Plünnecke's inequality for different summands

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Overview

Superadditivity of sumsets I.

Let $A_1, A_2, \dots A_n$ be finite sets of integers. How does the cardinality of the n-fold sumset $S = A_1 + A_2 + \cdots + A_n$ compare to the cardinalities of the $n-1$-fold sums $S_i = A_1 + \cdots + A_{i-1} + A_{i+1} + \cdots + A_n$?

If all sets are equal, $A_j = A$, then Vsevolod Lev observed that the quantity $\frac{|kA|-1}{k}$ is increasing (notation: $A + A + \cdots + A = kA$). The first cases of this result assert that

$$|2A| \geq 2|A| - 1, \quad \text{and} \tag{1}$$

$$|3A| \geq \frac{3}{2}|2A| - \frac{1}{2}. \tag{2}$$

Inequality (1) can be extended to different summands as

$$|A + B| \geq |A| + |B| - 1, \tag{3}$$

which also holds modulo a prime p, by Cauchy-Davenport:

$$|A + B| \geq \min(|A| + |B| - 1, p). \tag{4}$$

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Superadditivity of sumsets II.

Question

Do we have the superadditivity property for more than two summands, i.e.

$$|A+B+C| \geq \frac{|A+B|+|B+C|+|A+C|-1}{2}? \quad (5)$$

Do we have it modulo p in some form, e.g.

$$|3A| \geq \min\left(\frac{3}{2}|2A| - \frac{1}{2}, p\right) \quad (6)$$

Lev noticed that (5) is true in the case when the sets have the same diameter.
It turns out that (6) **is not true** unless $|A|$ is very small compared to p (Gyarmati, Konyagin, Ruzsa, 2007).
However, (5) **is true** for arbitrary finite sets and an arbitrary number of summands:

Theorem

For $S = A_1 + A_2 + \cdots + A_n$ and $S_i = A_1 + \cdots + A_{i-1} + A_{i+1} + \cdots + A_n$ we have

$$(n-1)|S| \geq -1 + \sum_{j=1}^{n} |S_j|. \quad (7)$$

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Proof of superadditivity

$S = A_1 + A_2 + \cdots + A_n$, $S_i = A_1 + \cdots + A_{i-1} + A_{i+1} + \cdots + A_n$, and we want to prove $(n-1)|S| \geq -1 + \sum_{j=1}^{n} |S_j|$.

We can assume that every A_i starts with 0 (translation invariance). Let a_i denote the largest element of A_i. Then $S \subset [0, a_1 + \ldots a_n]$.

Make $n-1$ copies of the interval $S \subset [0, a_1 + \ldots a_n]$ and in the ith copy mark the elements of the form $0 + (A_1 + A_2 + \cdots + A_{n-i} + A_{n-i+2} + \cdots + A_n)_{\leq a_1 + \cdots + a_{n-i}}$, and $a_{n-i} + (A_1 + \cdots + A_{n-i-1} + A_{n-i+1} + \cdots + A_n)_{> a_1 + \ldots a_{n-i-1}}$.

Let M denote the set of marked elements. Then

$$(n-1)|S| \geq |M| = \sum_{i=1}^{k} |S_i| - 1 \quad (8)$$

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Submultiplicativity of sumsets I.

Question

Can we give an upper bound on $|S|$ in terms of $|S_i|$?

Let us start with a useful Projection Lemma:

Lemma

Let $B \subset X_1 \times \cdots \times X_d$ be a finite subset of a Cartesian product. Let $B_i \subset X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_d$ be the corresponding „projection" of B. Then

$$|B|^{d-1} \leq \prod_{i=1}^{d} |B_i| . \qquad (9)$$

Proof: fairly straightforward induction on d.

This is not new. (Loomis-Whitney, 1949: $|K|^{d-1} \leq \prod_{i=1}^{d} |K_i|$ for any body $K \subset \mathbb{R}^d$; or the stronger Box Theorem of Bollobás and Thomason: for every body K there exists a rectangular box such that all $d-k$-dimensional projections of the box have smaller area than those of the body.)

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Now, with the help of the Projection Lemma we prove:

Theorem

For $S = A_1 + A_2 + \cdots + A_n$, $S_i = A_1 + \cdots + A_{i-1} + A_{i+1} + \cdots + A_n$ we have

$$|S| \leq \left(\prod_{i=1}^{k} |S_i| \right)^{\frac{1}{k-1}}. \tag{10}$$

Outline of proof: List the elements of A_i in some order. For each $s \in S$ let us consider the decomposition

$$s = a_{1,i_1} + a_{2,i_2} + \cdots + a_{n,i_n}, \tag{11}$$

where the finite sequence $(i_1, i_2, \ldots, i_n)$, is minimal in lexicographical order.

