Borel Conjecture(s)

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countable versus uncountable

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A continuous function $f : K \longrightarrow \mathbf{R}$ is nowhere constant if f is not constant on any non-empty open subset of K.

Definition

A set $X \subset \mathbf{R}$ is universally meager if $f^{-1}(X)$ is meager in K for any continuous nowhere constant function $f : K \longrightarrow \mathbf{R}$, where K is a Baire space.

This is a variation on the notion of universally Baire in which we require that $f^{-1}(X)$ has the Baire property. All universally meager sets are universally Baire, and so they have the usual regularity properties.

Theorem (Todorcevic)

Assume that there exists a compact cardinal. Then $X \subset \mathbf{R}$ is universally meager iff X is countable.

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Assume that there exists a compact cardinal. Then $X \subset \mathbf{R}$ is universally meager iff X is countable.

Suppose that $\mathcal{J}$ is a translation invariant σ-ideal on $\mathbf{R}$. Define

$$\mathcal{J}^\star = \left\{ X \subset \mathbf{R} : \forall A \in \mathcal{J}\ X + A = \bigcup_{x \in X} (A + x) \neq \mathbf{R} \right\}.$$

Clearly all countable sets of reals are in $\mathcal{J}^\star$.

Theorem (Solecki)

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Suppose that $\mathbf{A} = (A_-, A_+, A)$, where A is a binary relation between A_- and A_+.

Let

$$\mathfrak{d}(\mathbf{A}) = \{Z \subseteq A_+ : \forall x \in A_- \; \exists z \in Z \; A(x,z)\}.$$

$$\mathfrak{b}(\mathbf{A}) = \{Z \subseteq A_- : \forall y \in A_+ \; \exists z \in Z \; \neg A(z,y)\}.$$

$$\|\mathbf{A}\| = \min\{|Z| : Z \in \mathfrak{d}(\mathbf{A})\}.$$

Define $\mathbf{A}^\perp = (A_+, A_-, A^\perp)$, where $A^\perp = \{(z,x) : \neg A(x,z)\}$. Note that $\mathfrak{b}(\mathbf{A}) = \mathfrak{d}(\mathbf{A}^\perp)$.

Note that $\|\mathbf{A}\|$ is the smallest size of the "dominating" family in A_+ and $\|\mathbf{A}^\perp\|$ is the smallest size of the "unbounded" family in A_-.

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For an ideal $\mathcal{J}$ of subsets of $\mathbf{R}$ we have:

- $\mathrm{cof}(\mathcal{J}) = \|(\mathcal{J}, \mathcal{J}, \subseteq)\|$,
- $\mathrm{add}(\mathcal{J}) = \|(\mathcal{J}, \mathcal{J}, \subseteq)^{\perp}\| = \|(\mathcal{J}, \mathcal{J}, \not\supseteq)\|$,
- $\mathrm{cov}(\mathcal{J}) = \|(\mathbf{R}, \mathcal{J}, \in)\|$,
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For $f, g \in \omega^\omega$ we define $f \leq^\star g$ if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$.

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Suppose that $\mathbf{A} = (A_-, A_+, A)$ is given.

$X \subset \mathbf{R}$ is big if there is $f : X \longrightarrow A_+$ such that $f[X] \in \mathfrak{d}(\mathbf{A}) = \{Z \subseteq A_+ : \forall x \in A_- \ \exists z \in Z \ A(x, z)\}$.

The following observation is obvious:

In $\mathbf{ZFC} + \|\mathbf{A}\| = \aleph_1$ we have

$$X \text{ is big} \iff X \text{ is uncountable.}$$

To make it interesting we will require that f is Borel/continuous or otherwise definable.

Definition

Suppose that $\mathbf{A} = (A_-, A_+, A)$ is given.

A Borel Conjecture for $\mathbf{A}$ ($\mathrm{BC}(\mathbf{A})$) is the statement:

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Let $\mathcal{M}$ and $\mathcal{N}$ be the ideals of meager and Lebesgue measure zero subsets of $\mathbf{R}$.
The following diagram show that status of Borel Conjecture for the cardinal characteristics from the Cichon's diagram.

$$
\begin{tikzcd}
\mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \\
 & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & \\
\mathrm{add}(\mathcal{N}) \arrow[uu] \arrow[r] & \mathrm{add}(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{cov}(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{non}(\mathcal{N}) \arrow[uu]
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Green means that Borel Conjecture is consistent, red that it is not and yellow that the question is open.

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\begin{tikzcd}
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 & \textcolor{green}{\mathfrak{b}} \arrow[u] \arrow[r] & \textcolor{yellow}{\mathfrak{d}} \arrow[u] & \\
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BC(cov($\mathcal{M}$)) is Borel Conjecture (Laver)

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BC($\mathfrak{b}$) (Miller)

Theorem (Pawlikowski)

Borel Conjecture for $(\mathcal{M}, \mathcal{M}, \subset)$ is false.

If $\mathrm{cof}(\mathcal{M}) > \aleph_1$ then no $\aleph_1$ set is in $\eth((\mathcal{M}, \mathcal{M}, \subset)$. If $\mathrm{cof}(\mathcal{M}) = \aleph_1$ then there is a Lusin set. No Borel image of a Lusin set is a dominating family (in ω^ω) and so it is also not in $\eth((\mathcal{M}, \mathcal{M}, \subset)$.

