Monotonic Sequence Games

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The Game on an Interval

The Game on the Rationals

Eine Kleine Game Theory

Open Questions

Outline

Let

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and

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Ex. If $m = 2$ and $n = 3$ then $mn + 1 = 7$. A permutation in S_7, please!!

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Theorem

The winner of the game on $[mn+1]$ where $m \leq n$ is:

$m \backslash n$	0	1	2	3	4	5	6	7
0	A	A	A	A	A	A	A	A
1		B	A	B	A	B	A	B
2			A	A	A	A	A	A
3				A	A	A	A	?
4					A	?	?	?

where the patterns continue in each of the first 3 rows. ■

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If x_i is placed in column j of I,

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If x_i is placed in column j of I, and in column k of D

$$\begin{aligned} j &= \textit{length of a longest increasing subsequence ending at } x_i, \\ k &= \textit{length of a longest decreasing subsequence ending at } x_i. \end{aligned}$$

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$$

Build a *combined list C* by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{llll} x_i \in I & \text{and} \quad x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\ x_i \notin I & \text{and} \quad x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\ x_i \in I & \text{and} \quad x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } P. \end{array}$$

Build a *combined list C* by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{llll} x_i \in I & \text{and} & x_i \notin D & \Longrightarrow \text{color } x_i \text{ with } R, \\ x_i \notin I & \text{and} & x_i \in D & \Longrightarrow \text{color } x_i \text{ with } B, \\ x_i \in I & \text{and} & x_i \in D & \Longrightarrow \text{color } x_i \text{ with } P. \end{array}$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

$$\begin{array}{lccccccc} I: & \epsilon, & 4, & 2, & 2\ 5, & 2\ 3, & 1\ 3, & 1\ 3\ 6 \\ D: & \epsilon, & 4, & 4\ 2, & 5\ 2, & 5\ 3, & 5\ 3\ 1, & 6\ 3\ 1 \end{array}$$

Build a *combined list C* by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{llll} x_i \in I & \text{and} & x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\ x_i \notin I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\ x_i \in I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } P. \end{array}$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

$$\begin{array}{llllllll} I: & \epsilon, & 4, & 2, & 2\ 5, & 2\ 3, & 1\ 3, & 1\ 3\ 6 \\ D: & \epsilon, & 4, & 4\ 2, & 5\ 2, & 5\ 3, & 5\ 3\ 1, & 6\ 3\ 1 \end{array}$$

$C: \quad \epsilon,$

Build a *combined list C* by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{llcl} x_i \in I \text{ and } x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\ x_i \notin I \text{ and } x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\ x_i \in I \text{ and } x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } P. \end{array}$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

$$\begin{array}{lccccccc} I: & \epsilon, & 4, & 2, & 2\,5, & 2\,3, & 1\,3, & 1\,3\,6 \\ D: & \epsilon, & 4, & 4\,2, & 5\,2, & 5\,3, & 5\,3\,1, & 6\,3\,1 \end{array}$$

$$C: \quad \epsilon, \quad \overset{4}{P},$$

Build a *combined list C* by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{llcll} x_i \in I & \text{and} & x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\ x_i \notin I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\ x_i \in I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } P. \end{array}$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

$$\begin{array}{lccccccc} I: & \epsilon, & 4, & 2, & 2\ 5, & 2\ 3, & 1\ 3, & 1\ 3\ 6 \\ D: & \epsilon, & 4, & 4\ 2, & 5\ 2, & 5\ 3, & 5\ 3\ 1, & 6\ 3\ 1 \end{array}$$

$$C: \quad \epsilon, \quad \overset{4}{P}, \quad \overset{2}{P}\ \overset{4}{B}\ ,$$

Build a *combined list* C by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{llll} x_i \in I & \text{and} & x_i \notin D & \Longrightarrow \quad \text{color } x_i \text{ with } R, \\ x_i \notin I & \text{and} & x_i \in D & \Longrightarrow \quad \text{color } x_i \text{ with } B, \\ x_i \in I & \text{and} & x_i \in D & \Longrightarrow \quad \text{color } x_i \text{ with } P. \end{array}$$

Ex. $\pi = 4\,2\,5\,3\,1\,6$

$$\begin{array}{lccccccc} I: & \epsilon, & 4, & 2, & 2\,5, & 2\,3, & 1\,3, & 1\,3\,6 \\ D: & \epsilon, & 4, & 4\,2, & 5\,2, & 5\,3, & 5\,3\,1, & 6\,3\,1 \end{array}$$

$$C: \quad \epsilon, \quad \overset{4}{P}, \quad \overset{2}{P}\,\overset{4}{B}, \quad \overset{2}{P}\,\overset{5}{P},$$