Define a function $f : S \to A_1 \times A_2 \times \cdots \times A_n$, by

$$f(s) = (a_{1,i_1}, a_{2,i_2}, \ldots, a_{n,i_n}). \tag{12}$$

Then $|f(S)| = |S|$, and apply the projection lemma.

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Restricted sumsets

What if we restrict the addition of elements to a prescribed graph G?

Do we have $|A \overset{G}{+} A \overset{G}{+} A|^2 \leq |A \overset{G}{+} A|^3$,
where the left hand side is understood as addition over triangles and the right hand side is addition over edges of the graph.

No. There are easy counterexamples... However, we have:

Theorem

For A, B_1, B_2 and an arbitrary $S \subset B_1 + B_2$ we have

$$|S + A|^2 \leq |S||A + B_1||A + B_2| \tag{13}$$

The proof goes via a Plünnecke-type inequality, where we turn next.

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Plünnecke-type inequalities

A summary of some Plünnecke-type inequalities:
Let $i < k$ be integers, A, B sets in a commutative group and write $|A| = m$, $|A + iB| = \alpha m$. There is an $X \subset A$, $X \neq \emptyset$ such that

$$|X + kB| \leq \alpha^{k/i}|X|. \tag{14}$$

A more general form reads as follows:
Let $A, B_1, \dots, B_h$ be finite sets in a commutative group and write $|A| = m$, $|A + B_i| = \alpha_i m$, for $1 \leq i \leq h$. There exists an $X \subset A$, $X \neq \emptyset$ such that

$$|X + B_1 + \cdots + B_h| \leq \alpha_1\alpha_2 \dots \alpha_h|X|. \tag{15}$$

It is sometimes also useful to know that X is not only non-empty but also "large", i.e. $|A| = m$, and $\prod |A + B_i| = s$, $B_1 + \cdots + B_h = B$. For an arbitrary real number $0 \leq t < m$ there is an $X \subset A$, $|X| > t$ such that

$$|X + B| \leq \frac{s}{h-1}\left(\frac{1}{(m-t)^{h-1}} - \frac{1}{m^{h-1}}\right) + (|X| - t)\frac{s}{(m-t)^h}. \tag{16}$$

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$$|X + kB| \leq \alpha^{k/i}|X|. \tag{14}$$

A more general form reads as follows:
Let $A, B_1, \dots, B_h$ be finite sets in a commutative group and write $|A| = m$, $|A + B_i| = \alpha_i m$, for $1 \leq i \leq h$. There exists an $X \subset A$, $X \neq \emptyset$ such that

$$|X + B_1 + \dots + B_h| \leq \alpha_1\alpha_2 \dots \alpha_h|X|. \tag{15}$$

It is sometimes also useful to know that X is not only non-empty but also "large", i.e. $|A| = m$, and $\prod |A + B_i| = s$, $B_1 + \dots + B_h = B$. For an arbitrary real number $0 \leq t < m$ there is an $X \subset A$, $|X| > t$ such that

$$|X + B| \leq \frac{s}{h-1}\left(\frac{1}{(m-t)^{h-1}} - \frac{1}{m^{h-1}}\right) + (|X| - t)\frac{s}{(m-t)^h}. \tag{16}$$

Proof of restricted submultiplicativity I.

Now we prove

$$|S+A|^2 \le |S||A+B_1||A+B_2| \tag{17}$$

for all $S \subset B_1 + B_2$, with the help of the Plünnecke-type inequality above.
Notation $|A| = m$, $|A+B_1||A+B_2| = s$.

If $|S|$ is small, $|S| \le s/m^2$, then $|S+A| \le |S||A| \le \sqrt{s|S|}$, trivial.

If S is large, $|S| > s/m^2$, then let $t = m - \sqrt{s/|S|}$ and find $X \subset A$ such that $|X| = r > t$ and

$$|S+X| \le |B_1+B_2+X| \le \textit{small by Plunnecke} \tag{18}$$

and

$$|S+(A\setminus X)| \le |S||A\setminus X|. \tag{19}$$

We conclude that

$$|S+A| \le |S+X| + |S+(A\setminus X)| \le 2\sqrt{s|S|}. \tag{20}$$

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So we obtained $|S+A| \le |S+X| + |S+(A\setminus X)| \le 2\sqrt{s|S|}$.

We can dispose of the factor 2 by the method of exponentiation: $A' = A^k$, $B_1' = B_1^k$, $B_2' = B_2^k$ and $S' = S^k$ in the k'th direct power of the original group.

