Menger-bounded subgroups of the Baer-Specker group

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Outline

Non-dominating subgroups of the Baer-Specker group

- Definitions
- Studying the sufficient conditions
- A proposed simplification
- Bounds on the new cardinals
- Studying the construction
- Combinatorial possibilities for k-dominating families

Menger-boundedness

Definition

Let $k \geq 1$ and let $G \subseteq \mathbb{Z}^\omega$ be a subgroup. G^k is called Menger-bounded if

$$(\exists f \in \omega^{\uparrow\omega})(\forall g_1, \ldots g_k \in G)$$
$$\{n : \max_{1 \leq i \leq k, 0 \leq m \leq n} |g_i(m)| \leq f(n)\} \text{ is infinite}$$

Question

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A related question

Definition

Let $k \geq 1$. $D \subseteq \omega^\omega$ is called k-dominating if $\{\max(d_1, \ldots, d_k) : d_i \in D\}$ is $\leq^*$-dominating. For every $f \in \omega^\omega$, there are $d_1, \ldots, d_k \in D$ such that for all but finitely many n, $f(n) \leq \max(d_1(n), \ldots, d_k(n))$.

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Sharp and not so sharp dividing lines

Proposition

There is a 2-dominating not dominating family in the subsets of $\omega^{\uparrow\omega}$, namely $H = \{f \in \omega^{\uparrow\omega} : (\exists^\infty n)(f(n) \leq n)\}$.

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The condition on the non-existence side

Definition

$\mathfrak{u}$ is the smallest cardinality of a basis of a non-principal ultrafilter over ω.

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The conditions on the positive side

Definition

Let $\mathscr{U}$ and $\mathscr{V}$ be non-principal ultrafilters on ω. We say $\mathscr{U}$ and $\mathscr{V}$ are nearly coherent if there is some finite-to-one function $f : \omega \to \omega$ such that $f(\mathscr{U}) = f(\mathscr{V})$. $f(\mathscr{U}) = \{X \subseteq \omega : f^{-1}[X] \in \mathscr{U}\}$.

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More cardinals

Definition

$\mathfrak{r} = \min\{\mathscr{R} \subseteq [\omega]^\omega \ : \ (\forall f : \omega \to \{0,1\})(\exists R \in \mathscr{R})$ $f \restriction R$ is (almost) constant$\}$ is called the reaping number or refining number or unsplitting number.

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The dominating number is $\mathfrak{d} = \min\{\mathscr{D} \subseteq {}^\omega\omega \ : \ (\forall g \in {}^\omega\omega)(\exists f \in \mathscr{D})(g \leq^* f)\}$ is called the dominating number.

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$\mathfrak{u}$ and $\mathfrak{d}$ can be in any order.

$\mathfrak{u} \geq \mathfrak{r}$, Balcar and Simon.

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$\mathfrak{d} = \mathfrak{u} < \mathfrak{r}$ is consistent relative to ZFC.

Theorem, Aubrey

Aubrey: If $\mathfrak{r} < \mathfrak{d}$, then $\mathfrak{u} = \mathfrak{r}$.

So $\mathfrak{r} < \mathfrak{d}$ is as strong as $\mathfrak{u} < \mathfrak{d}$. We can write $\mathfrak{r}$ instead of $\mathfrak{u}$ all the time in this talk.

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First sort: $\mathfrak{r} \geq \mathfrak{d}$. Many construction possibilities.

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Partial answers to the question about the k-Menger-bounded not $(k+1)$-Menger-bounded groups

The following follows from Blass' result, Theorem 1, but people did not read ...

Obsolete Theorem, Banakh, Zdomskyy, Mildenberger

Under $\mathfrak{r} < \mathfrak{g}$, the answer is "no" for $k \geq 2$.

Since the not dominating but 2-dominating H is so easy to describe, it is a bit astonishing that the following is open.

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Under a weakening of CH, for every $k \geq 1$, there is a group whose k-th power is Menger-bounded but whose $(k+1)$-st power is not.

Remark: The construction for $k = 1$ is not even a bit easier than the construction for other k. There does not seem to be a hint to convert it in a ZFC construction.

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Good partitions of ω and a cardinal

Definition

A good partition of ω is a partition $P = \{A_n : n \in \omega\}$ into such that for all n, there are infinitely many i with $i, i+1 \in A_n$.

Definition

Let P be a good partition. We define a cardinal with no name yet

$$\mathfrak{d}'(P) = \min\{|\mathscr{F}| : \mathscr{F} \subseteq \omega^{\uparrow\omega} \wedge (\forall g \in \omega^{\uparrow\omega})(\exists A \in P)(\exists f \in \mathscr{F})$$

$$(\forall^\infty n \in A)$$

$$(f(g(n)) \geq g(n+1) \vee f(g(n+1)) \geq g(n+2) \vee n+1 \notin A)\}$$

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The sufficient condition

The weakening of CH used in Machura, Shelah and Tsaban's theorem

A sufficient condition is: There is a good partition P such that $\mathfrak{d}'(P) \geq \mathfrak{d}$.