Build a *combined list C* by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{llll} x_i \in I & \text{and} \quad x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\ x_i \notin I & \text{and} \quad x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\ x_i \in I & \text{and} \quad x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } P. \end{array}$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

$$\begin{array}{lllllll} I: & \epsilon, & 4, & 2, & 2\ 5, & 2\ 3, & 1\ 3, & 1\ 3\ 6 \\ D: & \epsilon, & 4, & 4\ 2, & 5\ 2, & 5\ 3, & 5\ 3\ 1, & 6\ 3\ 1 \end{array}$$

$$C: \quad \epsilon, \quad \overset{4}{P}, \quad \overset{2}{P}\ \overset{4}{B}\ , \quad \overset{2}{P}\ \overset{5}{P}, \quad \overset{2}{R}\ \overset{3}{P}\ \overset{5}{B},$$

Build a *combined list C* by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{llllll}
x_i \in I & \text{and} & x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\
x_i \notin I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\
x_i \in I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } P.
\end{array}$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

$$\begin{array}{lccccccc}
I: & \epsilon, & 4, & 2, & 2\ 5, & 2\ 3, & 1\ 3, & 1\ 3\ 6 \\
D: & \epsilon, & 4, & 4\ 2, & 5\ 2, & 5\ 3, & 5\ 3\ 1, & 6\ 3\ 1 \\
C: & \epsilon, & \overset{4}{P}, & \overset{2}{P}\ \overset{4}{B}\ , & \overset{2}{P}\ \ \overset{5}{P}, & \overset{2}{R}\ \overset{3}{P}\ \overset{5}{B}, & \overset{1}{P}\ \overset{3}{P}\ \overset{5}{B}\ , &
\end{array}$$

Build a *combined list* C by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{lllll} x_i \in I & \text{and} & x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\ x_i \notin I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\ x_i \in I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } P. \end{array}$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

$$\begin{array}{lccccccc} I: & \epsilon, & 4, & 2, & 2\ 5, & 2\ 3, & 1\ 3, & 1\ 3\ 6 \\ D: & \epsilon, & 4, & 4\ 2, & 5\ 2, & 5\ 3, & 5\ 3\ 1, & 6\ 3\ 1 \\ & & & & & & & \\ C: & \epsilon, & \overset{4}{P}, & \overset{2}{P}\ \overset{4}{B}, & \overset{2}{P}\ \overset{5}{P}, & \overset{2}{R}\ \overset{3}{P}\ \overset{5}{B}, & \overset{1}{P}\ \overset{3}{P}\ \overset{5}{B}, & \overset{1}{P}\ \overset{3}{P}\ \overset{6}{P} \end{array}$$

Build a *combined list C* by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{llll} x_i \in I & \text{and} & x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\ x_i \notin I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\ x_i \in I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } P. \end{array}$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

$$\begin{array}{lllllll} I: & \epsilon, & 4, & 2, & 2\ 5, & 2\ 3, & 1\ 3, & 1\ 3\ 6 \\ D: & \epsilon, & 4, & 4\ 2, & 5\ 2, & 5\ 3, & 5\ 3\ 1, & 6\ 3\ 1 \end{array}$$

$$C: \quad \epsilon, \quad \overset{4}{P}, \quad \overset{2}{P}\ \overset{4}{B}, \quad \overset{2}{P}\ \overset{5}{P}, \quad \overset{2}{R}\ \overset{3}{P}\ \overset{5}{B}, \quad \overset{1}{P}\ \overset{3}{P}\ \overset{5}{B}, \quad \overset{1}{P}\ \overset{3}{P}\ \overset{6}{P}$$

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.

Build a *combined list* C by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{llcll} x_i \in I & \text{and} & x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\ x_i \notin I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\ x_i \in I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } P. \end{array}$$

Ex. $\pi = 4\,2\,5\,3\,1\,6$

$$\begin{array}{lcccccccc} I: & \epsilon, & 4, & 2, & 2\,5, & 2\,3, & 1\,3, & 1\,3\,6 \\ D: & \epsilon, & 4, & 4\,2, & 5\,2, & 5\,3, & 5\,3\,1, & 6\,3\,1 \\ C: & \epsilon, & \overset{4}{P}, & \overset{2}{P}\,\overset{4}{B}, & \overset{2}{P}\,\overset{5}{P}, & \overset{2}{R}\,\overset{3}{P}\,\overset{5}{B}, & \overset{1}{P}\,\overset{3}{P}\,\overset{5}{B}, & \overset{1}{P}\,\overset{3}{P}\,\overset{6}{P} \end{array}$$

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.
B and P are called *bluish* and *draining blue* is $B \leftarrow \epsilon$, $P \leftarrow R$.

