A_{i-1}+A_{i+1}+\cdots+A_{n}$ ?

If all sets are equal, $A_{j}=A$, then $V$ sevolod Lev observed that the quantity $\frac{|k A|-1}{k}$ is increasing (notation: $A+A+\cdots+A=k A$ ). The first cases of this result assert that

$$
\begin{gathered}
|2 A| \geq 2|A|-1, \quad \text { and } \\
|3 A| \geq \frac{3}{2}|2 A|-\frac{1}{2} .
\end{gathered}
$$

Inequality (1) can be extended to different summands as

$$
|A+B| \geq|A|+|B|-1
$$

which also holds modulo a prime p, by Cauchy-Davenport:

$$
|A+B| \geq \min (|A|+|B|-1, p) .
$$

## Superadditivity of sumsets I.

Let $A_{1}, A_{2}, \ldots A_{n}$ be finite sets of integers. How does the cardinality of the $n$-fold sumset $S=A_{1}+A_{2}+\cdots+A_{n}$ compare to the cardinalities of the $n-1$-fold sums $S_{i}=A_{1}+\cdots+A_{i-1}+A_{i+1}+\cdots+A_{n}$ ?

If all sets are equal, $A_{j}=A$, then Vsevolod Lev observed that the quantity $\frac{|k A|-1}{k}$ is increasing (notation: $A+A+\cdots+A=k A$ ). The first cases of this result assert that

$$
\begin{gather*}
|2 A| \geq 2|A|-1, \quad \text { and }  \tag{1}\\
\quad|3 A| \geq \frac{3}{2}|2 A|-\frac{1}{2} \tag{2}
\end{gather*}
$$

Inequality (1) can be extended to different summands as

$$
|A+B| \geq|A|+|B|-1
$$

which also holds modulo a prime p, by Cauchy-Davenport:

$$
|A+B| \geq \min (|A|+|B|-1, p)
$$

## Superadditivity of sumsets I.

Let $A_{1}, A_{2}, \ldots A_{n}$ be finite sets of integers. How does the cardinality of the $n$-fold sumset $S=A_{1}+A_{2}+\cdots+A_{n}$ compare to the cardinalities of the $n-1$-fold sums $S_{i}=A_{1}+\cdots+A_{i-1}+A_{i+1}+\cdots+A_{n}$ ?

If all sets are equal, $A_{j}=A$, then Vsevolod Lev observed that the quantity $\frac{|k A|-1}{k}$ is increasing (notation: $A+A+\cdots+A=k A$ ). The first cases of this result assert that

$$
\begin{gather*}
|2 A| \geq 2|A|-1, \quad \text { and }  \tag{1}\\
\quad|3 A| \geq \frac{3}{2}|2 A|-\frac{1}{2} \tag{2}
\end{gather*}
$$

Inequality (1) can be extended to different summands as

$$
\begin{equation*}
|A+B| \geq|A|+|B|-1 \tag{3}
\end{equation*}
$$

which also holds modulo a prime p, by Cauchy-Davenport:

$$
|A+B| \geq \min (|\wedge|+|B|-1, p)
$$

## Superadditivity of sumsets I.

Let $A_{1}, A_{2}, \ldots A_{n}$ be finite sets of integers. How does the cardinality of the $n$-fold sumset $S=A_{1}+A_{2}+\cdots+A_{n}$ compare to the cardinalities of the $n-1$-fold sums $S_{i}=A_{1}+\cdots+A_{i-1}+A_{i+1}+\cdots+A_{n}$ ?

If all sets are equal, $A_{j}=A$, then Vsevolod Lev observed that the quantity $\frac{|k A|-1}{k}$ is increasing (notation: $A+A+\cdots+A=k A$ ). The first cases of this result assert that

$$
\begin{gather*}
|2 A| \geq 2|A|-1, \quad \text { and }  \tag{1}\\
\quad|3 A| \geq \frac{3}{2}|2 A|-\frac{1}{2} \tag{2}
\end{gather*}
$$

Inequality (1) can be extended to different summands as

$$
\begin{equation*}
|A+B| \geq|A|+|B|-1 \tag{3}
\end{equation*}
$$

which also holds modulo a prime p, by Cauchy-Davenport:

$$
\begin{equation*}
|A+B| \geq \min (|A|+|B|-1, p) \tag{4}
\end{equation*}
$$

## Superadditivity of sumsets II．

## Question

Do we have the superadditivity property for more than two summands，i．e．

$$
\begin{equation*}
|A+B+C| \geq \frac{|A+B|+|B+C|+|A+C|-1}{2} ? \tag{5}
\end{equation*}
$$

Do we have it modulo $p$ in some form，e．g．

$$
\begin{equation*}
|3 A| \geq \min \left(\frac{3}{2}|2 A|-\frac{1}{2}, p\right) \tag{6}
\end{equation*}
$$

[^0]
## Theorem

For $S=A_{1}+A_{2}+\cdots+A_{n}$ and $S_{i}=A_{1}+\cdots+A_{i-1}+A_{i+1}+\cdots+A_{n}$ we have

## Superadditivity of sumsets II．

## Question

Do we have the superadditivity property for more than two summands，i．e．

$$
\begin{equation*}
|A+B+C| \geq \frac{|A+B|+|B+C|+|A+C|-1}{2} ? \tag{5}
\end{equation*}
$$

Do we have it modulo $p$ in some form，e．g．

$$
\begin{equation*}
|3 A| \geq \min \left(\frac{3}{2}|2 A|-\frac{1}{2}, p\right) \tag{6}
\end{equation*}
$$

Lev noticed that（5）is true in the case when the sets have the same diameter．
It turns out that（6）is not true unless $|A|$ is very small compared to $p$（Gyarmati Konyagin，Ruzsa，2007），
However，（5）is true for arbitrary finite sets and an arbitrary number of summands：
Theerem
For $S=A_{1}+A_{2}+\cdots+A_{n}$ and $S_{i}=A_{1}+\cdots+A_{i-1}+A_{i+1}+\cdots+A_{n}$ we have

## Superadditivity of sumsets II.

## Question

Do we have the superadditivity property for more than two summands, i.e.

$$
\begin{equation*}
|A+B+C| \geq \frac{|A+B|+|B+C|+|A+C|-1}{2} ? \tag{5}
\end{equation*}
$$

Do we have it modulo $p$ in some form, e.g.

$$
\begin{equation*}
|3 A| \geq \min \left(\frac{3}{2}|2 A|-\frac{1}{2}, p\right) \tag{6}
\end{equation*}
$$

Lev noticed that (5) is true in the case when the sets have the same diameter. It turns out that (6) is not true unless $|A|$ is very small compared to $p$ (Gyarmati, Konyagin, Ruzsa, 2007).
However, (5) is true for arbitrary finite sets and an arbitrary number of summands:
Theorem
For $S=A_{1}+A_{2}+\cdots+A_{n}$ and $S_{i}=A_{1}+\cdots+A_{i-1}+A_{i+1}+\cdots+A_{n}$ we have

## Superadditivity of sumsets II.

## Question

Do we have the superadditivity property for more than two summands, i.e.

$$
\begin{equation*}
|A+B+C| \geq \frac{|A+B|+|B+C|+|A+C|-1}{2} ? \tag{5}
\end{equation*}
$$

Do we have it modulo $p$ in some form, e.g.

