Some Ramsey properties of the n-cube

(with Joszef Solymosi)

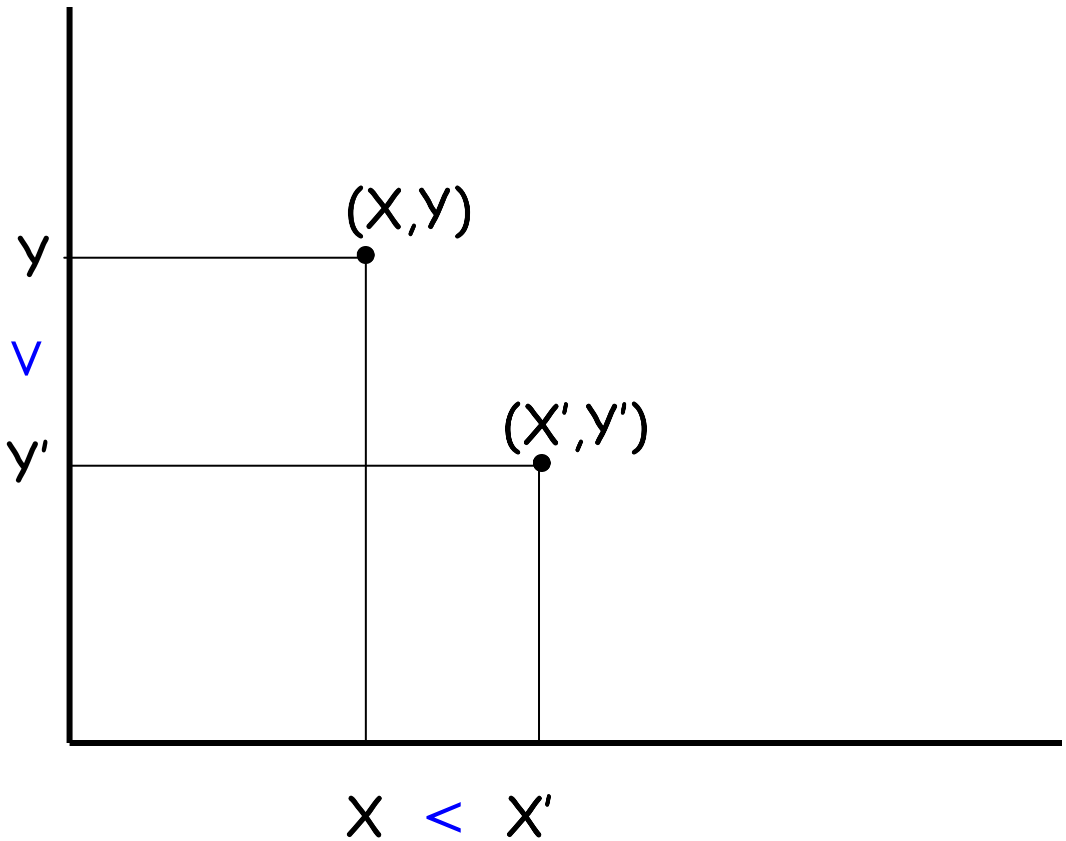
Some definitions.

$$\{0,1\}^n := \{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) : \varepsilon_i = 0 \text{ or } 1\}$$

$$D(n) := \{0,1\}^n \times \{0,1\}^n$$

(Can think of these as vertices and generalized diagonals of an n-cube).

Represent $(X, Y) \in D(n)$ schematically by :



$X < X'$ means $w(X) < w(X')$ where $w(X) = \#$ of 1's in X.

With $[n]:=\{1, 2,\ldots,n\}$, for $I \subseteq [n]$, we say that

$X = (x_1,\ldots,x_n)$ and $Y = (y_1,\ldots,y_n)$ are I-complementary

if $x_i = y_i$ iff $i \notin I$.