Build a *combined list* C by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{llcl}
x_i \in I & \text{and} \quad x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\
x_i \notin I & \text{and} \quad x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\
x_i \in I & \text{and} \quad x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } P.
\end{array}$$

Ex. $\pi = 4\,2\,5\,3\,1\,6$

$$\begin{array}{lccccccc}
I: & \epsilon, & 4, & 2, & 2\,5, & 2\,3, & 1\,3, & 1\,3\,6 \\
D: & \epsilon, & 4, & 4\,2, & 5\,2, & 5\,3, & 5\,3\,1, & 6\,3\,1 \\
C: & \epsilon, & \overset{4}{P}, & \overset{2}{P}\ \overset{4}{B}, & \overset{2}{P}\ \overset{5}{P}, & \overset{2}{R}\ \overset{3}{P}\ \overset{5}{B}, & \overset{1}{P}\ \overset{3}{P}\ \overset{5}{B}, & \overset{1}{P}\ \overset{3}{P}\ \overset{6}{P}
\end{array}$$

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.
B and P are called *bluish* and *draining blue* is $B \leftarrow \epsilon$, $P \leftarrow R$.
Algorithm for C. 1. Initially $C = \epsilon$.

Build a *combined list* C by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{llllll} x_i \in I & \text{and} & x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\ x_i \notin I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\ x_i \in I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } P. \end{array}$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

$$\begin{array}{lccccccc} I: & \epsilon, & 4, & 2, & 2\ 5, & 2\ 3, & 1\ 3, & 1\ 3\ 6 \\ D: & \epsilon, & 4, & 4\ 2, & 5\ 2, & 5\ 3, & 5\ 3\ 1, & 6\ 3\ 1 \\ C: & \epsilon, & \overset{4}{P}, & \overset{2}{P}\ \overset{4}{B}, & \overset{2}{P}\ \overset{5}{P}, & \overset{2}{R}\ \overset{3}{P}\ \overset{5}{B}, & \overset{1}{P}\ \overset{3}{P}\ \overset{5}{B}, & \overset{1}{P}\ \overset{3}{P}\ \overset{6}{P} \end{array}$$

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.
B and P are called *bluish* and *draining blue* is $B \leftarrow \epsilon$, $P \leftarrow R$.
Algorithm for C. 1. Initially $C = \epsilon$.
2. Each x_i inserts a P into the corresponding space of C.

Build a *combined list* C by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$
\begin{array}{llllll}
x_i \in I & \text{and} & x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\
x_i \notin I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\
x_i \in I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } P.
\end{array}
$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

$$
\begin{array}{llllllll}
I: & \epsilon, & 4, & 2, & 2\,5, & 2\,3, & 1\,3, & 1\,3\,6 \\
D: & \epsilon, & 4, & 4\,2, & 5\,2, & 5\,3, & 5\,3\,1, & 6\,3\,1 \\
C: & \epsilon, & \overset{4}{P}, & \overset{2}{P}\ \overset{4}{B}, & \overset{2}{P}\ \overset{5}{P}, & \overset{2}{R}\ \overset{3}{P}\ \overset{5}{B}, & \overset{1}{P}\ \overset{3}{P}\ \overset{5}{B}, & \overset{1}{P}\ \overset{3}{P}\ \overset{6}{P}
\end{array}
$$

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.
B and P are called *bluish* and *draining blue* is $B \leftarrow \epsilon$, $P \leftarrow R$.
Algorithm for C. 1. Initially $C = \epsilon$.
2. Each x_i inserts a P into the corresponding space of C.
3. Drain red from the closest redish element to the right of the new P (if any), and drain blue from the closest bluish element to the left of the new P (if any).

Build a *combined list* C by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{llll} x_i \in I & \text{and} & x_i \notin D & \Longrightarrow \text{color } x_i \text{ with } R, \\ x_i \notin I & \text{and} & x_i \in D & \Longrightarrow \text{color } x_i \text{ with } B, \\ x_i \in I & \text{and} & x_i \in D & \Longrightarrow \text{color } x_i \text{ with } P. \end{array}$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

I :	ϵ,	4,	2,	2 5,	2 3,	1 3,	1 3 6
D :	ϵ,	4,	4 2,	5 2,	5 3,	5 3 1,	6 3 1

C : ϵ,

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.
B and P are called *bluish* and *draining blue* is $B \leftarrow \epsilon$, $P \leftarrow R$.
Algorithm for C. 1. Initially $C = \epsilon$.
2. Each x_i inserts a P into the corresponding space of C.
3. Drain red from the closest redish element to the right of the new P (if any), and drain blue from the closest bluish element to the left of the new P (if any).

Build a *combined list* C by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{lllll} x_i \in I & \text{and} & x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\ x_i \notin I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\ x_i \in I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } P. \end{array}$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

$$\begin{array}{llllllll} I: & \epsilon, & 4, & 2, & 2\ 5, & 2\ 3, & 1\ 3, & 1\ 3\ 6 \\ D: & \epsilon, & 4, & 4\ 2, & 5\ 2, & 5\ 3, & 5\ 3\ 1, & 6\ 3\ 1 \end{array}$$

$$C: \quad \epsilon, \quad \overset{4}{P},$$

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.
B and P are called *bluish* and *draining blue* is $B \leftarrow \epsilon$, $P \leftarrow R$.
Algorithm for C. 1. Initially $C = \epsilon$.
2. Each x_i inserts a P into the corresponding space of C.
3. Drain red from the closest redish element to the right of the new P (if any), and drain blue from the closest bluish element to the left of the new P (if any).