$$
\begin{equation*}
|3 A| \geq \min \left(\frac{3}{2}|2 A|-\frac{1}{2}, p\right) \tag{6}
\end{equation*}
$$

Lev noticed that (5) is true in the case when the sets have the same diameter. It turns out that (6) is not true unless $|A|$ is very small compared to $p$ (Gyarmati, Konyagin, Ruzsa, 2007).
However, (5) is true for arbitrary finite sets and an arbitrary number of summands:

## Theorem

For $S=A_{1}+A_{2}+\cdots+A_{n}$ and $S_{i}=A_{1}+\cdots+A_{i-1}+A_{i+1}+\cdots+A_{n}$ we have

$$
\begin{equation*}
(n-1)|S| \geq-1+\sum_{j=1}^{n}\left|S_{j}\right| \tag{7}
\end{equation*}
$$

## Proof of superadditivity

$S=A_{1}+A_{2}+\cdots+A_{n}, S_{i}=A_{1}+\cdots+A_{i-1}+A_{i+1}+\cdots+A_{n}$, and we want to prove $(n-1)|S| \geq-1+\sum_{j=1}^{n}\left|S_{j}\right|$.

We can assume that every $A_{i}$ starts with 0 (translation invariance). Let $a_{i}$ denote the largest element of $A_{i}$. Then $S \subset\left[0, a_{1}+\ldots a_{n}\right]$.

Make $n-1$ conies of the interval $S \subset\left[0 a_{1}+\ldots a_{n}\right]$ and in the ith copy mark the elements of the form $0+\left(A_{1}+A_{2}+\cdots+A_{n-i}+A_{n-i+2}+\cdots+A_{n}\right) \leq a_{1}+\cdots+a_{n-i}$, and $a_{n-i}+\left(A_{1}+\cdots+A_{n-i-1}+A_{n-i+1}+\cdots+A_{n}\right)_{>a_{1}+\ldots a_{n-i-1}}$

Let $M$ denote the set of marked elements. Then

$$
(n-1)|S| \geq|M|=\sum_{i=1}^{k}\left|S_{i}\right|-1
$$

and we are done.

## Proof of superadditivity

$S=A_{1}+A_{2}+\cdots+A_{n}, S_{i}=A_{1}+\cdots+A_{i-1}+A_{i+1}+\cdots+A_{n}$ ，and we want to prove $(n-1)|S| \geq-1+\sum_{j=1}^{n}\left|S_{j}\right|$ ．

We can assume that every $A_{i}$ starts with 0 （translation invariance）．Let $a_{i}$ denote the largest element of $A_{i}$ ．Then $S \subset\left[0, a_{1}+\ldots a_{n}\right]$ ．

Make $n-1$ copies of the interval $S \subset\left[0, a_{1}+\ldots a_{n}\right]$ and in the ith copy mark the elements of the form 0


Let $M$ denote the set of marked elements．Then

$$
(n-1)|S| \geq|M|=\sum_{i=1}^{k}\left|S_{i}\right|-1
$$

and we are done．

## Proof of superadditivity

$S=A_{1}+A_{2}+\cdots+A_{n}, S_{i}=A_{1}+\cdots+A_{i-1}+A_{i+1}+\cdots+A_{n}$, and we want to prove $(n-1)|S| \geq-1+\sum_{j=1}^{n}\left|S_{j}\right|$.

We can assume that every $A_{i}$ starts with 0 (translation invariance). Let $a_{i}$ denote the largest element of $A_{i}$. Then $S \subset\left[0, a_{1}+\ldots a_{n}\right]$.

Make $n-1$ copies of the interval $S \subset\left[0, a_{1}+\ldots a_{n}\right]$ and in the $i$ th copy mark the elements of the form $0+\left(A_{1}+A_{2}+\cdots+A_{n-i}+A_{n-i+2}+\cdots+A_{n}\right)_{\leq a_{1}+\cdots+a_{n-i}}$, and $a_{n-i}+\left(A_{1}+\cdots+A_{n-i-1}+A_{n-i+1}+\cdots+A_{n}\right)_{>a_{1}+\ldots a_{n-i-1}}$.

Let $M$ denote the set of marked elements. Then


## Proof of superadditivity

$S=A_{1}+A_{2}+\cdots+A_{n}, S_{i}=A_{1}+\cdots+A_{i-1}+A_{i+1}+\cdots+A_{n}$, and we want to prove $(n-1)|S| \geq-1+\sum_{j=1}^{n}\left|S_{j}\right|$.

We can assume that every $A_{i}$ starts with 0 (translation invariance). Let $a_{i}$ denote the largest element of $A_{i}$. Then $S \subset\left[0, a_{1}+\ldots a_{n}\right]$.

Make $n-1$ copies of the interval $S \subset\left[0, a_{1}+\ldots a_{n}\right]$ and in the ith copy mark the elements of the form $0+\left(A_{1}+A_{2}+\cdots+A_{n-i}+A_{n-i+2}+\cdots+A_{n}\right)_{\leq a_{1}+\cdots+a_{n-i}}$, and $a_{n-i}+\left(A_{1}+\cdots+A_{n-i-1}+A_{n-i+1}+\cdots+A_{n}\right)_{>a_{1}+\ldots a_{n-i-1}}$.
Let $M$ denote the set of marked elements. Then

$$
\begin{equation*}
(n-1)|S| \geq|M|=\sum_{i=1}^{k}\left|S_{i}\right|-1 \tag{8}
\end{equation*}
$$

and we are done.

Superadditivity and submultiplicativity of sumsets
Plünnecke's inequality revisited Follow-up's and open problems

## Submultiplicativity of sumsets I.

## Question

## Can we give an upper bound on $|S|$ in terms of $\left|S_{i}\right|$ ?

Let us start with a useful Projection Lemma:

## Lemma

Let $B \subset X_{1} \times \cdots \times X_{d}$ be a finite subset of a Cartesian product. Let
$B_{i} \subset X_{1} \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_{d}$ be the corresponding „projection" of $B$. Then

(9)

Proof: fairly straightforward induction on $d$.
This is not new. (Loomis-Whitney, 1949: $|K|^{d-1} \leq \prod_{i=1}^{d}\left|K_{i}\right|$ for any body $K \subset \mathbb{R}^{d}$; or
the stronger Box Theorem of Bollobás and Thomason: for every body K there exists a rectangular box such that all $d-k$-dimensional projections of the box have smaller area than those of the body.)

## Submultiplicativity of sumsets I.

## Question

Can we give an upper bound on $|S|$ in terms of $\left|S_{i}\right|$ ?
Let us start with a useful Projection Lemma:

## Lemma

Let $B \subset X_{1} \times \cdots \times X_{d}$ be a finite subset of a Cartesian product. Let $B_{i} \subset X_{1} \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_{d}$ be the corresponding „projection" of B. Then

$$
\begin{equation*}
|B|^{d-1} \leq \prod_{i=1}^{d}\left|B_{i}\right| \tag{9}
\end{equation*}
$$

[^1]
## Submultiplicativity of sumsets I.