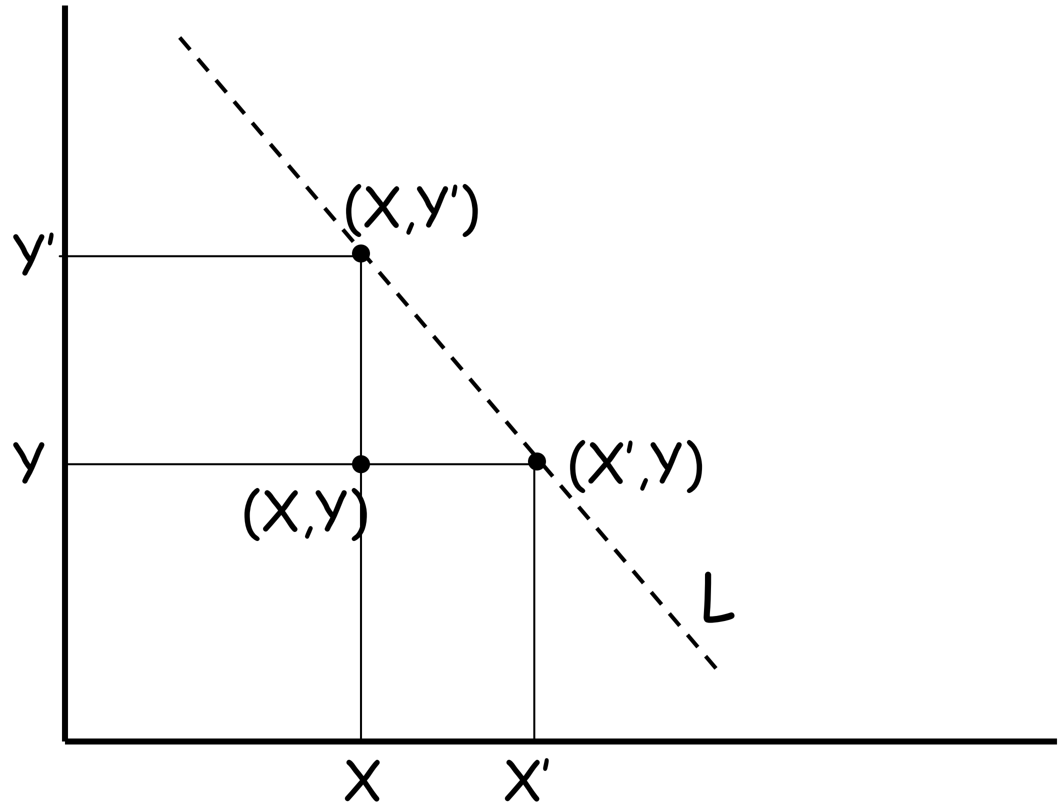
Define the line $L(I):=\{(X,Y)$: X and Y are I-complementary$\}$

Thus, $|L(I)| = 2^{|I|}$. We say that $L(I)$ has dimension $|I|$.

Fact: Every point (X,Y) in $D(n)$ lies on a unique line $L(I)$.

