Decompositions of reflexive groups
A talk dedicated to Andreas Blass

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The Definition of Reflexivity

Definition

M is a dual group if there is a group X with $M \cong X^* := \mathrm{Hom}(X, \mathbb{Z})$.

Definition

If $M^{**} = (M^*)^*$, then let σ_M be the evaluation map

$$\sigma_M : M \to M^{**} (x \mapsto x\sigma_M)$$

defined by

$$y(x\sigma_M) = xy \quad \forall y \in M^*.$$

Then M is reflexive if σ_M is an isomorphism.

Dual Groups versus Reflexivity

Examples: $\mathbb{Z}^\kappa$ and $\mathbb{Z}^{(\kappa)}$ are reflexive for any cardinal $\kappa < \aleph_m$ ($\aleph_m$ = first (ω-)measurable cardinal)
[apply Łoś theorem [1958] on slender groups]
Counterexamples: Essentially every group you can think of.

Observation

- *Reflexive groups are dual group.*
- [Mekler-Schlitt, 1986 and Eda-Ohta, 1987]
  There are dual groups which are not reflexive.

Mekler-Schlitt-idea: Construct a direct limit of an inverse-direct system of subgroups of $\mathbb{Z}^{\omega_1}$.
Eda-Ohta-idea: Construct a topological space X and consider $C(X, \mathbb{Z})$.
Reference: Eklof-Mekler *Almost Free Modules, Set-theoretic Methods*, North-Holland 2002.

More Results on Reflexive Groups

Theorem

Eda [1982, 1983] and Łoś [1958]:
$\mathbb{Z}^\kappa$ is reflexive if and only if κ is not measurable.

Problem 6 in Eklof-Mekler [1989]: *Is there a reflexive group of measurable cardinality?*
The answer:

Theorem

Shelah [2008]:
If μ is a measurable cardinal, then there are reflexive subgroups of $\mathbb{Z}^\mu$ of cardinality μ.

see Shelah, *Reflexive abelian groups and measurable cardinals and saturated mad families, to appear in Algebra Universalis.*

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The class of reflexive groups is closed under direct summands, direct sums and products.

Thus the groups in the Reid-class $\mathfrak{R}$ (obtained by transfinite iterated applications of products, direct sums and direct summands from $\mathbb{Z}$) are reflexive. [Dugas-Huisgen-Zimmermann, 1981] $\Rightarrow$ $\mathfrak{R}$ is large.

But:

Theorem

[Eda-Kamo-Ohta, 1993]: $C(\mathbb{Q}, \mathbb{Z}) \notin \mathfrak{R}$ is reflexive.

Moreover:

Theorem

The sentence:

'$\aleph_1$-separable groups of cardinality $\aleph_1$ are reflexive'

is independent of ZFC.

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A choice of contributions by A. Blass towards reflexivity

A.B. and C. Laflamme: *Consistency results about filters and the number of inequivalent growth types*, Journ. Symb. Logic **54** (1989) 50 – 56.

A.B.: *Cardinal characteristics and the product of countable many infinite cyclic groups*, Journ. Algebra **169** (1994) 512 – 540.

A.B. and J. Irwin: *Baer meets Baire: applications of category arguments and descriptive set theory to* $\mathbb{Z}^{\aleph_0}$, Colorado Proceedings, Dekker, New York (1996) 193–202.

A.B. and R. Göbel: *Subgroups of the Baer-Specker group with few endomorphisms but large dual*, Fund. Math. **149** (1996) 19–29.

A.B. and J. Irwin: *Special families of sets and Baer-Specker groups*, Comm. Algebra **33** (2005) 1733–1744.

A.B.: *Specker's theorem for Nöbeling's group*, Proc. Amer. Math. Soc. **130** (2005) 1581 – 1587.