## Question

Can we give an upper bound on $|S|$ in terms of $\left|S_{i}\right|$ ?
Let us start with a useful Projection Lemma:

## Lemma

Let $B \subset X_{1} \times \cdots \times X_{d}$ be a finite subset of a Cartesian product. Let $B_{i} \subset X_{1} \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_{d}$ be the corresponding „projection" of B. Then

$$
\begin{equation*}
|B|^{d-1} \leq \prod_{i=1}^{d}\left|B_{i}\right| \tag{9}
\end{equation*}
$$

Proof: fairly straightforward induction on $d$.

> This is not new. (Loomis-Whitney, 1949: $|K|^{d-1} \leq \prod_{i=1}^{d}\left|K_{i}\right|$ for any body $K \subset \mathbb{R}^{d}$; or the stronger Box Theorem of Bollobás and Thomason: for every body K there exists a rectangular box such that all $d-k$-dimensional projections of the box have smaller area than those of the body.)

## Submultiplicativity of sumsets I.

## Question

Can we give an upper bound on $|S|$ in terms of $\left|S_{i}\right|$ ?
Let us start with a useful Projection Lemma:

## Lemma

Let $B \subset X_{1} \times \cdots \times X_{d}$ be a finite subset of a Cartesian product. Let $B_{i} \subset X_{1} \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_{d}$ be the corresponding „projection" of B. Then

$$
\begin{equation*}
|B|^{d-1} \leq \prod_{i=1}^{d}\left|B_{i}\right| \tag{9}
\end{equation*}
$$

Proof: fairly straightforward induction on $d$.
This is not new. (Loomis-Whitney, 1949: $|K|^{d-1} \leq \prod_{i=1}^{d}\left|K_{i}\right|$ for any body $K \subset \mathbb{R}^{d}$; or the stronger Box Theorem of Bollobás and Thomason: for every body $K$ there exists a rectangular box such that all $d-k$-dimensional projections of the box have smaller area than those of the body.)

## Submultiplicativity of sumsets II.

Now, with the help of the Projection Lemma we prove:

## Theorem

For $S=A_{1}+A_{2}+\cdots+A_{n}, S_{i}=A_{1}+\cdots+A_{i-1}+A_{i+1}+\cdots+A_{n}$ we have

$$
\begin{equation*}
|S| \leq\left(\prod_{i=1}^{k}\left|S_{i}\right|\right)^{\frac{1}{k-1}} \tag{10}
\end{equation*}
$$

Outline of proof: List the elements of $A_{i}$ in some order. For each $s \in S$ let us consider the decomposition

$$
s=a_{1, i_{1}}+a_{2, i_{2}}+\cdots+a_{n, i_{n}},
$$

where the finite sequence $\left(i_{1}, i_{2}, \ldots, i_{n}\right)$, is minimal in lexicographical order.
Define a function $f: S \rightarrow A_{1} \times A_{2} \times \cdots \times A_{n}$, by

$$
\begin{equation*}
f(s)=\left(a_{1, i_{1}}, a_{2, i_{2}}, \ldots, a_{n, i_{n}}\right) . \tag{12}
\end{equation*}
$$

Then $|f(S)|=|S|$, and apply the projection lemma.

## Submultiplicativity of sumsets II.

Now, with the help of the Projection Lemma we prove:

## Theorem

For $S=A_{1}+A_{2}+\cdots+A_{n}, S_{i}=A_{1}+\cdots+A_{i-1}+A_{i+1}+\cdots+A_{n}$ we have

$$
\begin{equation*}
|S| \leq\left(\prod_{i=1}^{k}\left|S_{i}\right|\right)^{\frac{1}{k-1}} \tag{10}
\end{equation*}
$$

Outline of proof: List the elements of $A_{i}$ in some order. For each $s \in S$ let us consider the decomposition

$$
\begin{equation*}
s=a_{1, i_{1}}+a_{2, i_{2}}+\cdots+a_{n, i_{n}} \tag{11}
\end{equation*}
$$

where the finite sequence $\left(i_{1}, i_{2}, \ldots, i_{n}\right)$, is minimal in lexicographical order.
Define a function $f: S \rightarrow A_{1} \times A_{2} \times \cdots \times A_{n}$, by

$$
\begin{equation*}
f(s)=\left(a_{1, i_{1}}, a_{2, i_{2}}, \ldots, a_{n, i_{n}}\right) \tag{12}
\end{equation*}
$$

Then $|f(S)|=|S|$, and apply the projection lemma.

## Submultiplicativity of sumsets II.

Now, with the help of the Projection Lemma we prove:

## Theorem

For $S=A_{1}+A_{2}+\cdots+A_{n}, S_{i}=A_{1}+\cdots+A_{i-1}+A_{i+1}+\cdots+A_{n}$ we have

$$
\begin{equation*}
|S| \leq\left(\prod_{i=1}^{k}\left|S_{i}\right|\right)^{\frac{1}{k-1}} \tag{10}
\end{equation*}
$$

Outline of proof: List the elements of $A_{i}$ in some order. For each $s \in S$ let us consider the decomposition

$$
\begin{equation*}
s=a_{1, i_{1}}+a_{2, i_{2}}+\cdots+a_{n, i_{n}} \tag{11}
\end{equation*}
$$

where the finite sequence $\left(i_{1}, i_{2}, \ldots, i_{n}\right)$, is minimal in lexicographical order.
Define a function $f: S \rightarrow A_{1} \times A_{2} \times \cdots \times A_{n}$, by

$$
\begin{equation*}
f(s)=\left(a_{1, i_{1}}, a_{2, i_{2}}, \ldots, a_{n, i_{n}}\right) \tag{12}
\end{equation*}
$$

Then $|f(S)|=|S|$, and apply the projection lemma.

## Restricted sumsets

What if we restrict the addition of elements to a prescribed graph $G$ ？
Do we have $|A \stackrel{G}{+} A \stackrel{G}{+} A|^{2} \leq|A \stackrel{G}{+} A|^{3}$ ，
where the left hand side is understood as addition over triangles and the right hand
side is addition over edges of the graph．
No．There are easy counterexamples．．．However，we have：

## Theorem

For $\wedge, \boldsymbol{B}_{1}, \boldsymbol{B}_{2}$ and an arbitrary $S \subset B_{1}+B_{2}$ we have

$$
|S+A|^{2} \leq|S|\left|A+B_{1}\right|\left|A+B_{2}\right|
$$

The proof goes via a Plünnecke－type inequality，where we turn next．

## Restricted sumsets

What if we restrict the addition of elements to a prescribed graph $G$ ?
Do we have $|A \stackrel{G}{+} A+A|^{2} \leq|A \stackrel{G}{+} A|^{3}$, where the left hand side is understood as addition over triangles and the right hand side is addition over edges of the graph.

No. There are easy counterexamples... However, we have:

## Theorem

For $A, D_{1}, B_{2}$ and an arbitrary $S \subset B_{1}+B_{2}$ we have


The proof goes via a Plünnecke-type inequality, where we turn next.

## Restricted sumsets

What if we restrict the addition of elements to a prescribed graph $G$ ?
Do we have $|A \stackrel{G}{+} A+A|^{2} \leq|A \stackrel{G}{+} A|^{3}$, where the left hand side is understood as addition over triangles and the right hand side is addition over edges of the graph.

No. There are easy counterexamples... However, we have:

## Theorem

For $A, B_{1}, B_{2}$ and an arbitrary $S \subset B_{1}+B_{2}$ we have


The proof goes via a Plünnecke-type inequality, where we turn next.