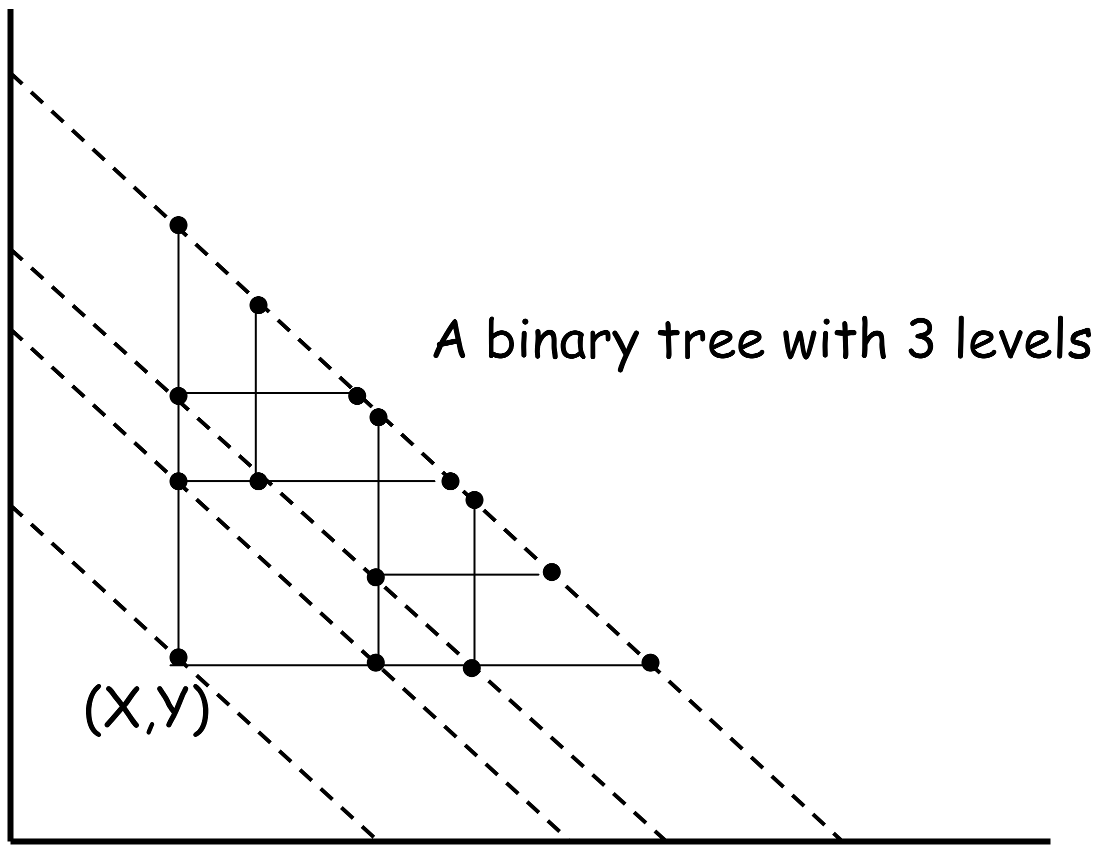
(Just take $I = \{i \in [n] : x_i \neq y_i\}$).

By a "corner" in D(n) we mean a set of 3 points of the form (X,Y), (X',Y), (X,Y') where (X,Y') and (X',Y) are on a common line L.



We can think of a corner as binary tree with one level and root (X,Y).

More generally, a binary tree $B(m)$ with m levels and root (X,Y) is defined recursively by joining the root to two binary trees with m-1 levels. All of the 2^k points at level k are required to be on the same line.



Theorem. For all r and m, there is an $n_0 = n_0(r,m)$ such that if $n \geq n_0$ and the points of D(n) are arbitrarily r-colored, then there is always a monochromatic binary tree B(m) with m levels formed.