Build a *combined list* C by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{llll} x_i \in I & \text{and} \quad x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\ x_i \notin I & \text{and} \quad x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\ x_i \in I & \text{and} \quad x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } P. \end{array}$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

$$\begin{array}{llllllll} I: & \epsilon, & 4, & 2, & 2\,5, & 2\,3, & 1\,3, & 1\,3\,6 \\ D: & \epsilon, & 4, & 4\,2, & 5\,2, & 5\,3, & 5\,3\,1, & 6\,3\,1 \end{array}$$

$$C: \quad \epsilon, \quad {\uparrow}\overset{4}{P},$$

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.
B and P are called *bluish* and *draining blue* is $B \leftarrow \epsilon$, $P \leftarrow R$.
Algorithm for C. 1. Initially $C = \epsilon$.
2. Each x_i inserts a P into the corresponding space of C.
3. Drain red from the closest redish element to the right of the new P (if any), and drain blue from the closest bluish element to the left of the new P (if any).

Build a *combined list* C by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{llllll} x_i \in I & \text{and} & x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\ x_i \notin I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\ x_i \in I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } P. \end{array}$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

I :	ϵ,	4,	2,	2 5,	2 3,	1 3,	1 3 6
D :	ϵ,	4,	4 2,	5 2,	5 3,	5 3 1,	6 3 1

$$C : \quad \epsilon, \quad {}_{\uparrow}\overset{4}{P}, \quad \overset{2}{P}\ \overset{4}{B}\ ,$$

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.
B and P are called *bluish* and *draining blue* is $B \leftarrow \epsilon$, $P \leftarrow R$.
Algorithm for C. 1. Initially $C = \epsilon$.
2. Each x_i inserts a P into the corresponding space of C.
3. Drain red from the closest redish element to the right of the new P (if any), and drain blue from the closest bluish element to the left of the new P (if any).

Build a *combined list* C by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{llllll} x_i \in I & \text{and} & x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\ x_i \notin I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\ x_i \in I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } P. \end{array}$$

Ex. $\pi = 4\,2\,5\,3\,1\,6$

$$\begin{array}{lccccccc} I: & \epsilon, & 4, & 2, & 2\,5, & 2\,3, & 1\,3, & 1\,3\,6 \\ D: & \epsilon, & 4, & 4\,2, & 5\,2, & 5\,3, & 5\,3\,1, & 6\,3\,1 \end{array}$$

$$C: \quad \epsilon, \quad \uparrow\overset{4}{P}, \quad \overset{2}{P}\,\overset{4}{B}\uparrow,$$

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.
B and P are called *bluish* and *draining blue* is $B \leftarrow \epsilon$, $P \leftarrow R$.
Algorithm for C. 1. Initially $C = \epsilon$.
2. Each x_i inserts a P into the corresponding space of C.
3. Drain red from the closest redish element to the right of the new P (if any), and drain blue from the closest bluish element to the left of the new P (if any).

Build a *combined list* C by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{lllll} x_i \in I & \text{and} & x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\ x_i \notin I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\ x_i \in I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } P. \end{array}$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

$$\begin{array}{llllllll} I: & \epsilon, & 4, & 2, & 2\ 5, & 2\ 3, & 1\ 3, & 1\ 3\ 6 \\ D: & \epsilon, & 4, & 4\ 2, & 5\ 2, & 5\ 3, & 5\ 3\ 1, & 6\ 3\ 1 \end{array}$$

$$C: \quad \epsilon, \quad \uparrow\overset{4}{P}, \quad \overset{2}{P}\ \overset{4}{B}\uparrow, \quad \overset{2}{P}\ \ \overset{5}{P},$$

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.
B and P are called *bluish* and *draining blue* is $B \leftarrow \epsilon$, $P \leftarrow R$.

Algorithm for C. 1. Initially $C = \epsilon$.

2. Each x_i inserts a P into the corresponding space of C.

3. Drain red from the closest redish element to the right of the new P (if any), and drain blue from the closest bluish element to the left of the new P (if any).

Build a *combined list* C by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{llllll} x_i \in I & \text{and} & x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\ x_i \notin I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\ x_i \in I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } P. \end{array}$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

$$\begin{array}{lccccccc} I: & \epsilon, & 4, & 2, & 2\ 5, & 2\ 3, & 1\ 3, & 1\ 3\ 6 \\ D: & \epsilon, & 4, & 4\ 2, & 5\ 2, & 5\ 3, & 5\ 3\ 1, & 6\ 3\ 1 \end{array}$$

$$C: \quad \epsilon, \quad \uparrow \overset{4}{P}, \quad \overset{2}{P}\ \overset{4}{B} \uparrow, \quad \overset{2}{P} \uparrow \overset{5}{P},$$

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.
B and P are called *bluish* and *draining blue* is $B \leftarrow \epsilon$, $P \leftarrow R$.
Algorithm for C. 1. Initially $C = \epsilon$.
2. Each x_i inserts a P into the corresponding space of C.
3. Drain red from the closest redish element to the right of the new P (if any), and drain blue from the closest bluish element to the left of the new P (if any).