## Restricted sumsets

What if we restrict the addition of elements to a prescribed graph $G$ ?
Do we have $|A \stackrel{G}{+} A+A|^{2} \leq|A \stackrel{G}{+} A|^{3}$, where the left hand side is understood as addition over triangles and the right hand side is addition over edges of the graph.

No. There are easy counterexamples... However, we have:

## Theorem

For $A, B_{1}, B_{2}$ and an arbitrary $S \subset B_{1}+B_{2}$ we have

$$
\begin{equation*}
|S+A|^{2} \leq|S|\left|A+B_{1}\right|\left|A+B_{2}\right| \tag{13}
\end{equation*}
$$

The proof goes via a Plünnecke-type inequality, where we turn next.

## Restricted sumsets

What if we restrict the addition of elements to a prescribed graph $G$ ?
Do we have $|A \stackrel{G}{+} A+A|^{2} \leq|A \stackrel{G}{+} A|^{3}$, where the left hand side is understood as addition over triangles and the right hand side is addition over edges of the graph.

No. There are easy counterexamples... However, we have:

## Theorem

For $A, B_{1}, B_{2}$ and an arbitrary $S \subset B_{1}+B_{2}$ we have

$$
\begin{equation*}
|S+A|^{2} \leq|S|\left|A+B_{1}\right|\left|A+B_{2}\right| \tag{13}
\end{equation*}
$$

The proof goes via a Plünnecke-type inequality, where we turn next.

## Plünnecke-type inequalities

## A summary of some Plünnecke-type inequalities:

Let $i<k$ be integers, $A, B$ sets in a commutative group and write $|A|=m$, $|A+i B|=\alpha m$. There is an $X \subset A, X \neq \emptyset$ such that

$$
|X+k B| \leq \alpha^{k / i}|X|
$$

A more general form reads as follows:
Let $A, B_{1}, \ldots, B_{h}$ be finite sets in a commutative group and write $|A|=m$, $\left|A+B_{i}\right|=\alpha_{i} m$, for $i \leq i \leq h$. There exists an $X \subset A, X \neq \emptyset$ such that

$$
\begin{equation*}
\left|X+B_{1}+\cdots+B_{h}\right| \leq \alpha_{1} \alpha_{2} \ldots \alpha_{h}|X| . \tag{15}
\end{equation*}
$$

It is sometimes also useful to know that $X$ is not only non-empty but also "large", i.e. $|A|=m$, and $\Pi\left|A+B_{i}\right|=s, B_{1}+\cdots+B_{h}=B$. For an arbitrary real number $0 \leq t<m$ there is an $X \subset A,|X|>t$ such that

$$
\begin{equation*}
|X+B| \leq \frac{s}{h-1}\left(\frac{1}{(m-t)^{h-1}}-\frac{1}{m^{h-1}}\right)+(|X|-t) \frac{s}{(m-t)^{h}} \tag{16}
\end{equation*}
$$

## Plünnecke-type inequalities

A summary of some Plünnecke-type inequalities:
Let $i<k$ be integers, $A, B$ sets in a commutative group and write $|A|=m$, $|A+i B|=\alpha m$. There is an $X \subset A, X \neq \emptyset$ such that

$$
\begin{equation*}
|X+k B| \leq \alpha^{k / i}|X| \tag{14}
\end{equation*}
$$

A more general form reads as follows:
Let $A, B_{1}, \ldots, B_{h}$ be finite sets in a commutative group and write $|A|=m$,
$\left|A+B_{j}\right|=\alpha_{i} m$, for $1 \leq i \leq h$. There exists an $X \subset A, X \neq \emptyset$ such that

$$
\left|X+B_{1}+\cdots+B_{h}\right| \leq \alpha_{1} \alpha_{2} \ldots \alpha_{h}|X|
$$

It is sometimes also useful to know that $X$ is not only non-empty but also "large ', i.e. $|A|=m$, and $\prod\left|A+B_{i}\right|=s, B_{1}+\cdots+B_{h}=B$. For an arbitrary real number $0 \leq t<m$ there is an $X \subset A,|X|>t$ such that


## Plünnecke-type inequalities

A summary of some Plünnecke-type inequalities:
Let $i<k$ be integers, $A, B$ sets in a commutative group and write $|A|=m$, $|A+i B|=\alpha m$. There is an $X \subset A, X \neq \emptyset$ such that

$$
\begin{equation*}
|X+k B| \leq \alpha^{k / i}|X| \tag{14}
\end{equation*}
$$

A more general form reads as follows:
Let $A, B_{1}, \ldots, B_{h}$ be finite sets in a commutative group and write $|A|=m$, $\left|A+B_{i}\right|=\alpha_{i} m$, for $1 \leq i \leq h$. There exists an $X \subset A, X \neq \emptyset$ such that

$$
\begin{equation*}
\left|X+B_{1}+\cdots+B_{h}\right| \leq \alpha_{1} \alpha_{2} \ldots \alpha_{h}|X| . \tag{15}
\end{equation*}
$$



## Plünnecke-type inequalities

A summary of some Plünnecke-type inequalities:
Let $i<k$ be integers, $A, B$ sets in a commutative group and write $|A|=m$, $|A+i B|=\alpha m$. There is an $X \subset A, X \neq \emptyset$ such that

$$
\begin{equation*}
|X+k B| \leq \alpha^{k / i}|X| \tag{14}
\end{equation*}
$$

A more general form reads as follows:
Let $A, B_{1}, \ldots, B_{h}$ be finite sets in a commutative group and write $|A|=m$, $\left|A+B_{i}\right|=\alpha_{i} m$, for $1 \leq i \leq h$. There exists an $X \subset A, X \neq \emptyset$ such that

$$
\begin{equation*}
\left|X+B_{1}+\cdots+B_{h}\right| \leq \alpha_{1} \alpha_{2} \ldots \alpha_{h}|X| \tag{15}
\end{equation*}
$$

It is sometimes also useful to know that $X$ is not only non-empty but also "large", i.e. $|A|=m$, and $\prod\left|A+B_{i}\right|=s, B_{1}+\cdots+B_{h}=B$. For an arbitrary real number $0 \leq t<m$ there is an $X \subset A,|X|>t$ such that

$$
\begin{equation*}
|X+B| \leq \frac{s}{h-1}\left(\frac{1}{(m-t)^{h-1}}-\frac{1}{m^{h-1}}\right)+(|X|-t) \frac{s}{(m-t)^{h}} . \tag{16}
\end{equation*}
$$

## Proof of restricted submultiplicativity I．

Now we prove

$$
\begin{equation*}
|S+A|^{2} \leq|S|\left|A+B_{1}\right|\left|A+B_{2}\right| \tag{17}
\end{equation*}
$$

for all $S \subset B_{1}+B_{2}$ ，with the help of the Plünnecke－type inequality above．
Notation $|A|=m,\left|A+B_{1}\right|\left|A+B_{2}\right|=s$ ．
If $|S|$ is small，$|S| \leq s / m^{2}$ ，then $|S+A| \leq|S||A| \leq \sqrt{s|S|}$ ，trivial．
If $S$ is large，$|S|>s / m^{2}$ ，then let $t=m-\sqrt{s /|S|}$ and find $X \subset A$ such that
$|X|=r>t$ and

$$
|S+X| \leq\left|B_{1}+B_{2}+X\right| \leq \text { small by Plunnecke }
$$

and

$$
|S+(A \backslash X)| \leq|S||A \backslash X| .
$$

We conclude that

$$
|S+A| \leq|S+X|+|S+(A \backslash X)| \leq 2 \sqrt{S|S|} .
$$

## Proof of restricted submultiplicativity I．

Now we prove

$$
\begin{equation*}
|S+A|^{2} \leq|S|\left|A+B_{1}\right|\left|A+B_{2}\right| \tag{17}
\end{equation*}
$$

for all $S \subset B_{1}+B_{2}$ ，with the help of the Plünnecke－type inequality above． Notation $|A|=m,\left|A+B_{1}\right|\left|A+B_{2}\right|=s$ ．

