Ohio University

Club-guessing and Coloring Theorems

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- Introduction: What is a coloring theorem?
- Successors of Singular Cardinals
- Club-guessing and a Theorem
- Applications, Issues, and Open Questions

A Problem

Show that at any party with at least six guests, either there are three people who are total strangers, or there are three people, all of whom know each other.

Ordinary Partition Symbol

- $[A]^n$ denotes the set of all n-element subsets of A.
- $\beta \to (\alpha)^\gamma_\delta$ means for every $F : [\beta]^\gamma \to \delta$, there is an $H \in [\beta]^\alpha$ *homogeneous* for f, i.e., $f \upharpoonright [H]^\alpha$ is constant.

Ramsey's Theorem

For any positive integers k, r, and m, there is an N such that

$$N \longrightarrow (m)^k_r.$$

The Case of ω

At a party with infinitely many guests, there are either infinitely many people all of whom know each other, or there are infinitely many all of whom are strangers to each other.

Thus $\omega \longrightarrow (\omega)^2_2$.

In fact, for any natural numbers n and m, we have $\omega \longrightarrow (\omega)^n_m$.

The Uncountable

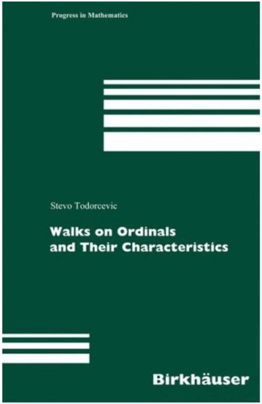
If κ is an uncountable cardinal such that $\kappa \longrightarrow (\kappa)^2_2$, then κ is weakly compact.

In particular, the naive generalization of Ramsey's Theorem fails at an awful lot of cardinals.

General Question

Suppose κ isn't weakly compact. How *badly* does Ramsey's Theorem fail?

Glib Answer: Ask Stevo



Square-brackets notation

$$\kappa \longrightarrow [\sigma]^2_\theta$$

means that for any $F : [\kappa]^2 \longrightarrow \theta$, there is a set $H \subseteq \kappa$ of size σ such that $F \upharpoonright [H]^2$ omits at least one color.

Breakdowns of Ramsey Theory

And therefore...

$$\kappa \nrightarrow [\sigma]^2_\theta$$

means that there is a function $F : [\kappa]^2 \longrightarrow \theta$ such that F assumes every color on any set $A \subseteq \kappa$ of cardinality σ.

We have a "coloring theorem" at κ.

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- Coloring theorems for successors of regular cardinals

What's left?

Suppose $\lambda = \mu^+$ for μ singular. What sorts of coloring theorems hold for λ?

In particular, does $\lambda \nrightarrow [\lambda]^2_\lambda$ hold?

Some old results

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We will be combining these ideas.

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We define $\mathrm{Tr}(\alpha, \beta)$ to be the ordinals appearing in this walk.

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We will use "braking technology" obtained from a scale for μ.

A Simplifying Assumption

For simplicity, we will temporarily assume that μ is singular of countable cofinality.

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  - $\alpha < \beta \implies f_\alpha <^* f_\beta$, and
  - for every $f \in \prod_{n<\omega} \mu_n$, there is an $\alpha < \mu^+$ such that $f <^* f_\alpha$.

Some Theorems

1. Scales always exist. [Shelah]

2. If in our scale we have $\mu_n \nrightarrow [\mu_n]^2_{\mu_n}$ for all n, then $\mu^+ \nrightarrow [\mu^+]^2_{\mu^+}$. [Todorcevic])

The Δ Map

If $(\vec{\mu}, \vec{f})$ is a scale for μ, then we define

$$\Delta : [\mu^+]^2 \to \omega$$

by (for $\alpha < \beta$)

$$\Delta(\alpha, \beta) = \max\{n < \omega : f_\beta(n) \leq f_\alpha(n)\}.$$

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(In general, for singular μ we have $\mu^+ \nrightarrow [\mu^+]^2_{\mathrm{cf}(\mu)}$.)

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Given $\alpha < \beta < \mu^+$, let $\beta = \beta_0 > \beta_1 > \cdots > \beta_n = \alpha$ enumerate $\mathrm{Tr}(\alpha, \beta)$, and define

$$c(\alpha, \beta) = \beta_k,$$

where k is least such that

$$\Delta(\alpha, \beta_k) \neq \Delta(\alpha, \beta).$$

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We will show that the function $c : [\mu^+] \to \mu^+$ just defined has some quite strong properties IF the C-system we use is suitably chosen.

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For that, we need to talk about club-guessing.

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Let S be a stationary subset $\{\delta < \mu^+ : \mathrm{cf}(\delta) = \aleph_0\}$. Assume there is a sequence $\bar{C} = \langle C_\delta : \delta \in S \rangle$ such that

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2. $\langle \mathrm{cf}(\alpha) : \alpha \in C_\delta\rangle$ increases to μ, and
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3. for every closed unbounded $E \subseteq \mu^+$,

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We say that $\bar{C}$ is a nice club-guessing sequence.

"The Club-Guessing Ideal"

$A \in I$ if and only if there is a closed unbounded $E \subseteq \mu^+$ such that

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I^* denotes the dual filter, and I^+ denotes the I-positive sets.