Build a *combined list* C by assigning colors red (R), blue (B), and purple (P) to the x_i as follows:

$$\begin{array}{llllll} x_i \in I & \text{and} & x_i \notin D & \Longrightarrow & \text{color } x_i \text{ with } R, \\ x_i \notin I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } B, \\ x_i \in I & \text{and} & x_i \in D & \Longrightarrow & \text{color } x_i \text{ with } P. \end{array}$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

$$\begin{array}{lccccccc} I: & \epsilon, & 4, & 2, & 2\ 5, & 2\ 3, & 1\ 3, & 1\ 3\ 6 \\ D: & \epsilon, & 4, & 4\ 2, & 5\ 2, & 5\ 3, & 5\ 3\ 1, & 6\ 3\ 1 \\ & & & & & & & \\ C: & \epsilon, & \uparrow\overset{4}{P}, & \overset{2}{P}\ \overset{4}{B}\uparrow, & \overset{2}{P}\uparrow\overset{5}{P}, & \overset{2}{R}\ \overset{3}{P}\ \overset{5}{B}, & & \end{array}$$

R and P are called *redish* and *draining red* is $R \leftarrow \epsilon$, $P \leftarrow B$.
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Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

$$\begin{array}{lccccccc} I: & \epsilon, & 4, & 2, & 2\ 5, & 2\ 3, & 1\ 3, & 1\ 3\ 6 \\ D: & \epsilon, & 4, & 4\ 2, & 5\ 2, & 5\ 3, & 5\ 3\ 1, & 6\ 3\ 1 \end{array}$$

$$C: \quad \epsilon, \quad \uparrow\overset{4}{P}, \quad \overset{2}{P}\ \overset{4}{B}\uparrow, \quad \overset{2}{P}\uparrow\overset{5}{P}, \quad \uparrow\overset{2}{R}\ \overset{3}{P}\ \overset{5}{B},$$

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\end{array}$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

$$\begin{array}{lccccccc}
I: & \epsilon, & 4, & 2, & 2\,5, & 2\,3, & 1\,3, & 1\,3\,6 \\
D: & \epsilon, & 4, & 4\,2, & 5\,2, & 5\,3, & 5\,3\,1, & 6\,3\,1
\end{array}$$

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\end{array}
$$

Ex. $\pi = 4\ 2\ 5\ 3\ 1\ 6$

I :	ϵ,	4,	2,	2 5,	2 3,	1 3,	1 3 6
D :	ϵ,	4,	4 2,	5 2,	5 3,	5 3 1,	6 3 1
C :	ϵ,	$\uparrow \overset{4}{P}$,	$\overset{2}{P}\ \overset{4}{B} \uparrow$,	$\overset{2}{P} \uparrow \overset{5}{P}$,	$\uparrow \overset{2}{R}\ \overset{3}{P}\ \overset{5}{B}$,	$\overset{1}{P}\ \overset{3}{P}\ \overset{5}{B} \uparrow$,	$\overset{1}{P}\ \overset{3}{P}\ \overset{6}{P}$

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Theorem (Otago-S)

The winner of the game on $\mathbb{Q}$ where $m \leq n$ is:

$m \backslash n$	0	1	2	3	4	5	6	7
0	A	A	A	A	A	A	A	A
1		B	B	B	B	B	B	B
2			A	B	A	B	A	B
3				A	A	A	A	A
4					A	A	A	A

where the patterns continue in each of the first 5 rows. ∎

Outline

Let G be a 2-person, “last player to move wins” game with no draw positions.

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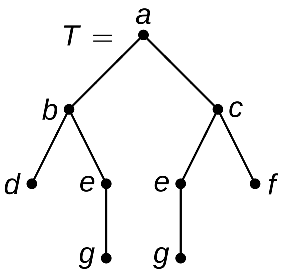

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If the same position w is a child of both v and v' then identify the copies of w to get a (di)graph $\overline{T}$.

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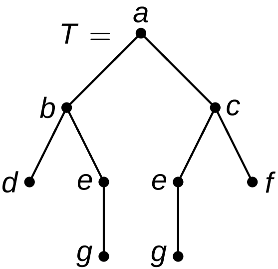

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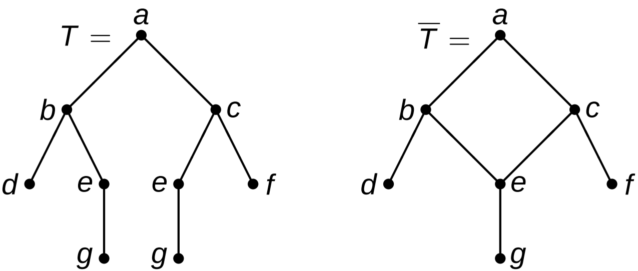