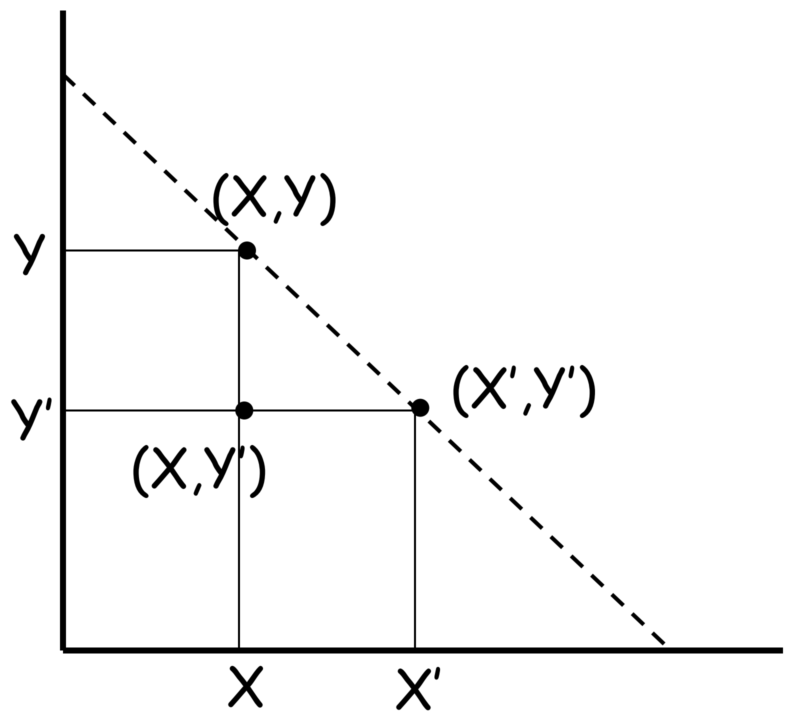
(In fact, we will show that we can take $n_0(r,m) = c \cdot 6^{rm}$).

Sketch of proof: Let n be large (to be specified later) and suppose that D(n) is r-colored. Consider the 2^n points on the line $L_0 = L([n])$.

Let $S_0 \subseteq L_0$ be the set of points having the "most popular" color. Thus, $|S_0| \geq \frac{2^n}{r}$.

Consider the "grid" G_1 (≈ Cartesian product) defined by:

$$G_1 = \{(X, Y') : (X, Y) \in S_0, (X', Y') \in S_0, \text{with } X < X'\}.$$



Thus,

$$|G_1| \geq \binom{S_0}{2} > \frac{1}{4}|S_0|^2 \geq \frac{1}{4r^2} \cdot 4^n := a_1 4^n.$$

Let us call a line L of dimension t small if $t < n/3$

and deficient if $|L \cap G_1| \leq a_1 2^t$.

Thus, the total number of points on small or deficient lines is at most

$$\sum_{t < n/3} 2^t \binom{n}{t} \cdot 2^{n-t} + \sum_{t \geq n/3} a_1 2^t \binom{n}{t} \cdot 2^{n-t}$$

But

$$\sum_{t<n/3} 2^t \binom{n}{t} \cdot 2^{n-t} + \sum_{t\geq n/3} a_1 2^t \binom{n}{t} \cdot 2^{n-t}$$

$$= 4^n - 2^n(1-a_1)(2^n - \sum_{t<n/3} \binom{n}{t})$$

$$\leq 4^n - 2^n(1-a_1)(2^n - 1.96^n)$$

$$(\text{since} \sum_{t<n/3} \binom{n}{t} < 2^n(\tfrac{3}{4}e^{\frac{n}{4}})^{2n/3} < 1.96^n)$$

$$\leq \frac{a_1}{2} 4^n$$

provided $a_1 \geq 2 \cdot (.98^n)$.

Thus, if we discard these points, we still have at least $\frac{a_1}{2}4^n$ points remaining in G_1, and all these points are on "good" lines, i.e., not small and not deficient.

Let $L_1 = L(I_1)$ be such a good line, say of dimension $|I_1| = n_1 \geq n/3$.