If $|S|$ is small，$|S| \leq s / m^{2}$ ，then $|S+A| \leq|S||A| \leq \sqrt{s|S|}$ ，trivial．
If $S$ is large，$|S|>s / m^{2}$ ，then let $t=m-\sqrt{s /|S|}$ and find $X \subset A$ such that $|X|=r>t$ and

$$
|S+X| \leq\left|B_{1}+B_{2}+X\right| \leq \text { small by Plunnecke }
$$

and

$$
|S+(A \backslash X)| \leq|S||A \backslash X| .
$$

We conclude that

$$
|S+A| \leq|S+X|+|S+(A \backslash X)| \leq 2 \sqrt{S|S|} .
$$

## Proof of restricted submultiplicativity I.

Now we prove

$$
\begin{equation*}
|S+A|^{2} \leq|S|\left|A+B_{1}\right|\left|A+B_{2}\right| \tag{17}
\end{equation*}
$$

for all $S \subset B_{1}+B_{2}$, with the help of the Plünnecke-type inequality above. Notation $|A|=m,\left|A+B_{1}\right|\left|A+B_{2}\right|=s$.

If $|S|$ is small, $|S| \leq s / m^{2}$, then $|S+A| \leq|S||A| \leq \sqrt{s|S|}$, trivial.
If $S$ is large, $|S|>s / m^{2}$, then let $t=m-\sqrt{s /|S|}$ and find $X \subset A$ such that
$|X|=r>t$ and

$$
|S+X| \leq\left|B_{1}+B_{2}+X\right| \leq \text { small by Plunnecke }
$$

and

$$
|S+(A \backslash X)| \leq|S||A \backslash X| .
$$

We conclude that

$$
|S+A| \leq|S+X|+|S+(A \backslash X)| \leq 2 \sqrt{S|S|} .
$$

## Proof of restricted submultiplicativity I.

Now we prove

$$
\begin{equation*}
|S+A|^{2} \leq|S|\left|A+B_{1}\right|\left|A+B_{2}\right| \tag{17}
\end{equation*}
$$

for all $S \subset B_{1}+B_{2}$, with the help of the Plünnecke-type inequality above. Notation $|A|=m,\left|A+B_{1}\right|\left|A+B_{2}\right|=s$.

If $|S|$ is small, $|S| \leq s / m^{2}$, then $|S+A| \leq|S||A| \leq \sqrt{s|S|}$, trivial.
If $S$ is large, $|S|>s / m^{2}$, then let $t=m-\sqrt{s /|S|}$ and find $X \subset A$ such that $|X|=r>t$ and

$$
|S+X| \leq\left|B_{1}+B_{2}+X\right| \leq \text { small by Plunnecke }
$$

and

$$
|S+(A \backslash X)| \leq|S||A \backslash X| .
$$

We conclude that

$$
|S+A| \leq|S+X|+|S+(A \backslash X)| \leq 2 \sqrt{S|S|} .
$$

## Proof of restricted submultiplicativity I.

Now we prove

$$
\begin{equation*}
|S+A|^{2} \leq|S|\left|A+B_{1}\right|\left|A+B_{2}\right| \tag{17}
\end{equation*}
$$

for all $S \subset B_{1}+B_{2}$, with the help of the Plünnecke-type inequality above. Notation $|A|=m,\left|A+B_{1}\right|\left|A+B_{2}\right|=s$.

If $|S|$ is small, $|S| \leq s / m^{2}$, then $|S+A| \leq|S||A| \leq \sqrt{s|S|}$, trivial.
If $S$ is large, $|S|>s / m^{2}$, then let $t=m-\sqrt{s /|S|}$ and find $X \subset A$ such that $|X|=r>t$ and

$$
\begin{equation*}
|S+X| \leq\left|B_{1}+B_{2}+X\right| \leq \text { small by Plunnecke } \tag{18}
\end{equation*}
$$

and

$$
\begin{equation*}
|S+(A \backslash X)| \leq|S||A \backslash X| . \tag{19}
\end{equation*}
$$

We conclude that

$$
\begin{equation*}
|S+A| \leq|S+X|+|S+(A \backslash X)| \leq 2 \sqrt{s|S|} \tag{20}
\end{equation*}
$$

## Proof of restricted submultiplicativity II.

So we obtained $|S+A| \leq|S+X|+|S+(A \backslash X)| \leq 2 \sqrt{s|S|}$.
We can dispose of the factor 2 by the method of exponentiation: $A^{\prime}=A^{k}, B_{1}^{\prime}=B_{1}^{k}$ $B_{2}^{\prime}=B_{2}^{k}$ and $S^{\prime}=S^{k}$ in the $k^{\prime}$ th direct power of the original group.

Then

$$
\left|S^{\prime}+A^{\prime}\right| \leq 2 \sqrt{s^{\prime}\left|S^{\prime}\right|} .
$$

Since $\left|S^{\prime}+A^{\prime}\right|=|S+A|^{k}, s^{\prime}=s^{k}$ and $\left|S^{\prime}\right|=|S|^{k}$, we get

$$
|S+A| \leq 2^{1 / k} \sqrt{S|S|}
$$

## Proof of restricted submultiplicativity II.

So we obtained $|S+A| \leq|S+X|+|S+(A \backslash X)| \leq 2 \sqrt{s|S|}$.
We can dispose of the factor 2 by the method of exponentiation: $A^{\prime}=A^{k}, B_{1}^{\prime}=B_{1}^{k}$, $B_{2}^{\prime}=B_{2}^{k}$ and $S^{\prime}=S^{k}$ in the $k^{\prime}$ th direct power of the original group.

$$
\left|S^{\prime}+A^{\prime}\right| \leq 2 \sqrt{s^{\prime}\left|S^{\prime}\right|} .
$$

Since $\left|S^{\prime}+A^{\prime}\right|=|S+A|^{k}, s^{\prime}=s^{k}$ and $\left|S^{\prime}\right|=|S|^{k}$, we get

$$
|S+A| \leq 2^{1 / k} \sqrt{S^{\prime} \mid}
$$

## Proof of restricted submultiplicativity II.

So we obtained $|S+A| \leq|S+X|+|S+(A \backslash X)| \leq 2 \sqrt{s|S|}$.
We can dispose of the factor 2 by the method of exponentiation: $A^{\prime}=A^{k}, B_{1}^{\prime}=B_{1}^{k}$, $B_{2}^{\prime}=B_{2}^{k}$ and $S^{\prime}=S^{k}$ in the $k^{\prime}$ th direct power of the original group.