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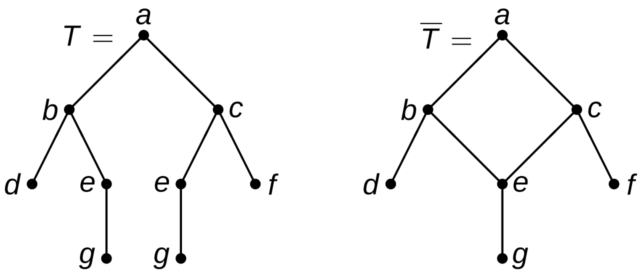

Let $\ell(v) = \begin{cases} \mathcal{N} & \text{if the next player wins from position } v, \\ \mathcal{P} & \text{if the previous player wins from position } v. \end{cases}$

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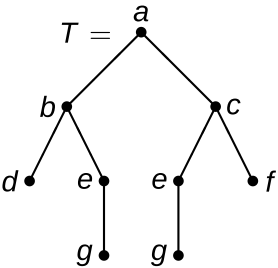
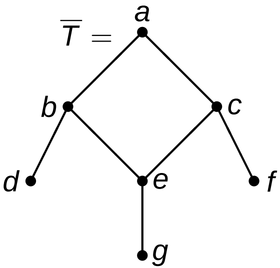
Ex. $T =$ 

$\overline{T} =$ 

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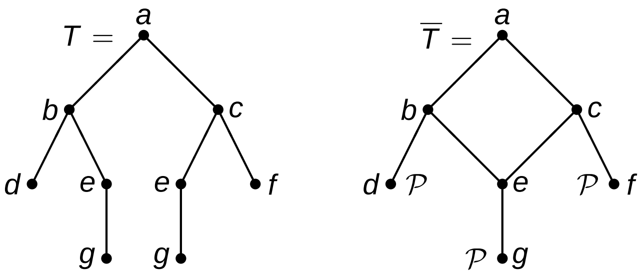

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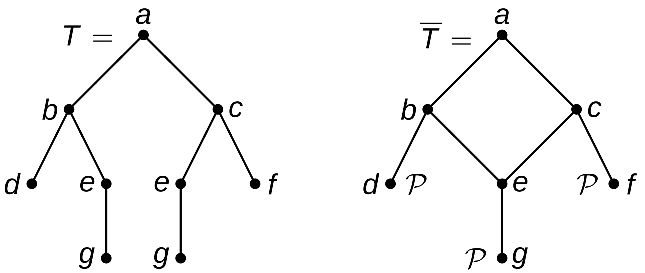

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Now label all terminal $v \in \overline{T}$ with $\mathcal{P}$, and work upwards using:

(i) if there is a $\mathcal{P}$-child of v then let $\ell(v) = \mathcal{N}$

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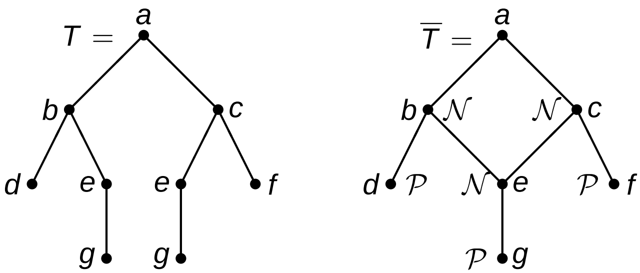

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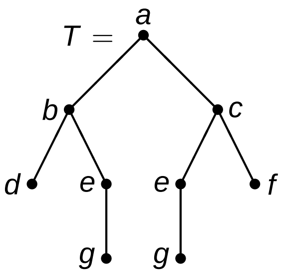

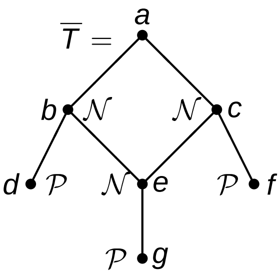

Let $\ell(v) = \begin{cases} \mathcal{N} & \text{if the next player wins from position } v, \\ \mathcal{P} & \text{if the previous player wins from position } v. \end{cases}$

Now label all terminal $v \in \overline{T}$ with $\mathcal{P}$, and work upwards using:

(i) if there is a $\mathcal{P}$-child of v then let $\ell(v) = \mathcal{N}$

(ii) if all children of v are $\mathcal{N}$ then let $\ell(v) = \mathcal{P}$

Let G be a 2-person, "last player to move wins" game with no draw positions. The *game tree*, T of G has

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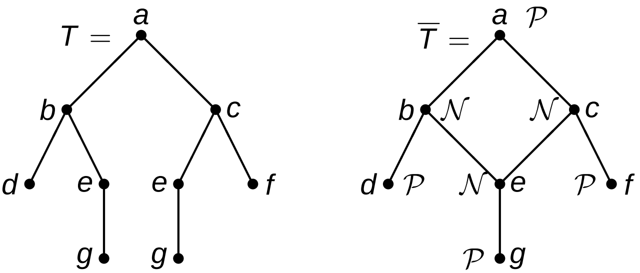

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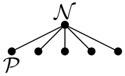
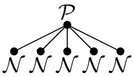
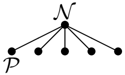
(i) 
$\mathcal{N}$
$\mathcal{P}$

(ii) 
$\mathcal{P}$
$\mathcal{N}$ $\mathcal{N}$ $\mathcal{N}$ $\mathcal{N}$ $\mathcal{N}$

(i)



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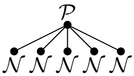
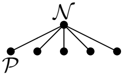
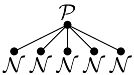


Theorem (Otago-S)

The winner of the game on $\mathbb{Q}$ where $m = n \geq 3$ is A.