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Decompositions of Abelian Groups

History of Particlar Decompositions of Abelian Groups

- *The Sabbagh Problem, 1970*
  answered by Eklof and Shelah [Proceedings Oberwolfach 1975, Gordon and Breach, London 1978]:
  For any natural number m there is a locally free abelian group M with
  $$M \cong M \oplus \mathbb{Z}^n \iff m \mid n.$$
- *Eklof-Mekler Problem 12, 1989*
  Can we find a dual abelian group M (so $M = G^*$) such that $M \ncong M \oplus \mathbb{Z}$?
- *Göbel and Shelah*, Proceedings of the Perth Conference, 2001:
  Assuming MA: *There is a reflexive group M with $M \ncong M \oplus \mathbb{Z}$.*

Combining Eklof-Mekler's and Sabbagh's Problem

Central topic of my talk:

Theorem

Assuming MA: *For any natural number* m *there is a reflexive abelian group* M *with*

$$M \cong M \oplus \mathbb{Z}^n \iff m \mid n.$$

Göbel –Agnes Paras: *Decompositions of reflexive groups and Martin's axiom*, to appear Houston Journal of Math. 2008

Shift maps on $P = \mathbb{Z}^\omega$

Let

$$S := \bigoplus_{i<\omega} \mathbb{Z}\mathbf{e}_i \subseteq P := \prod_{i<\omega} \mathbb{Z}\mathbf{e}_i$$

and put

$$\mathbf{x} = \sum_{i\in\omega} x_i\mathbf{e}_i, \quad \text{with } x_i \in \mathbb{Z}.$$

Let m be a fixed natural number and define $\varphi, \varphi^{-1} \in \operatorname{End} P$ by

$$\mathbf{x}\varphi = \sum_{i\in\omega} x_i\mathbf{e}_{i+m} \quad \text{and} \quad \mathbf{x}\varphi^{-1} = \sum_{i\in\omega} x_i\mathbf{e}_{i-m}$$

where $\mathbf{e}_{i-m} = 0$ if $i < m$ and let

$$R = \mathbb{Z}[\langle\varphi\rangle] \subseteq \operatorname{End} P, \quad \text{hence } P \text{ is an } R\text{-module.}$$

Scalar product and $\mathbb{Z}$-adic closure

Let $\mathbb{D}$ be the $\mathbb{Z}$-adic closure of S in P.
The *scalar product*

$$\Phi : S \times S \to \mathbb{Z} \text{ with } \Phi(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij}$$

extends uniquely to $\Phi : \mathbb{D} \times \mathbb{D} \longrightarrow \widehat{\mathbb{Z}}$ and

$$\Phi(\mathbf{e}_i\varphi^k, \mathbf{e}_j) = \delta_{i+mk,j} = \Phi(\mathbf{e}_i, \mathbf{e}_j\varphi^{-k}).$$

If $\theta = \sum n_k\varphi^k \in R$, $\theta' = \sum n_k\varphi^{-k}$, then

$$\Phi(\mathbf{x}\theta, \mathbf{y}) = \Phi(\mathbf{x}, \mathbf{y}\theta').$$

Goal: Find a reflexive R-module G such that

$$\boxed{S \subset G \subset_* \mathbb{D} \text{ and } (\eta \in \mathrm{Mon}\, G \text{ and } G/G\eta \cong \mathbb{Z}^n) \Rightarrow m \mid n.}$$

The set $\mathfrak{P}$ for MA

Let $\mathfrak{P}$ be the set of pairs (H_1, H_2) where, for $i = 1, 2$,

1. $S \subseteq H_i \subseteq_* \mathbb{D}$,
2. H_i is an R-module,
3. $|H_i| < 2^{\aleph_0}$ and
4. $\Phi : H_1 \times H_2 \to \mathbb{Z}$

If $\mathbb{H} = (H_1, H_2)$ and $\mathbb{H}' = (H_1', H_2')$ are elements of $\mathfrak{P}$, define $\mathbb{H} \subseteq \mathbb{H}'$ if $H_i \subseteq H_i'$ $(i = 1, 2)$. Thus $\mathfrak{P}$ is a poset.
We say that a homomorphism $\eta : S \to \mathbb{D}$ is *essentially in* R if for some

$$\theta \in R \quad \mathbf{e}_i\eta = \mathbf{e}_i\theta \text{ for almost all } i \in \omega.$$

On the other hand a homomorphism $\eta : S \to \mathbb{D}$ which is not essentially in R is *undesirable*. We must get rid of those. The next lemma explains how this works.

The Main Lemma

Main Lemma

(ZFC + MA) *Let* $\mathbb{H} = (H_1, H_2) \in \mathfrak{P}$, $\mathbf{b} = \sum_{i\in\omega} b_i \mathbf{e}_i \in P \setminus H_2$, $\eta : S \to \mathbb{D}$ *be undesirable and* $0 \in U < \mathbb{D}$ *such that* $|U| < 2^{\aleph_0}$. *Then there exists* $\mathbf{a} = \sum_{i\in\omega} a_i \mathbf{e}_i \in \mathbb{D}$ *such that*

1. $\Phi(\mathbf{a}, \mathbf{b}) \in \widehat{\mathbb{Z}} \setminus \mathbb{Z}$,
2. $\mathbf{a}\eta \notin H_1' := \langle H_1, R\mathbf{a}\rangle_*$,
3. $(H_1', H_2) \in \mathfrak{P}$ *and*
4. $U \cap R\mathbf{a} = 0$.

