Szemerédi's theorem, Hardy sequences, and nilmanifolds

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Universal patterns

$\Lambda \subset \mathbb{N}$ has positive density if

$$d(\Lambda) = \lim_{N \to \infty} \frac{|\Lambda \cap [0, N]|}{N} > 0.$$

General question: What kind of patterns occur within every $\Lambda \subset \mathbb{N}$ with $d(\Lambda) > 0$?

We restrict to shift invariant patterns: So we are equally happy to find $\{n_1, \ldots, n_k\}$ or $\{m + n_1, \ldots, m + n_k\}$ within Λ.

General problem: If $a_1(n), \ldots, a_k(n)\colon \mathbb{N} \to \mathbb{Z}$ determine if every $\Lambda \subset \mathbb{N}$ with $d(\Lambda) > 0$ contains a pattern of the form

$$m,\ m + a_1(n), \ldots,\ m + a_k(n)$$

for some $m, n \in \mathbb{N}$.

Examples of universal patterns

Szemerédi (75): If $d(\Lambda) > 0$ then Λ contains arbitrarily long arithmetic progressions:

$$m,\ m+n, \ldots, m+kn.$$

Bergelson and Leibman (96): $p_1, \ldots, p_k \in \mathbb{Z}[t]$ with $p_i(0) = 0$. If $d(\Lambda) > 0$ then Λ contains patterns of the form:

$$m,\ m+p_1(n), \ldots, m+p_k(n).$$

This gives patterns of the form

$$m,\ m+n^2, \ldots, m+kn^2,$$

and

$$m,\ m+n,\ m+n^2, \ldots, m+n^k.$$

Proof uses ergodic theory, relies heavily on techniques developed by Furstenberg.

Arithmetic progressions

Problem: Let $a(n)\colon \mathbb{N} \to \mathbb{Z}$. Determine whether every $\Lambda \subset \mathbb{N}$ with $d(\Lambda) > 0$ contains AP's of the form

$$m,\ m+a(n),\ldots,\ m+ka(n).$$

Examples: $a(n) = n,\ n^2,\ n^2 - 1$.

Non-examples: $a(n) = 2n+1,\ n^2+1,\ n!$.

Obvious obstruction: $a(n) \equiv 0 \pmod r$ must have solution for every $r \in \mathbb{N}$. Take $\Lambda = r\mathbb{Z}$.

Less obvious obstruction: $a(n)$ cannot be lacunary. If it is, then for some $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ we have $\|a(n)\alpha\|$ stays away from zero. Take $\Lambda = \{n \in \mathbb{N}\colon \|n\alpha\| \leq \varepsilon\}$ for ε small enough.

New (non-polynomial) examples

What can we say about

$$a(n) = [n^{\sqrt{2}}] \text{ ?}$$

Fr., Wierdl (08): If $a \in \mathbb{R}$ is positive, then every $\Lambda \subset \mathbb{N}$ with positive density contains APs of the form

$$m, m + [n^a], \ldots, m + k[n^a].$$

Proof uses ergodic theory, but traditional machinery of Furstenberg does not apply.

Why? We do not see how to show the corresponding coloristic result directly.

Logarithmico-exponential functions

We prove a much more general result involving the logarithmico-exponential functions of Hardy:

$$\mathcal{LE} = \{\text{functions builded using } +, -, \times, /, \text{ the functions } c, e^x, \log x \text{ and composition.}\}$$

Examples:

$$x^a (= e^{a \log x}),\ \ x \log x,\ \ x^2 / \log x,$$

$$x^{2008} e^{-\sqrt{\log x}} + \sqrt{x^{\sqrt{2}} + 1}.$$

New examples in $\mathcal{LE}$

$a(x) \prec b(x)$ if $a(x)/b(x) \to 0$ as $x \to +\infty$.

Fr., Wierdl (08): Suppose that $a \in \mathcal{LE}$ satisfies $x^d \prec a(x) \prec x^{d+1}$ for some $d \in \mathbb{N}$. Then every $\Lambda \subset \mathbb{N}$ with positive density contains APs of the form

$$m, \; m + [a(n)], \ldots, \; m + k[a(n)].$$

So the following sequences work

$$[n^{\sqrt{2}}], \; [n \log n], \; [n^2/\log n],$$

$$\left[n^{2008} e^{-\sqrt{\log n}} + \sqrt{n^{\sqrt{2}} + 1} \right].$$

New examples in $\mathcal{H}$

Hardy field: A subfield of the field of (germs) of continuous functions, that is closed under differentiation.

$\mathcal{H}$: The set of functions that belong to some Hardy field.

The set $\mathcal{LE}$ in our main result can be replaced by $\mathcal{H}$.

This allows us to deal with expressions involving (for example) the functions

$$\Gamma(x),\ \zeta(x),\ \mathrm{Li}(x).$$

The following sequences are also good for Szemerédi's theorem

$$[\log(n!)],\ [n^{\sqrt{5}}\zeta(n)],\ [\mathrm{Li}(n^{3/2})].$$

Ideas in the proof

Key observation: The range of $[a(n)]$ contains lots of long polynomial progressions.

Example: If $a \in \mathcal{LE}$ and $x \prec a(x) \prec x^2$, then for every $m \in \mathbb{N}$ it contains APs of the form

$$\{c_m + mn, 1 \leq n \leq N_m\}, \quad \text{with} \quad N_m \to \infty.$$

Crucial property of $\mathcal{LE}$ used: It is closed under differentiation and its elements have finitely many zeros.