(i) 

(ii) 

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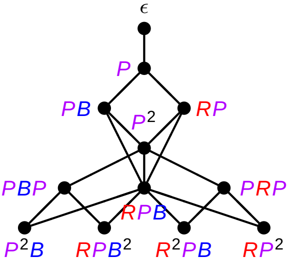
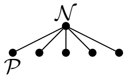
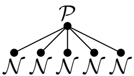


(i) 

(ii) 

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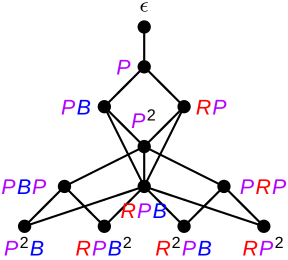
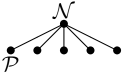
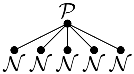


Suppose B wins, so $\ell(\epsilon) = \mathcal{P}$.

(i) 

(ii) 

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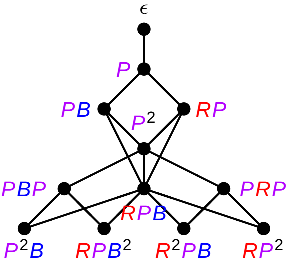

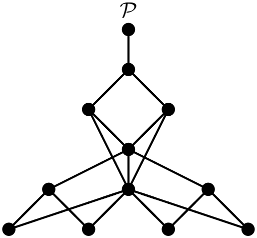

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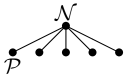
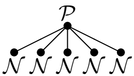
(i) 

(ii) 

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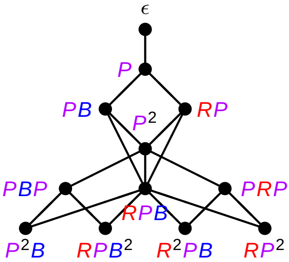

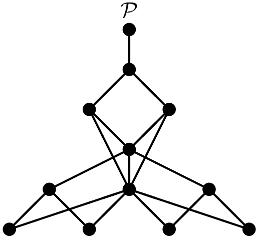

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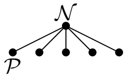
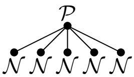
(i) 

(ii) 

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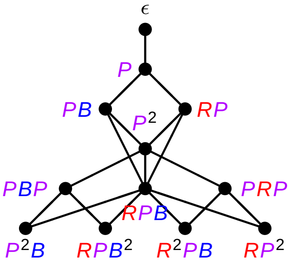

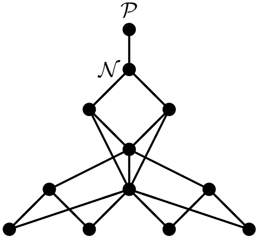

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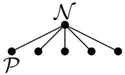
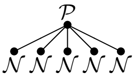
(i) 

(ii) 

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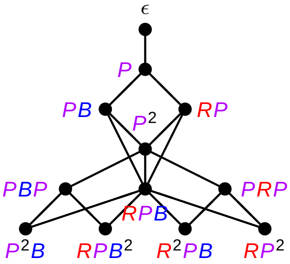

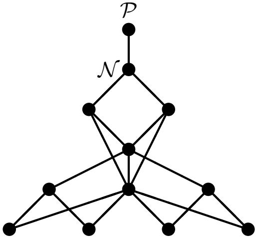

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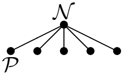
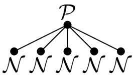
(i) 

(ii) 

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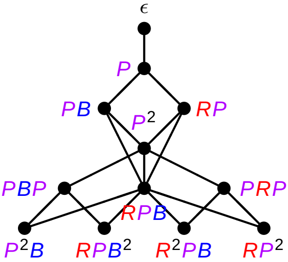

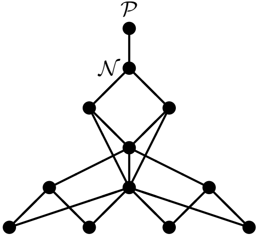

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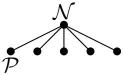
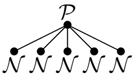
(i) 

(ii) 

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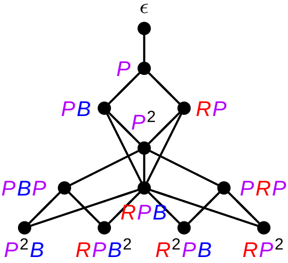