Then

$$
\begin{equation*}
\left|S^{\prime}+A^{\prime}\right| \leq 2 \sqrt{s^{\prime}\left|S^{\prime}\right|} . \tag{21}
\end{equation*}
$$

Since $\left|S^{\prime}+A^{\prime}\right|=|S+A|^{k}, S^{\prime}=s^{k}$ and $\left|S^{\prime}\right|=|S|^{k}$, we get

$$
|S+A| \leq 2^{1 / k} \sqrt{s|S|} .
$$

## Proof of restricted submultiplicativity II.

So we obtained $|S+A| \leq|S+X|+|S+(A \backslash X)| \leq 2 \sqrt{s|S|}$.
We can dispose of the factor 2 by the method of exponentiation: $A^{\prime}=A^{k}, B_{1}^{\prime}=B_{1}^{k}$, $B_{2}^{\prime}=B_{2}^{k}$ and $S^{\prime}=S^{k}$ in the $k^{\prime}$ th direct power of the original group.

Then

$$
\begin{equation*}
\left|S^{\prime}+A^{\prime}\right| \leq 2 \sqrt{s^{\prime}\left|S^{\prime}\right|} . \tag{21}
\end{equation*}
$$

Since $\left|S^{\prime}+A^{\prime}\right|=|S+A|^{k}, s^{\prime}=s^{k}$ and $\left|S^{\prime}\right|=|S|^{k}$, we get

$$
\begin{equation*}
|S+A| \leq 2^{1 / k} \sqrt{s|S|} . \tag{22}
\end{equation*}
$$

## An arbitrary number of summands

Is it true for more than 3 summands? That is

## Question

For $A, B_{1}, \ldots B_{k}$ and $S \subset B_{1}+\cdots+B_{k}$, is it true that

$$
\begin{equation*}
|S+A|^{k} \leq|S| \prod_{i=1}^{k}\left|A+B_{1}+\cdots+B_{i-1}+B_{i+1}+\cdots+B_{k}\right| ? \tag{23}
\end{equation*}
$$

## Plünnecke＇s inequality for different summands $I$ ．

To follow the method of proof above we need a more general form of Plünnecke＇s inequality．

Theorem
Let $I<k$ be integers，and let $A, B_{1}, \ldots, B_{k}$ be finite sets in a commutative group $G$ ．Let $K=\{1,2, \ldots, k\}$ ，and for any $I \subset K$ put

（This is compatible with the previous notation if we identify a one－element subset of $K$ with its element．）Write

$$
\left(\prod_{L \subset K,|L|=1} \alpha_{L}\right)^{\prime}
$$

There exists an $X \subset A, X \neq \emptyset$ such that

（25）

## Plünnecke's inequality for different summands $I$.

To follow the method of proof above we need a more general form of Plünnecke's inequality.

## Theorem

Let $l<k$ be integers, and let $A, B_{1}, \ldots, B_{k}$ be finite sets in a commutative group $G$. Let $K=\{1,2, \ldots, k\}$, and for any $I \subset K$ put

$$
\begin{gathered}
B_{l}=\sum_{i \in I} B_{i} \\
|A|=m,\left|A+B_{l}\right|=\alpha_{l} m .
\end{gathered}
$$

(This is compatible with the previous notation if we identify a one-element subset of $K$ with its element.) Write

$$
\begin{equation*}
\beta=\left(\prod_{L \subset K,|L|=I} \alpha_{L}\right)^{(I-1)!(k-l)!/(k-1)!} \tag{24}
\end{equation*}
$$

There exists an $X \subset A, X \neq \emptyset$ such that

$$
\begin{equation*}
\left|X+B_{K}\right| \leq \beta|X| \tag{25}
\end{equation*}
$$

## Plünnecke's inequality for different summands II.

The outline of proof:
First let $k=I+1$, and $\left|A+B_{K \backslash\{i\}}\right|=\alpha_{i^{*}}$. Then there is an $X \subset A, X \neq \emptyset$ such that

$$
\begin{equation*}
\left|X+B_{K}\right| \leq c_{k} \beta|X| \tag{26}
\end{equation*}
$$

with a constant $c_{k}$ depending on $k$. This is done by reduction to the original Plünnecke:
Consider the groun $G^{\prime}=G \times H_{1} \times \cdots \times H_{k}$ where $H_{1} \quad . \quad H_{k}$, are cyclic grouns of order $n_{i}=\alpha_{i^{*}} q$, with some large integer $q$. (NOTE: we use here that $k=I+1$, so that $n_{i}$ can be defined.)

Let $B_{i}^{\prime}=B_{i} \times\{0\} \times \cdots \times\{0\} \times H_{i} \times\{0\} \times \cdots \times\{0\}$ and $B^{\prime}=\bigcup_{i=1}^{k} B_{i}^{\prime} \cdot$ We abuse notation: $A=A \times\{0\}$ $\times \cdots \times\{0\}$.

We can prove that

$$
\begin{equation*}
\left|A+(k-1) B^{\prime}\right| \leq 2 k m(\beta q)^{\prime} \tag{27}
\end{equation*}
$$

if $q$ is chosen large enough.
Then anply Plünnecke to the sets $A$ and $B^{\prime}$ in $G^{\prime}$. We conclude that there exists an $X \subset A$ such that

$$
\begin{equation*}
\left|X+B_{K}\right|=c_{k} \beta|X| . \tag{28}
\end{equation*}
$$

## Plünnecke's inequality for different summands II.

The outline of proof:
First let $k=I+1$, and $\left|A+B_{K \backslash\{i\}}\right|=\alpha_{i^{*}}$. Then there is an $X \subset A, X \neq \emptyset$ such that

$$
\begin{equation*}
\left|X+B_{K}\right| \leq c_{k} \beta|X| \tag{26}
\end{equation*}
$$

with a constant $c_{k}$ depending on $k$. This is done by reduction to the original Plünnecke:
Consider the group $G^{\prime}=G \times H_{1} \times \cdots \times H_{k}$, where $H_{1}, \ldots H_{k}$ are cyclic groups of order $n_{i}=\alpha_{i^{*}} q$, with some large integer $q$. (NOTE: we use here that $k=I+1$, so that $n_{i}$ can be defined.)

Let $B_{i}^{\prime}=B_{i} \times\{0\} \times \cdots \times\{0\} \times H_{i} \times\{0\} \times \cdots \times\{0\}$ and $B^{\prime}=\bigcup_{i=1}^{k} B_{i}^{\prime}$. We abuse notation: $A=A \times\{0\} \times \cdots \times\{0\}$.

## We can nrove that

$$
\left|A+(k-1) B^{\prime}\right| \leq 2 k m(\beta q)^{\prime}
$$

if $q$ is chosen large enough.
Then apply Plünnecke to the se ts $A$ and $B^{\prime}$ in $G^{\prime}$. We conclude that there exists an $X \subset A$ such that

$$
\left|X+B_{K}\right|=c_{K} \beta|X| .
$$

## Plünnecke's inequality for different summands II.

The outline of proof:
First let $k=I+1$, and $\left|A+B_{K \backslash\{i\}}\right|=\alpha_{i^{*}}$. Then there is an $X \subset A, X \neq \emptyset$ such that

$$
\begin{equation*}
\left|X+B_{K}\right| \leq c_{k} \beta|X| \tag{26}
\end{equation*}
$$

with a constant $c_{k}$ depending on $k$. This is done by reduction to the original Plünnecke:
Consider the group $G^{\prime}=G \times H_{1} \times \cdots \times H_{k}$, where $H_{1}, \ldots H_{k}$ are cyclic groups of order $n_{i}=\alpha_{i^{*}} q$, with some large integer $q$. (NOTE: we use here that $k=I+1$, so that $n_{i}$ can be defined.)