In particular,

$$|L_1 \cap G_1| \geq a_1 2^{n_1}.$$

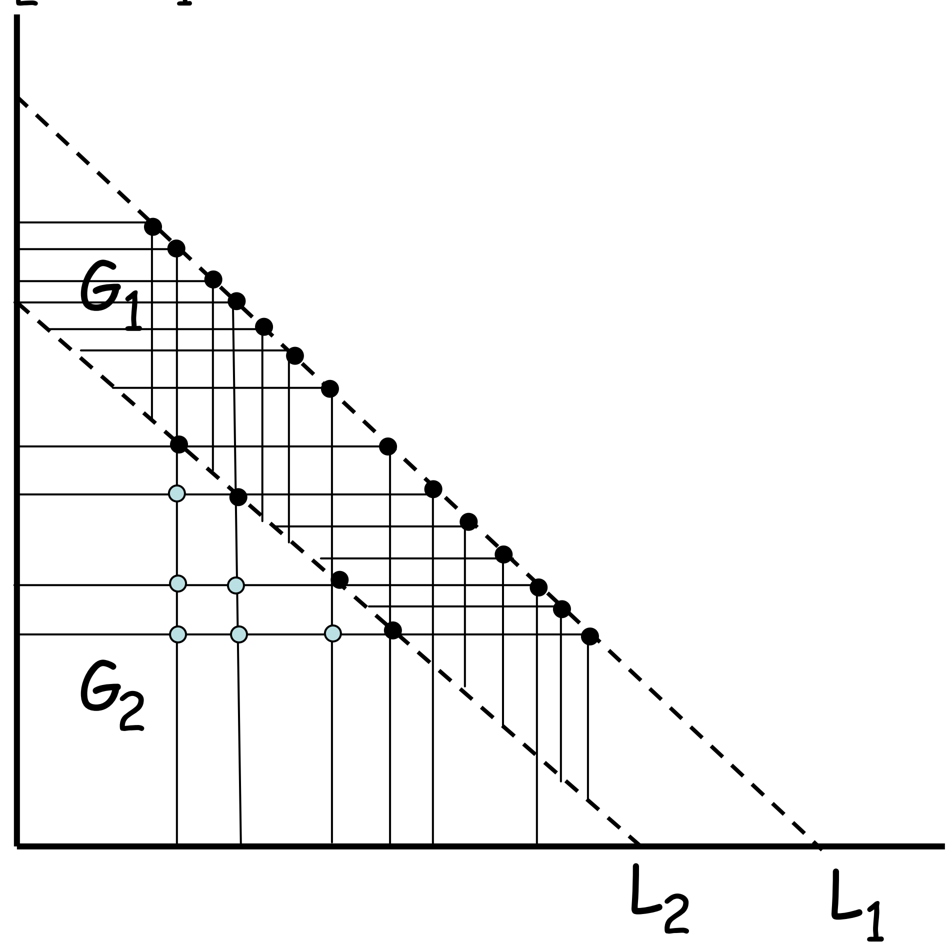
Let S_1 be the set of points of $L_1 \cap G_1$ with the most popular color c_1. Therefore,

$$|S_1| \geq \frac{a_1}{r} 2^{n_1}.$$

Now, let G_2 be the "grid" formed by S_1, i.e.,

$$G_2 = \{(X, Y') : (X, Y) \in S_1, (X', Y') \in S_1, \text{ with } X < X'\}.$$

Observe that $G_2 \subset G_1$.



Now, let G_2 be the "grid" formed by S_1, i.e.,

$$G_2 = \{(X, Y') : (X, Y) \in S_1, (X', Y') \in S_1, \text{ with } X < X'\}.$$

Observe that $G_2 \subset G_1$.

Therefore, we have

$$|G_2| \geq \binom{|S_1|}{2} \geq \left(\tfrac{a_1}{2r}\right)^2 4^n := a_2 4^n.$$

As before, let us classify a line L of dimension t small if $t < n_1/3$ and deficient if $|L \cap G_2| \le a_2 2^t$.

A similar calculation as before shows that if we remove from G_2 all the points on small or deficient lines, then at least $\frac{a_2}{2} 4^{n_1}$ points will remain in G_2, provided $a_2 \ge 2 \cdot (.98^{n_1})$.

Now take some "good" line $L_2 = L(I_2)$ with $I_2 \subset I_1$, with dimension $|I_2| = n_2 \ge n_1/3$ and with $|L_2 \cap G_2| \ge a_2 2^{n_2}$.