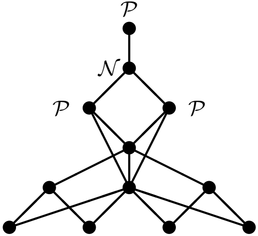

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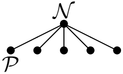
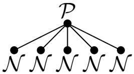
(i) 

(ii) 

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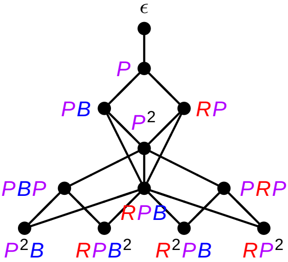

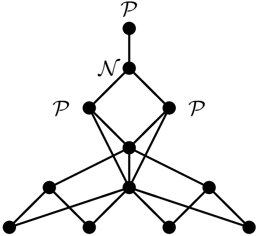

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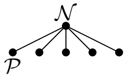
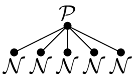
(i) 

(ii) 

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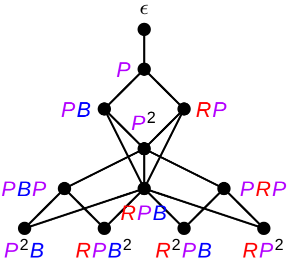

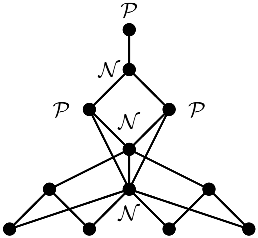

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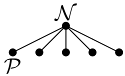
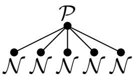
(i) 

(ii) 

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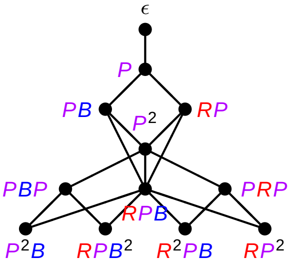

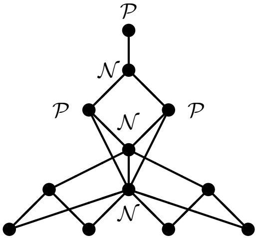

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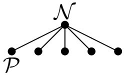
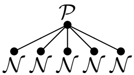
(i) 

(ii) 

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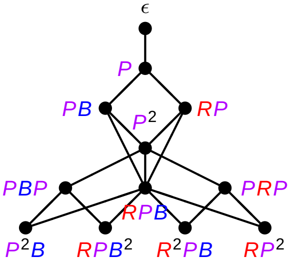

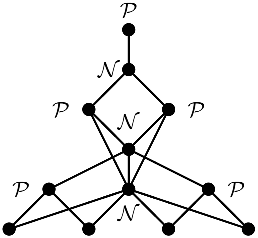

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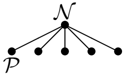
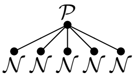
(i) 

(ii) 

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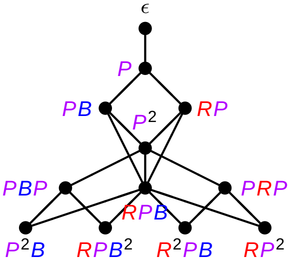

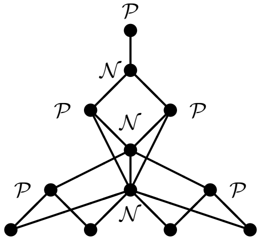

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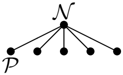
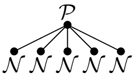
(i) 

(ii) 

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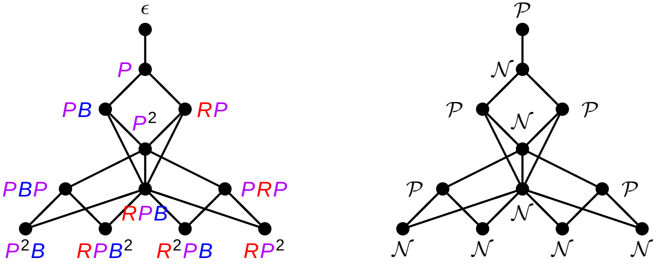

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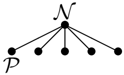
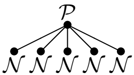
(i) 

(ii) 

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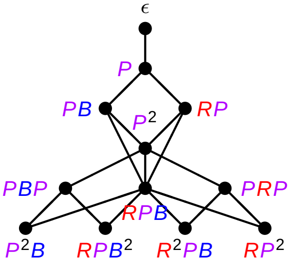

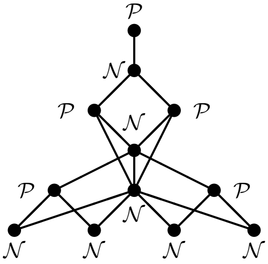

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HAPPY
BIRTHDAY
ANDREAS!!