We can prove that

$$
\left|A+(k-1) B^{\prime}\right| \leq 2 k m(\beta q)^{\prime}
$$

if $q$ is chosen large enough.
Then apply Dlünnecke to the sets $A$ and $B^{\prime}$ in $G^{\prime}$. We conclude that there exists an $X \subset A$ such that


## Plünnecke's inequality for different summands II.

The outline of proof:
First let $k=I+1$, and $\left|A+B_{K \backslash\{i\}}\right|=\alpha_{i^{*}}$. Then there is an $X \subset A, X \neq \emptyset$ such that

$$
\begin{equation*}
\left|X+B_{K}\right| \leq c_{k} \beta|X| \tag{26}
\end{equation*}
$$

with a constant $c_{k}$ depending on $k$. This is done by reduction to the original Plünnecke:
Consider the group $G^{\prime}=G \times H_{1} \times \cdots \times H_{k}$, where $H_{1}, \ldots H_{k}$ are cyclic groups of order $n_{i}=\alpha_{i^{*}} q$, with some large integer $q$. (NOTE: we use here that $k=I+1$, so that $n_{i}$ can be defined.)

Let $B_{i}^{\prime}=B_{i} \times\{0\} \times \cdots \times\{0\} \times H_{i} \times\{0\} \times \cdots \times\{0\}$ and $B^{\prime}=\bigcup_{i=1}^{k} B_{i}^{\prime}$. We abuse notation: $A=A \times\{0\} \times \cdots \times\{0\}$.

We can prove that

$$
\left|A+(k-1) B^{\prime}\right| \leq 2 k m(\beta q)^{\prime}
$$

if $q$ is chosen large enough.
Then apply Plünnecke to the sets $A$ and $B^{\prime}$ in $G^{\prime}$. We conclude that there exists an $X \subset A$ such that

## Plünnecke's inequality for different summands II.

The outline of proof:
First let $k=I+1$, and $\left|A+B_{K \backslash\{i\}}\right|=\alpha_{i^{*}}$. Then there is an $X \subset A, X \neq \emptyset$ such that

$$
\begin{equation*}
\left|X+B_{K}\right| \leq c_{k} \beta|X| \tag{26}
\end{equation*}
$$

with a constant $c_{k}$ depending on $k$. This is done by reduction to the original Plünnecke:
Consider the group $G^{\prime}=G \times H_{1} \times \cdots \times H_{k}$, where $H_{1}, \ldots H_{k}$ are cyclic groups of order $n_{i}=\alpha_{i *} q$, with some large integer $q$. (NOTE: we use here that $k=I+1$, so that $n_{i}$ can be defined.)
Let $B_{i}^{\prime}=B_{i} \times\{0\} \times \cdots \times\{0\} \times H_{i} \times\{0\} \times \cdots \times\{0\}$ and $B^{\prime}=\bigcup_{i=1}^{k} B_{i}^{\prime}$. We abuse notation: $A=A \times\{0\} \times \cdots \times\{0\}$.

We can prove that

$$
\begin{equation*}
\left|A+(k-1) B^{\prime}\right| \leq 2 k m(\beta q)^{\prime} \tag{27}
\end{equation*}
$$

if $q$ is chosen large enough.
Then apply Plünnecke to the sets $A$ and $B^{\prime}$ in $G^{\prime}$. We conclude that there exists an $X \subset A$ such that

## Plünnecke's inequality for different summands II.

The outline of proof:
First let $k=I+1$, and $\left|A+B_{K \backslash\{i\}}\right|=\alpha_{i^{*}}$. Then there is an $X \subset A, X \neq \emptyset$ such that

$$
\begin{equation*}
\left|X+B_{K}\right| \leq c_{k} \beta|X| \tag{26}
\end{equation*}
$$

with a constant $c_{k}$ depending on $k$. This is done by reduction to the original Plünnecke:
Consider the group $G^{\prime}=G \times H_{1} \times \cdots \times H_{k}$, where $H_{1}, \ldots H_{k}$ are cyclic groups of order $n_{i}=\alpha_{i *} q$, with some large integer $q$. (NOTE: we use here that $k=I+1$, so that $n_{i}$ can be defined.)
Let $B_{i}^{\prime}=B_{i} \times\{0\} \times \cdots \times\{0\} \times H_{i} \times\{0\} \times \cdots \times\{0\}$ and $B^{\prime}=\bigcup_{i=1}^{k} B_{i}^{\prime}$. We abuse notation: $A=A \times\{0\} \times \cdots \times\{0\}$.

We can prove that

$$
\begin{equation*}
\left|A+(k-1) B^{\prime}\right| \leq 2 k m(\beta q)^{\prime} \tag{27}
\end{equation*}
$$

if $q$ is chosen large enough.
Then apply Plünnecke to the sets $A$ and $B^{\prime}$ in $G^{\prime}$. We conclude that there exists an $X \subset A$ such that

$$
\begin{equation*}
\left|X+B_{K}\right|=c_{k} \beta|X| . \tag{28}
\end{equation*}
$$

## Plünnecke's inequality for different summands III.

Still in the case $k=I+1$ we prove that $|X|$ can be chosen large, i.e. there exists $X \subset A$ such that
$|X|>(1-\varepsilon) m$, and

$$
\begin{equation*}
\left|X+B_{K}\right| \leq c(k, \varepsilon) \beta|X| \tag{29}
\end{equation*}
$$

Then the general case, $k=I+h$ follows by induction on $h$, and we get an $X \subset A$ such that $|X|>(1-\varepsilon) m$, and

$$
\left|X+B_{K}\right| \leq c(k, l, \varepsilon) \beta|X|
$$

Finally we remove the constant $c(k, I, \varepsilon)$ with the method of exponentiation, and obtain

$$
\left|X+B_{K}\right| \leq \beta|X|
$$

as desired.

## Plünnecke＇s inequality for different summands III．

Still in the case $k=I+1$ we prove that $|X|$ can be chosen large，i．e．there exists $X \subset A$ such that
$|X|>(1-\varepsilon) m$, and

$$
\begin{equation*}
\left|X+B_{K}\right| \leq c(k, \varepsilon) \beta|X| \tag{29}
\end{equation*}
$$

Then the general case，$k=I+h$ follows by induction on $h$ ，and we get an $X \subset A$ such that $|X|>(1-\varepsilon) m$ ，and

$$
\begin{equation*}
\left|X+B_{K}\right| \leq c(k, l, \varepsilon) \beta|X| \tag{30}
\end{equation*}
$$

Finally we remove the constant $c(k, I, \varepsilon)$ with the method of exponentiation，and obtain

$$
\left|X+B_{K}\right| \leq \beta|X|
$$

as desired．

## Plünnecke's inequality for different summands III.

Still in the case $k=I+1$ we prove that $|X|$ can be chosen large, i.e. there exists $X \subset A$ such that
$|X|>(1-\varepsilon) m$, and

$$
\begin{equation*}
\left|X+B_{K}\right| \leq c(k, \varepsilon) \beta|X| \tag{29}
\end{equation*}
$$

Then the general case, $k=I+h$ follows by induction on $h$, and we get an $X \subset A$ such that $|X|>(1-\varepsilon) m$, and

$$
\begin{equation*}
\left|X+B_{K}\right| \leq c(k, l, \varepsilon) \beta|X| \tag{30}
\end{equation*}
$$

Finally we remove the constant $c(k, l, \varepsilon)$ with the method of exponentiation, and obtain

$$
\begin{equation*}
\left|X+B_{K}\right| \leq \beta|X| \tag{31}
\end{equation*}
$$

as desired.