Let $S_2 \subseteq L_2 \cap G_2$ have the most popular color c_2 so that

$$|S_2| \geq \frac{a_2}{r} 2^{n_2}.$$

Then, with G_3 defined to be the "grid" formed by S_2, we have $|G_3| \geq (\frac{a_2}{2r})^2 4^{n_2}$, and so on.

We continue this process for rm steps.

In general, we define

$$a_{i+1} = (\frac{a_i}{2r})^2, 1 \leq i \leq rm - 1, \text{ with } a_1 = \frac{1}{4r^2}.$$

Also, we have $n_{i+1} \geq n_i/3$. In addition, we need to have

$$a_i \geq 2 \cdot (.98^{n_i}) \text{ for all } i.$$

In particular, this implies that in general

$$a_k = \frac{1}{2^{3\cdot 2^k - 2} r^{2^{k+1}-2}}$$

It is now straightforward to check that all the required inequalities are satisfied by choosing $n \geq n_0(r,m) = c \cdot 6^{rm}$ for some absolute constant c.

Hence, there must be m indices $i_1 < i_2 < \dots < i_m$ such that all the sets S_{i_k} have the same color c_0.

These m sets S_{i_k} contain the desired binary tree B(m). ■

Interpretation.

$$Q_n := \{(x_1, x_2, \ldots x_n) : x_i = 0 \text{ or } 1,\ 1 \le i \le n\} \qquad \text{n-cube}$$

$L_n :=$ set of all $\binom{2^n}{2}$ line segments joining two vertices of Q_n

L_n is the set of diagonals of Q_n

$$\{\mathbf{x}, \overline{\mathbf{x}}\} = \{(x_1, \ldots, x_n), (\overline{x}_1, \ldots, \overline{x}_n)\}, \qquad \overline{x}_i = 1 - x_i$$

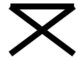
is a **main** diagonal of Q_n

An (affine) k-cube of Q_n is a subset of 2^k points of the form $\{(y_1,\ldots,y_n) : y_i = 0 \text{ or } 1 \text{ iff } i \in I\}$ for some k-subset $I \subseteq [n] := \{1,2,\ldots,n\}$.

(Thus, for $i \notin I$, the coordinate y_i is fixed).

We will say that three connected diagonals of the form $\{\mathbf{x},\mathbf{y}\}$, $\{\mathbf{y},\mathbf{z}\}$, $\{\mathbf{z},\mathbf{w}\}$ form a self-crossing path, denoted by $\overline{\times}$, if $\{\mathbf{x},\mathbf{y}\}$ and $\{\mathbf{z},\mathbf{w}\}$ are both main diagonals of the same subcube.

Corollary

In any r-coloring of the edges in L_n, there is always a monochromatic  , provided $n > c \cdot 6^r$.

The same argument should work for any subgraph G of (Q_n, L_n), provided that G has enough edges and for any pair of crossing main diagonals, G has all the edges between the pairs endpoints.

Another application

(Partial Hales-Jewett lines)

For every r there is an $n_0 = n_0(r)$ with the following property. In any r-coloring of $\{0,1,2,3\}^n$, with $n > n_0$, there is always a monochromatic set of 3 points of the form

$$(\ldots,a,\ldots 0,\ldots,b,\ldots,3,\ldots,0,\ldots,c,\ldots,3,\ldots,d,\ldots)$$
$$(\ldots,a,\ldots 1,\ldots,b,\ldots,2,\ldots,1,\ldots,c,\ldots,2,\ldots,d,\ldots)$$
$$(\ldots,a,\ldots 2,\ldots,b,\ldots,1,\ldots,2,\ldots,c,\ldots,1,\ldots,d,\ldots)$$

In other words, every column is either constant, increasing from 0, or decreasing from 3.