This forces a lot of nice properties on the derivatives of its elements, which enable us to estimate various exponential sums.

Correspondence principle

A measure preserving system is a finite measure space $(X, \mathcal{B}, \mu)$ together with a transformation T that satisfies $\mu(T^{-1}A) = \mu(A)$ for every $A \in \mathcal{B}$.

Furstenberg (77): If $\Lambda \subset \mathbb{N}$ has positive density, then there exists a mps $(X, \mathcal{B}, \mu, T)$ and $A \in \mathcal{B}$ with $\mu(A) > 0$ such that

$$d\Big(\Lambda \cap (\Lambda + n_1) \cap \cdots \cap (\Lambda + n_r)\Big) \geq \mu(A \cap T^{n_1}A \cap \cdots \cap T^{n_r}A)$$

for every $n_1, \ldots, n_r \in \mathbb{N}$.

Intuitively: $X \leftrightarrow \mathbb{N}$, $\mathcal{B} \leftrightarrow 2^{\mathbb{N}}$, $\mu \leftrightarrow d$, $A \leftrightarrow \Lambda$, $T \leftrightarrow T_{sh}$.

Translation to ergodic theory

For convenience: $d = 1$, $k = 2$. Suffices to show the following multiple recurrence property:

If $a \in \mathcal{LE}$ satisfies $x \prec a(x) \prec x^2$, then for all mps $(X, \mathcal{B}, \mu, T)$ and $A \in \mathcal{B}$ with $\mu(A) > 0$

$$\mu(A \cap T^{[a(n)]} A \cap T^{2[a(n)]} A) > 0 \quad (*)$$

for some $n \in \mathbb{N}$.

Host & Kra, Leibman (05): If $a(n)$ is polynomial, suffices to check $(*)$ for a special class of systems called nilsystems.

Nilsystem: A mps that is induced by a left translation on $X = G/\Gamma$ where G is k-step nilpotent group and Γ is a subgroup so that G/Γ is compact.

Reduction to nilsystems

$[a(n)]$ is not polynomial, but contains a lot of polynomial progressions:

$$\bigcup_{m\in\mathbb{N}} \{c_m+mn, 1 \leq n \leq N_m\} \subset \{[a(n)]\colon n \in \mathbb{N}\}$$

Enables us to reduce things to nilsystems.

But checking that

$$\mu(A \cap T^{c_m+mn}A \cap T^{2(c_m+mn)}A) > 0,$$

for nilsystems is not easy.

Dealing with nilsystems

We can compare the average (in m, n) of

$$\mu(A \cap T^{c_m+mn} A \cap T^{2(c_m+mn)} A)$$

with the average (in n) of

$$\mu(A \cap T^n A \cap T^{2n} A)$$

which we know is positive.

Key ingredient: If $X = G/\Gamma$ is a nilmanifold and $a \in G$ is "irrational", then the sequence

$$\{a^{c_m+mn}\Gamma,\ 1 \leq n \leq N_m, m \in \mathbb{N}\}$$

is equidistributed on X.

Need a quantitative equidistribution result of Green & Tao (07) to do this.

Criterion for 2-term APs

Problem: Find a useful criterion for checking that a sequence $a(n)$ of integers is good for Szemerédi's theorem.

Kamae, Mendés-France (80): Suppose

(i) $\{a(n)\alpha\}$ is equidistributed in $\mathbb{T}$ for all irrational α.

(ii) For every $r \in \mathbb{N}$ the set $\{n \in \mathbb{N}\colon r | a(n)\}$ has positive density.

Then every $\Lambda \subset \mathbb{N}$ with $d(\Lambda) > 0$ contains patterns

$$m, \ m + a(n).$$

Applicable to $a(n) = n^2, \ p_n - 1, \ [n^{\sqrt{2}}]$.

But life is tougher for 3-term or longer APs.

Conjecture for k-term APs

Stronger equidistribution properties on nilmanifolds needed.

Irrational nilrotation on X: $b \in G$ such that $(b^{rn}\Gamma)_{n\in\mathbb{N}}$ is equidistributed for every $r \in \mathbb{N}$.

Conjecture (Fr, Lesigne, Wierdl): Suppose that

(i) $(b^{a(n)}\Gamma)_{n\in\mathbb{N}}$ is equidistributed for every irrational b on a k-step nilmanifold X.

(ii) For every $r \in \mathbb{N}$ the set $\{n \in \mathbb{N}\colon r|a(n)\}$ has positive density.

Then every $\Lambda \subset \mathbb{N}$ with $d(\Lambda) > 0$ contains APs of the form

$$m,\, m + a(n), \ldots,\, m + ka(n).$$

True if the density of the set $\{a(1), a(2), \ldots\}$ is positive.

Open problems

Problem 1: Show that every $\Lambda \subset \mathbb{N}$ with positive density contains APs of the form

$$m,\, m+[p^{\sqrt{2}}], \ldots, m+k[p^{\sqrt{2}}],$$

where p is a prime number.

Problem 2: Let $a \in \mathcal{LE}$ have at most polynomial growth and for $p \in \mathbb{R}[t]$ we have $|a(x) - p(x)| \to \infty$. Then every $\Lambda \subset \mathbb{N}$ with positive density contains APs of the form

$$m,\, m+[a(n)], \ldots, m+k[a(n)].$$

Problem 3: Show that the primes contain the configurations of Problem 2.