## Application: submultiplicativity for restricted sumsets

With this generalized Plünnecke inequality at hand we can prove the restricted version of submultiplicativity:

Theorem
For $A, B_{1}, \ldots B_{k}$ and $S \subset B_{1}+\cdots+B_{k}$, is it true that

## Application: submultiplicativity for restricted sumsets

With this generalized Plünnecke inequality at hand we can prove the restricted version of submultiplicativity:

## Theorem

For $A, B_{1}, \ldots B_{k}$ and $S \subset B_{1}+\cdots+B_{k}$, is it true that

$$
\begin{equation*}
|S+A|^{k} \leq|S| \prod_{i=1}^{k}\left|A+B_{1}+\cdots+B_{i-1}+B_{i+1}+\cdots+B_{k}\right| \tag{32}
\end{equation*}
$$

## Follow-up's and open problems I

## Bollobás and Balister prove that

## Theorem

If $\mathcal{A}$ is a uniform $k$-cover of $[0, \ldots, n]$ then

$$
\begin{gather*}
|S|^{k} \leq \prod_{A \in \mathcal{A}}\left|S_{A}\right| \quad \text { and }  \tag{33}\\
k(|S|-1) \geq \sum_{A \in \mathcal{A}}\left(\left|S_{A}\right|-1\right) \tag{34}
\end{gather*}
$$

## Also the Loomis-Whitney "projection theorem"

for any body $K \subset \mathbb{R}^{d}$ is equivalent to the following entropy inequality: if
$X=\left(X_{1}, X_{2}, \ldots X_{n}\right)$ is a sequence of $n$ random variables then

$$
(n-1) H(X) \leq \sum H\left(X_{[n] \backslash\{i\}}\right)
$$

## Follow-up's and open problems I

## Bollobás and Balister prove that

## Theorem

If $\mathcal{A}$ is a uniform $k$-cover of $[0, \ldots, n]$ then

$$
\begin{gather*}
|S|^{k} \leq \prod_{A \in \mathcal{A}}\left|S_{A}\right| \quad \text { and }  \tag{33}\\
k(|S|-1) \geq \sum_{A \in \mathcal{A}}\left(\left|S_{A}\right|-1\right) \tag{34}
\end{gather*}
$$

Also the Loomis-Whitney "projection theorem",

$$
\begin{equation*}
|K|^{d-1} \leq \prod_{i=1}^{d}\left|K_{i}\right| \tag{35}
\end{equation*}
$$

for any body $K \subset \mathbb{R}^{d}$ is equivalent to the following entropy inequality: if $X=\left(X_{1}, X_{2}, \ldots X_{n}\right)$ is a sequence of $n$ random variables then

$$
\begin{equation*}
(n-1) H(X) \leq \sum_{i} H\left(X_{[n] \backslash\{i\}}\right) \tag{36}
\end{equation*}
$$

## Follow-up's and open problems II

## Question

Is it true that for independent random variables we have
$(n-1) H\left(X_{1}+\ldots X_{n}\right) \leq \sum_{i} H\left(X_{1}+\cdots+X_{i-1}+X_{i+1}+\cdots+X_{n}\right)$ ?

Question
Let $A_{1}, \ldots, A_{k}$ be finite, nonempty sets in an arbitrary noncommutative group. Put
$S=A_{1}+\cdots+A_{k}, n_{i}=\max _{a \in A_{i}}\left|A_{1}+\cdots+A_{i-1}+a+A_{i+1}+\cdots+A_{k}\right|$. Is it true
that

Also, can we improve on the submultiplicativity result if the sets are equal? That is, e.g.
Question
Is it true that

$$
|3 A|^{2} \leq\left(\frac{2}{9}+\varepsilon\right)|2 A|^{3} ?
$$

## Follow-up's and open problems II

## Question

Is it true that for independent random variables we have
$(n-1) H\left(X_{1}+\ldots X_{n}\right) \leq \sum_{i} H\left(X_{1}+\cdots+X_{i-1}+X_{i+1}+\cdots+X_{n}\right)$ ?

## Question

Let $A_{1}, \ldots, A_{k}$ be finite, nonempty sets in an arbitrary noncommutative group. Put $S=A_{1}+\cdots+A_{k}, n_{i}=\max _{a \in A_{i}}\left|A_{1}+\cdots+A_{i-1}+a+A_{i+1}+\cdots+A_{k}\right|$. Is it true that

$$
\begin{equation*}
|S| \leq\left(\prod_{i=1}^{k} n_{i}\right)^{\frac{1}{k-1}} ? \tag{37}
\end{equation*}
$$

Also, can we improve on the submultiplicativity result if the sets are equal? That is, e.g.
Question
Is it true that

$$
|3 A|^{2} \leq\left(\frac{2}{9}+\varepsilon\right)|2 A|^{3} ?
$$

## Follow-up's and open problems II

## Question

Is it true that for independent random variables we have

$$
(n-1) H\left(X_{1}+\ldots X_{n}\right) \leq \sum_{i} H\left(X_{1}+\cdots+X_{i-1}+X_{i+1}+\cdots+X_{n}\right) ?
$$

## Question

Let $A_{1}, \ldots, A_{k}$ be finite, nonempty sets in an arbitrary noncommutative group. Put $S=A_{1}+\cdots+A_{k}, n_{i}=\max _{a \in A_{i}}\left|A_{1}+\cdots+A_{i-1}+a+A_{i+1}+\cdots+A_{k}\right|$. Is it true that

$$
\begin{equation*}
|S| \leq\left(\prod_{i=1}^{k} n_{i}\right)^{\frac{1}{k-1}} ? \tag{37}
\end{equation*}
$$

Also, can we improve on the submultiplicativity result if the sets are equal? That is, e.g.

## Question

Is it true that

$$
\begin{equation*}
|3 A|^{2} \leq\left(\frac{2}{9}+\varepsilon\right)|2 A|^{3} ? \tag{38}
\end{equation*}
$$

## Question

```
Is the Kakeya conjecture true?
```


[^0]:    Lev noticed that（5）is true in the case when the sets have the same diameter． It turns out that（6）is not true unless $|A|$ is very small compared to $p$（Gyarmati， Konyagin，Ruzsa，2007）．
    However，（5）is true for arbitrary finnite seis and an arbitrary number of summands

[^1]:    Proof: fairly straightforward induction on $d$.
    This is not new. (Loomis-Whitney, 1949: $|K|^{d-1} \leq \prod_{i=1}^{d}\left|K_{i}\right|$ for any body $K \subset \mathbb{R}^{d}$; or the stronger Box Theorem of Bollobás and Thomason: for every body $K$ there exists a rectangular box such that all $d-k$-dimensional projections of the box have smaller area than those of the body.)