Idea of proof:

To each point $(x_1, x_2, \ldots, x_n)$ in $\{0,1,2,3\}^n$, we associate the point $((a_1, a_2, \ldots, a_n),(b_1, b_2, \ldots, b_n))$ in $\{0,1\}^n \times \{0,1\}^n$ by the following rule:

$x_k \longleftrightarrow$	a_k	b_k
0	0	0
1	0	1
2	1	0
3	1	1

Then it is not hard to verify that a monochromatic corner in D(n) corresponds to a monochromatic partial Hales-Jewett line.

For example, the "corner"

(1 1 0 0 0 1 1 0, 1 1 0 1 1 0 0 1)

(1 1 0 0 0 1 1 0, 1 1 0 1 0 1 0 0)

(1 1 0 0 1 0 1 1, 1 1 0 1 0 1 0 0)

For example, the "corner"

(3 3 0 1 1 2 2 1) corresponds to

(1 1 0 0 0 1 1 0, 1 1 0 1 1 0 0 1)

(1 1 0 0 0 1 1 0, 1 1 0 1 0 1 0 0) (1 1 0 0 1 0 1 1, 1 1 0 1 0 1 0 0)

(3 3 0 1 0 3 2 0) (3 3 0 1 2 1 2 2)

i.e.,

(3 3 0 1 0 3 2 0)
(3 3 0 1 1 2 2 1)
(3 3 0 1 2 1 2 2)

Observe that if we associate the point

$$(a_1, a_2, \ldots, a_i, \ldots, a_n)$$

with the integer $\prod_i p_i^{a_i}$

where p_i denotes the i^{th} prime, then the points

$$(\ldots, a, \ldots 0, \ldots, b, \ldots, 3, \ldots, 0, \ldots, c, \ldots, 3, \ldots, d, \ldots)$$
$$(\ldots, a, \ldots 1, \ldots, b, \ldots, 2, \ldots, 1, \ldots, c, \ldots, 2, \ldots, d, \ldots)$$
$$(\ldots, a, \ldots 2, \ldots, b, \ldots, 1, \ldots, 2, \ldots, c, \ldots, 1, \ldots, d, \ldots)$$

correspond to a 3-term *geometric* progression.

By mapping points $(a_1, a_2, \dots, a_i, \dots, a_n) \in \{0,1,2,3\}^n$

to points $(b_1, b_2, \dots, b_i, \dots, b_n) \in F_3^n$ by:

$a_i = 0 \text{ or } 3 \Rightarrow b_i = 0, \quad a_i = 1 \Rightarrow b_i = 1, \quad a_i = 2 \Rightarrow b_i = 2$

we obtain:

Theorem: If the points of F_3^n are colored with c log n colors then there is always a monochromatic affine line formed.

$$
\begin{aligned}
&(\dots,a,\dots 0,\dots,b,\dots,0,\dots,0,\dots,c,\dots,0,\dots,d,\dots)\\
&(\dots,a,\dots 1,\dots,b,\dots,2,\dots,1,\dots,c,\dots.,2,\dots,d,\dots)\\
&(\dots,a,\dots 2,\dots,b,\dots,1,\dots,2,\dots,c,\dots.,1,\dots,d,\dots)
\end{aligned}
$$

The same techniques can be used to prove the following:

Theorem. For every r, there exist $\delta = \delta(r)$ and $n_0 = n_0(r)$ with the following property:

If A and B are sets of real numbers with $|A| = |B| = n \geq n_0$ and $|A + B| \leq n^{1+\delta}$, then any r-coloring of A x B contains a monochromatic "corner", i.e., a set of 3 points of the form (a,b), (a',b), (a,b').

(Can choose $\delta = \frac{1}{2^{2^{2r}}}$).

By iterating these techniques, one can also show that with the same hypotheses on A and B (with appropriate $\delta = \delta(r,m)$ and $n_0(r,m) = n_0$), if $A \times B$ is r-colored then each set contains a monochromatic translate of a fairly large "Hilbert cube", i.e., sets of the form

$$H_m(a,a_1,\ldots,a_m\} = \{a + \sum_{1\le i\le m} \varepsilon_i a_i\} \subset A, \quad H_m(b,a_1,\ldots,a_m\} = \{b + \sum_{1\le i\le m} \varepsilon_i a_i\} \subset B$$

where $m = \Omega(\log\log(n))$ and $\varepsilon_i = 0$ or 1, $1 \le i \le m$.