Let $\mathbb{H} = (H_1, H_2) \in \mathfrak{P}$, $\mathbf{b} = \sum_{i\in\omega} b_i\mathbf{e}_i \in P \setminus H_2$ and $U < \mathbb{D}$ such that $|U| < 2^{\aleph_0}$. If $\eta : S \to \mathcal{D}$ is undesirable, then either

$$\eta - \theta \text{ has infinite rank, for all } \theta \in R, \tag{1}$$

so η is a *stray* or there exists $\theta \in R$ such that

$$f = \eta - \theta \text{ has finite rank and } f(\mathbf{e}_i) \neq 0 \text{ for almost all } i \in \omega. \tag{2}$$

We approximate $\mathbf{a} \in \mathbb{D}$:

1. $p = (u^p, A^p) \in \mathfrak{F}$ with:
2. $u^p \subseteq H_2$ finite
3. $A^p = \langle a_i^p \mid i < l^p \text{ and } i! \mid a_i^p \in \mathbb{Z}, l^p \in \mathbb{N}\rangle$.

$\mathfrak{F}$ becomes a poset by $p \leq q$, for some $p, q \in \mathfrak{F}$, if

1. $u^p \subseteq u^q$ and $a_i^p = a_i^q$ for $i < l^p$
2. $\Phi(\sum_{i<l^p} a_i^p\mathbf{e}_i, \mathbf{x}) = \Phi(\sum_{i<l^q} a_i^q\mathbf{e}_i, \mathbf{x})$, for all $\mathbf{x} \in u^p$.

First we apply MA to $\mathfrak{F}$ to get $\mathbf{a}$:

Step 1: $\mathfrak{F}$ is σ-centered.
Recall $p = (u^p, A^p)$. Define an equivalence relation $\sim$ on $\mathfrak{F}$:

$$p \sim q \quad \text{if} \quad A^p = A^q.$$

Thus $[p] := \{q \in \mathcal{F} \mid q \sim p\}$ is directed and $\mathfrak{F}$ is a countable union of $[p]$s. $\square$

The Main Lemma

Step 2: We define $< 2^{\aleph_0}$ dense subsets of $\mathfrak{F}$ which describe locally the desired properties of $\mathbf{a}$:
Given $(H_1, H_2) \in \mathfrak{P}$, let $\mathbf{x} \in H_2$, $\ell \in \mathbb{N}$, $k \in \mathbb{Z}$, $r \in R$ and $\mathbf{y} \in U$.

1. $D_\ell = \{p \in \mathfrak{F} \mid \ell < l^p\}$. This makes $\mathbf{a} : \omega \to \mathbb{Z}$ a total map.
2. $D_{\mathbf{x}} = \{p \in \mathfrak{F} \mid \mathbf{x} \in u^p\}$. Hence $\mathbf{a}$ can be added to H_1.
3. $D_k = \{p \in \mathfrak{F} \mid \sum_{i<l^p} a_i^p b_i \not\equiv k \mod l^p!\}$. Thus $\Phi(\mathbf{a}, \mathbf{b}) \notin \mathbb{Z}$.
4. $D_{r,\mathbf{y}} = \left\{p \in \mathfrak{F} \mid r(\sum_{i<l^p} a_i^p \mathbf{e}_i) \not\equiv \mathbf{y} \mod l^p!\mathbb{D}\right\}$, so $U \cap R\mathbf{a} = 0$.
5. If $n \in \mathbb{N}$, $\mathbf{d} \in H_1$, $\theta \in R$ and $\mathrm{rk}(\eta - \theta) = \infty$ then $D_{n,\theta,\mathbf{d}} = \{p \in \mathfrak{F} \mid (n\eta - \theta)(\sum_{i<l^p} a_i^p \mathbf{e}_i) \not\equiv \mathbf{d} \mod l^p!\mathbb{D}\}$. This kills any stray η [case (1)].
6. If $\mathbf{x} \in H_2, z \in \mathbb{Z}, \mathbf{e}_i(\eta - \theta) = \sum_{j=1}^k n_{ij}\mathbf{v}_j$ for $\oplus_{i=1}^k \mathbb{Z}\mathbf{v}_i \sqsubseteq \mathbb{D}$ and $n_i = n_{ij_0} \neq 0$ infinitely often, then
$D_z^{\mathbf{x}} = \{p \in \mathfrak{F} \mid \mathbf{x} \in u^p, |z| < l^p$ and $\sum_{i<l^p} a_i^p n_i \not\equiv z \mod l^p!\}$.
This kills an undesired η which is not a stray [case (2)].