In other words...

$A \in I^+$ if and only if $\langle C_\delta : \delta \in S \cap A \rangle$ is still a very nice club-guessing sequence.

Some Easy Facts

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2. if E is club in μ^+, then $\{\delta \in S : C_\delta \cap E \text{ is infinite}\}$ is in I^*.

“The *Other* Club-Guessing Ideal”

$A \in J$ if and only if there is a club $E \subseteq \mu^+$ such that

$$\{\delta \in S : A \cap E \cap C_\delta \text{ is infinite } \delta\} \text{ is non-stationary.}$$

"The *Other* Club-Guessing Ideal"

$A \in J$ if and only if there is a club $E \subseteq \mu^+$ such that

$$\{\delta \in S : A \cap E \cap C_\delta \text{ is infinite } \delta\} \text{ is non-stationary.}$$

So $A \in J^+$ means $\langle A \cap C_\delta : \delta \in S \rangle$ "is" a nice club-guessing sequence.

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This means that J is closed under *increasing* unions of length κ.

Goal Revisited

Where are we going with this? Let's go back a few slides....

Suitably chosen?

Assume $\bar{C} = \langle C_\delta \in S \rangle$ is a nice club-guessing sequence. Then there is a C-system $\bar{e} = \langle e_\alpha : \alpha < \lambda \rangle$ such that

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We say that $\bar{e}$ swallows $\bar{C}$.

Theorem [TE 2006]

If μ is singular, and if $\bar{e}$ is a C-system on μ^+ that swallows a nice club-guessing sequence with associated ideal J, then the function $c : [\mu^+]^2 \longrightarrow \mu^+$ defined by "walking along $\bar{e}$ until Δ changes" has the property that it takes on J-almost all values on any unbounded subset of μ^+.

Application: A Theorem of Shelah

Suppose μ is a singular cardinal, and let J be the "other" club-guessing ideal associated with a nice club-guessing sequence on μ^+. If μ^+ can be partitioned into θ disjoint J-positive sets, then

$$\mu^+ \nrightarrow [\mu^+]^2_\theta.$$

A stronger result [TE 2006]

If μ^+ can be partitioned into μ (not μ^+!) disjoint J-positive sets, then $\mathrm{Pr}_1(\mu^+, \mu^+, \mu^+, \mathrm{cf}(\mu))$ holds.

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If μ^+ can be partitioned into μ (not μ^+!) disjoint J-positive sets, then $\mathrm{Pr}_1(\mu^+, \mu^+, \mu^+, \mathrm{cf}(\mu))$ holds.

This is a much stronger version of $\mu^+ \nrightarrow [\mu^+]^2_{\mu^+}$ involving "blocks" of ordinals.

A converse

Suppose μ is a strong limit singular cardinal, and $\mu^+ \to [\mu^+]^2_{\mu^+}$. Then there is an ideal K on μ^+ such that

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2. K is κ-indecomposable for all regular κ satisfying $\mathrm{cf}(\mu) < \kappa < \mu$, and
3. K is "close to maximal" in the sense that $|\mathcal{P}(\mu^+)/K| < \mu$.

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- The existence of such an ideal implies that every stationary subset of $\{\delta < \mu^+ : \mathrm{cf}(\mu) \neq \mathrm{cf}(\delta)\}$ reflects.
- We can always partition μ^+ into $\mathrm{cf}(\mu)$ disjoint J-positive sets if μ is strong limit.

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- If μ is singular of uncountable cofinality, then nice club-guessing sequences exist on any stationary $S \subseteq \{\delta < \mu^+ : \mathrm{cf}(\delta) = \mathrm{cf}(\mu)\}$.

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- If μ is singular of uncountable cofinality, then nice club-guessing sequences exist on any stationary $S \subseteq \{\delta < \mu^+ : \mathrm{cf}(\delta) = \mathrm{cf}(\mu)\}$.
- The situation where μ has countable cofinality is still unresolved, but we get an approximation to this.

Theorem [TE and Shelah 2007]

The "approximately nice club-guessing sequences" can be used to get most of the above results.

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- We need to use a generalized version of minimal walks.
- We partially fix a significant error in *Cardinal Arithmetic*.

Open Questions

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2. How does one "saturate" an ideal of the form J?
3. Is it consistent that $\mathrm{pp}(\mu) > \mu^+$ and there is an indecomposable ultrafilter on μ^+? Can we get this an have all stationary subsets of μ^+ reflecting as well?

4. Can we have a Jonsson cardinal κ with the property that there is an $F : [\kappa]^{<\omega} \longrightarrow \kappa$ such that

$$\{\mathrm{ran}(F \restriction [A]^{<\omega}) : A \in [\kappa]^{\kappa}\}$$

has the finite intersection property? What about $F : [\kappa]^2 \longrightarrow \kappa$?

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has the finite intersection property? What about $F : [\kappa]^2 \longrightarrow \kappa$?

5. Suppose $(\vec{\mu}, \vec{f})$ is a scale for μ, and $\mu_n \nrightarrow [\mu_n]^{<\omega}_{\mu_n}$ for all $n < \omega$. Does $\mu^+ \nrightarrow [\mu^+]^2_{\mu^+}$?

Do I have time for some fun at the blackboard?

The End!