Then

$$|S'+A'| \le 2\sqrt{s'|S'|}. \qquad (21)$$

Since $|S'+A'| = |S+A|^k$, $s' = s^k$ and $|S'| = |S|^k$, we get

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An arbitrary number of summands

Is it true for more than 3 summands? That is

Question

For $A, B_1, \dots B_k$ and $S \subset B_1 + \cdots + B_k$, is it true that

$$|S+A|^k \leq |S| \prod_{i=1}^{k} |A + B_1 + \cdots + B_{i-1} + B_{i+1} + \cdots + B_k| \; ? \tag{23}$$

Plünnecke's inequality for different summands I.

To follow the method of proof above we need a more general form of Plünnecke's inequality.

Theorem

Let $l < k$ be integers, and let $A, B_1, \dots, B_k$ be finite sets in a commutative group G. Let $K = \{1, 2, \dots, k\}$, and for any $I \subset K$ put

$$B_I = \sum_{i \in I} B_i,$$

$$|A| = m, \quad |A + B_I| = \alpha_I m.$$

(This is compatible with the previous notation if we identify a one-element subset of K with its element.) Write

$$\beta = \left(\prod_{L \subset K, |L| = l} \alpha_L \right)^{(l-1)!(k-l)!/(k-1)!} . \tag{24}$$

There exists an $X \subset A$, $X \neq \emptyset$ such that

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The outline of proof:

First let $k = l + 1$, and $|A + B_{K\setminus\{i\}}| = \alpha_{i^*}$. Then there is an $X \subset A$, $X \neq \emptyset$ such that

$$|X + B_K| \leq c_k \beta |X| \tag{26}$$

with a constant c_k depending on k. This is done by reduction to the original Plünnecke:

Consider the group $G' = G \times H_1 \times \cdots \times H_k$, where $H_1, \ldots H_k$ are cyclic groups of order $n_i = \alpha_{i^*} q$, with some large integer q. (NOTE: we use here that $k = l + 1$, so that n_i can be defined.)

Let $B_i' = B_i \times \{0\} \times \cdots \times \{0\} \times H_i \times \{0\} \times \cdots \times \{0\}$ and $B' = \bigcup_{i=1}^k B_i'$. We abuse notation: $A = A \times \{0\} \times \cdots \times \{0\}$.

We can prove that

$$|A + (k-1)B'| \leq 2km(\beta q)^l \tag{27}$$

if q is chosen large enough.

Then apply Plünnecke to the sets A and B' in G'. We conclude that there exists an $X \subset A$ such that

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Still in the case $k = l + 1$ we prove that $|X|$ can be chosen large, i.e. there exists $X \subset A$ such that
$|X| > (1 - \varepsilon)m$, and

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Then the general case, $k = l + h$ follows by induction on h, and we get an $X \subset A$ such that $|X| > (1 - \varepsilon)m$, and

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Finally we remove the constant $c(k, l, \varepsilon)$ with the method of exponentiation, and obtain

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Application: submultiplicativity for restricted sumsets

With this generalized Plünnecke inequality at hand we can prove the restricted version of submultiplicativity:

Theorem

For $A, B_1, \ldots B_k$ and $S \subset B_1 + \cdots + B_k$, is it true that

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Follow-up's and open problems I

Bollobás and Balister prove that

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If $\mathcal{A}$ is a uniform k-cover of $[0, \ldots, n]$ then

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Also the Loomis-Whitney "projection theorem",

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for any body $K \subset \mathbb{R}^d$ is equivalent to the following entropy inequality: if $X = (X_1, X_2, \ldots X_n)$ is a sequence of n random variables then

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Is it true that for independent random variables we have $(n-1)H(X_1 + \dots X_n) \leq \sum_i H(X_1 + \cdots + X_{i-1} + X_{i+1} + \cdots + X_n)$?

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Let $A_1, \dots, A_k$ *be finite, nonempty sets in an arbitrary noncommutative group. Put* $S = A_1 + \cdots + A_k$, $n_i = \max_{a \in A_i} |A_1 + \cdots + A_{i-1} + a + A_{i+1} + \cdots + A_k|$. *Is it true that*

$$|S| \leq \left(\prod_{i=1}^{k} n_i \right)^{\frac{1}{k-1}} ? \tag{37}$$

Also, can we improve on the submultiplicativity result if the sets are equal? That is, e.g.

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Is it true that

$$|3A|^2 \leq (\frac{2}{9} + \varepsilon)|2A|^3 ? \tag{38}$$

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Is the Kakeya conjecture true?