Dispensing with the alternative

We define another cardinal without a name:

Definition

Now let P be any partition of ω into infinitely many infinite sets.

$$\mathfrak{d}_*(P) = \min\{|\mathscr{F}| \;:\; \mathscr{F} \subseteq \omega^{\uparrow\omega} \wedge (\forall g \in \omega^{\uparrow\omega})(\exists A \in P)(\exists f \in \mathscr{F}) \\ (\forall^\infty n \in A)(f(g(n)) \geq g(n+1))\}$$

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Is $\mathfrak{d}_*(P) = \mathfrak{d}'(P)$?

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Some estimates with other cardinals

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$\mathfrak{d}_*(P)$ does not depend on P.

Since $\mathfrak{d}'(P)$ has the disjunction in its requirement, which $\mathfrak{d}_*$ does not have, $\mathfrak{d}_*(P) \geq \mathfrak{d}'(P)$ for good P.

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$\mathfrak{d}'(P)$ and P

Slimmer A's in the partition give smaller $\mathfrak{d}'(P)$.

Recall the definition:

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Mathematically this is a pessimistic conjecture, because if it were false, then the construction of the k-Menger-bounded not $k+1$-Menger-bounded groups would also be possible in this hardly known land $\mathfrak{g} \leq \mathfrak{u} < \mathfrak{d}$.

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The conjecture is true

For the moment we also allow P that have only one or two parts. It is clear that this leads to larger cardinals. We are in the $\mathfrak{r} \geq \mathfrak{d}$ area:

Proposition

$\mathfrak{d}_*(\{A, \omega \setminus A\}) \leq \mathfrak{r}$ *and* $\mathfrak{d}'(\{\omega\}) \leq \mathfrak{r}$.

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For $R \in \mathscr{R}$ let $f_R : \omega \to \omega$ be the increasing enumeration of R, that is $f_R(n)$ is the n-th element of R.

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A reduction

Claim

$\{f_R : R \in \mathscr{R}\}$ is a family as in the computation of $\mathfrak{d}'(\{\omega\})$ and in the computation of $\mathfrak{d}_*(\{A, \omega \smallsetminus A\})$.

Proof of the claim

Proof of the claim: Assume that not. Then

$$(\exists g \in \omega^{\uparrow\omega})(\forall R \in \mathscr{R})\Big((\exists^\infty n \in A)(f_R(g(n)) < g(n+1))$$
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(written for a partition into two parts) or (for $\partial'(\{\omega\})$)

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Does the reaping family actually reap?

Set $A = A_0$ and $\omega \setminus A = A_1$. Enumerate the $n \in A_\ell$ such that $f_R(g(n)) < g(n+1)$ as $n^R_{\ell,k}$, $k \in \omega$, for $\ell = 0, 1$.
Since $f_R(g(n^R_{\ell,k})) \geq g(n^R_{\ell,k})$, we have

$$(\forall \ell \in 2)(\forall R \in \mathscr{R})(\forall k \in \omega)(R \cap [g(n^R_{\ell,k}), g(n^R_{\ell,k}+1)) \neq \emptyset).$$

Set

$$B_\ell = \bigcup_{k \in \omega, R \in \mathscr{R}} [g(n^R_{\ell,k}), g(n^R_{\ell,k}+1)).$$

Since $n^R_{\ell,k} \in A_\ell$ and $A_0 \cap A_1 = \emptyset$, $B_0 \cap B_1 = \emptyset$.

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The contradiction

So

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and hence $\mathscr{R}$ is not refining.

You see that for $\eth'$ we need only one part of the partition, since the negation gives two adjacent intervals that are hit by R.

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A modified construction

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$\mathfrak{r} \geq \mathfrak{d}$ is a sufficient condition for the existence of subgroups of $\mathbb{Z}^\omega$ whose k-th power is Menger-bounded but whose $(k+1)$-st power is not.

Proof: We look at the properties of a stratification of ω^ω that are used. Change the construction slightly.

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In the c.c.c. models of $\mathfrak{u} < \mathfrak{d}$ (from BsSh:257) there are groups with Menger-bounded k-th power but non-Menger-bounded $(k+1)$-st power.

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Comparing different k's

Observation

If there is a k-non-dominating family that is $k+1$-dominating and $k' < k$ and there is an n such that

$$k' \cdot n \leq k,$$

$$(k'+1) \cdot n \geq k+1,$$

then the family of all maxima over n elements of the first family is not k'-dominating but $k'+1$ dominating.

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An overall picture

We do not yet know any model of ZFC where a k-non-dominating not $k+1$-dominating family exists and no k-Menger-bounded not $k+1$-Menger-bounded group exists.
Could there be such a significant difference?

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