Applying MA for $\mathfrak{F}$

MA $\Rightarrow \exists \mathbb{G} \subseteq \mathfrak{F}$ generic with $D \cap \mathbb{G} \neq \emptyset$ for all constructed dense D.
Let $\mathbf{a} = \sum_{i \in \omega} a_i \mathbf{e}_i$, where $a_i = a_i^p$ for any $p \in \mathbb{G}$ with $i < l^p$.
Thus $\mathbf{a} \in \mathbb{D}$ well-defined.

Put $H_1'' := H_1 + R\mathbf{a} \subseteq H_1' := \langle H_1, R\mathbf{a} \rangle_* \subseteq \mathbb{D}$ and $\mathbb{H}' = (H_1', H_2)$.

And show $\mathbb{H}' \in \mathfrak{P}$:
Convention: If $r = \sum n_k \varphi^k \in R$ then $r' = \sum n_k \varphi^{-k} \in R$.
Let $\mathbf{c} = r\mathbf{a} + \mathbf{f} \in H_1''$ and $\mathbf{y} \in H_2$. Then
$\Phi(\mathbf{c}, \mathbf{y}) = \Phi(\mathbf{a}, r'\mathbf{y}) + \Phi(\mathbf{f}, \mathbf{y})$. $r'\mathbf{y} \in H_2$ and $D_{r'\mathbf{y}}$ is dense
$\Rightarrow \exists p \in D_{r'\mathbf{y}} \cap \mathbb{G}$ and $\Phi(\mathbf{a}, r'\mathbf{y}) = \Phi(\sum_{i<l^p} a_i^p \mathbf{e}_i, r'\mathbf{y}) \in \mathbb{Z}$.
$\Phi(H_1 \times H_2) \subseteq \mathbb{Z} \Rightarrow \Phi(\mathbf{f}, \mathbf{y}) \in \mathbb{Z} \Rightarrow \Phi(\mathbf{c}, \mathbf{y}) \in \mathbb{Z} \Rightarrow \Phi(H_1'' \times H_2) \subseteq \mathbb{Z}$.
If $0 \neq \mathbf{x} = \sum_{i \in \omega} x_i \mathbf{e}_i \in H_1' \Rightarrow \exists t \in \mathbb{N}\ t\mathbf{x} = \mathbf{h} := \sum_{i \in \omega} h_i \mathbf{e}_i \in H_1''$.
Thus $tx_i = h_i$, for all i, and by purity
$\Phi(\mathbf{h}, \mathbf{y}) = \Phi(t\mathbf{x}, \mathbf{y}) = \sum t x_i y_i = t\Phi(\mathbf{x}, \mathbf{y}) \in t\widehat{\mathbb{Z}} \cap \mathbb{Z} = t\mathbb{Z}. \Rightarrow$
$\Phi(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}$. Thus $\mathbb{H}' \in \mathfrak{P}$.

Contents

Main Theorem

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(ZFC + MA) Let $m > 1$. There are two subgroups H_i $(i = 1, 2)$ of the Baer-Specker group P with the following properties:

1. $S \subseteq H_i \subseteq_* \mathbb{D}$
2. *H_i is $\aleph_1$-free and slender.*
3. *There is a natural bilinear form $\Phi : H_1 \times H_2 \longrightarrow \mathbb{Z}$ induced by $\Phi(\mathbf{e}_i, \mathbf{e}_j) = \delta_{i,j}$ $(i, j \in \omega)$, which yields $H_1^* \cong H_2$ and $H_2^* \cong H_1$ such that H_1, H_2 are reflexive.*
4. *$H_i \oplus \mathbb{Z}^n \cong H_i$ if and only if n is a multiple of m (i = 1,2).*
5. $\mathrm{End}\, H_i = R \oplus \mathrm{Fin}\, H_i$ (i = 1,2).