Some questions.

Can we "complete the square" for some of these results?

For example, one can use these techniques to show that if the points of $[N] \times [N]$ are colored with fewer than $c \log \log N$ colors, then there is always a monochromatic "corner" formed, i.e., 3 points (a,b), (a',b), (a,b') with $a' + b = a + b'$. (By projection, this gives a 3-AP).

Is it the case that with these bounds (or even better ones), we can guarantee the 4^{th} point (a',b') to be monochromatic, as well ?

Similarly, if the diagonals on an n-cube are r-colored, with $r < c \log \log n$, is it true that a monochromatic ⧖ must be formed ?

What about a monochromatic □ ?

What are the density analogs of these results? (Shkredov)

We do know something about $\boxtimes$.

(the 6 diagonals spanned by 4 coplanar vertices of an n-cube).

For example, there is an N_0 so that if all the diagonals of an N-cube are 2-colored with $N \geq N_0$, then a monochromatic $\boxtimes$ must always be formed.

An estimate for N_0 .

<u>Arrow notation</u>

$$3\uparrow n := 3^n$$

$$3\uparrow\uparrow n := 3\uparrow(3\uparrow(3\uparrow \ldots(3\uparrow 3)\ldots)) = 3^{3^{3^{\cdot^{\cdot^{\cdot^{3^{3}}}}}}} \text{ (n 3's)}$$

$$3\uparrow\uparrow\uparrow n := 3\uparrow\uparrow(3\uparrow\uparrow(3\uparrow\uparrow \ldots (3\uparrow\uparrow 3)\ldots)), \text{ etc.}$$

(n 3's)

For example:

$$3\uparrow 3 = 3^3 = 27$$

$$3\uparrow\uparrow 3 = 3\uparrow(3\uparrow 3) = 3^{27} = 7625597484987$$

$$3\uparrow\uparrow\uparrow 3 = 3\uparrow\uparrow(3\uparrow\uparrow 3) = 3\uparrow\uparrow 7625597484987$$

$$= \underbrace{3^{3^{3^{\cdot^{\cdot^{\cdot^{3^{3^{\cdot^{\cdot^{\cdot^{3^{3}}}}}}}}}}}}}_{7625597484987\ \text{3's}}$$

Each additional arrow makes a very big difference!

$$N_0 \le \left.\begin{array}{c} 3\uparrow\uparrow\uparrow\uparrow 3 \\ 3\overbrace{\uparrow\uparrow\uparrow\ldots\uparrow\uparrow\uparrow}3 \\ 3\overbrace{\uparrow\uparrow\uparrow\uparrow\ldots\ldots\ldots\uparrow\uparrow\uparrow\uparrow}3 \\ \vdots \\ 3\overbrace{\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\ldots\ldots\ldots\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow}3 \end{array}\right] \text{64 levels}$$

It has just been shown that $N_0 \geq 11$.

There is clearly room for improvement here !

$$N_0 \leq \left.\begin{matrix} 3 \uparrow\uparrow\uparrow\uparrow 3 \\ \overbrace{3\uparrow\uparrow\uparrow \ldots \uparrow\uparrow\uparrow 3} \\ \overbrace{3\uparrow\uparrow\uparrow\uparrow \ldots\ldots\ldots \uparrow\uparrow\uparrow\uparrow 3} \\ \vdots \\ \overbrace{3\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow \ldots\ldots\ldots \uparrow\uparrow\uparrow\uparrow\uparrow\uparrow 3} \end{matrix}\right] \text{64 levels}$$

It has just been shown that $N_0 \geq 11$.

There is clearly room for improvement here !