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Theorem (Miller)

Borel Conjecture for $\mathfrak{b}$, that is $\mathrm{BC}((\omega^\omega, \omega^\omega, \leq^\star))$, is consistent with **ZFC**. *Specifically, it s consistent that whenever X is uncountable set of reals then there exists a Borel mapping of X onto an unbounded family in ω^ω.*

This holds in a model where every uncountable set has a subset which is a G_δ but not F_σ.

Conjecture (Hurewicz)

Suppose that $X \subset \mathbf{R}$. The following conditions are equivalent:

1. *For every continuous function $F : X \longrightarrow \omega^\omega$, $F[X]$ is $\leq^\star$-bounded,*
2. *X is σ-compact.*

Theorem (Just, Miller, Scheepers, Szeptycki)

Hurewicz Conjecture is false. In fact there is a set $X \subset \mathbf{R}$ of size $\mathfrak{b}$ whose every continuous image into ω^ω is bounded.

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Hurewicz Conjecture is false. In fact there is a set $X \subset \mathbf{R}$ *of size* $\mathfrak{b}$ *whose every continuous image into* ω^ω *is bounded.*

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Borel Conjecture for $\mathfrak{b}$, that is $\mathrm{BC}((\omega^\omega, \omega^\omega, \leq^\star))$, is consistent with **ZFC***. Specifically, it s consistent that whenever X is uncountable set of reals then there exists a Borel mapping of X onto an unbounded family in ω^ω.*

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Suppose that $X \subset \mathbf{R}$. The following conditions are equivalent:

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Definition (Borel)

A metric space X has strong measure zero if for every sequence of positive reals $\{\varepsilon_n : n \in \omega\}$ there exists a sequence $\{X_n : n \in \omega\}$ such that each set X_n has diameter $< \varepsilon_n$ and $X \subseteq \bigcup_{n\in\omega} X_n$.
Let $\mathcal{SN}$ be the collection of all strong measure zero sets.

Theorem (Laver)

Borel Conjecture is consistent with **ZFC**. *In particular* BC *implies* BC(cov($\mathcal{M}$)).

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The following are equivalent for $X \subset 2^\omega$:

1. $X \in \mathcal{SN}$,
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These properties make sense in the general context of an abelian Polish group.

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Let $\kappa_G = \min\{|X| : X \subset \mathbf{G}\ \&\ \exists F \in \mathcal{M}\ X + F = \mathbf{G}\}$. *It is consistent that* $\kappa_{2^\omega} < \kappa_{Z^\omega}$.

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Suppose that $\mathbf{G}$ *is locally compact Polish group and* $\mathcal{E}$ *is the ideal of compact null subsets of* $\mathbf{G}$. *Then* $\lambda_G = \min\{|X| : X \subset \mathbf{G}\ \&\ \exists E \in \mathcal{E}\ X + E = \mathbf{G}\}$ *does not depend on* $\mathbf{G}$.

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Let $\mathbf{m}$ *be Laver real over* $\mathbf{V}$*. Let* $\{s_n : n \in \omega\} \in \mathbf{V}[\mathbf{m}]$ *be such that for all* $n \in \omega$*,* $s_n \in 2^{[\mathbf{m}(n),\mathbf{m}(n+1))}$*.*
Then in $\mathbf{V}[\mathbf{m}]$*,* $|\{x \in \mathbf{V} \cap 2^\omega : \exists^\infty n \; s_n \subset x| \leq \aleph_0$*.*

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Question

Is is consistent with **ZFC** *that every uncountable set of reals can be Borel mapped onto a non-meager set?*

Theorem (Bartoszynski,Shelah)

It is consistent with **ZFC** *that every uncountable set of reals can be mapped onto a non-null set by a uniformly continuous function.*

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There exists a proper forcing notion $\mathbb{P}$ *which adds an uniformly continuous function* $F : 2^\omega \longrightarrow 2^\omega$ *such that if* $X \subseteq \mathbf{V} \cap 2^\omega$, $X \in \mathbf{V}$ *and* $X \notin \mathcal{SN}$ *then in* $\mathbf{V}^{\mathbb{P}}$, $F[X] + \mathbb{Q} = 2^\omega$.

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Definition

We say that a set of reals X is strongly meager ($X \in \mathcal{SM}$) if $X \in \mathcal{N}^\star$, that is for every $G \in \mathcal{N}$, $X + G \neq 2^\omega$.
Dual Borel Conjecture DBC says that $\mathcal{N}^\star = [\mathbf{R}]^{\leq \aleph_0}$.

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We say that a sequence of clopen subsets of 2^ω, $\{C_n : n \in \omega\}$ is big over N, if

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The following are used in all constructions of the models for DBC — one needs a forcing notion $\mathbb{P}$ which satisfies a strong form of ccc and adds a big sequence.

The following is the key observation.

Theorem (Lorenz)

For every $\varepsilon > 0$ and a sufficiently large finite set $I \subset \omega$ there exists $N_\varepsilon \in \omega$ (not depending on I) such that if $X \subseteq 2^I$, $|X| \geq N_\varepsilon$ then there exists a set $C \subseteq 2^I$, $\frac{|C|}{2^{|I|}} \leq \varepsilon$ and $C + X = 2^I$.

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Towards Borel Conjecture+ Dual Borel Conjecture consider a smaller goal: to construct a model for DBC without adding Cohen reals.

The key fact is the following strengthening of the Lorenz Theorem.

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For every $\varepsilon, \delta > 0$ and a sufficiently large finite set $I \subseteq \omega$ there exists $N_{\varepsilon,\delta} \in \omega$ (not depending on I) and a family $\mathcal{A}_I$ consisting of sets $C \subseteq 2^I$, $\frac{|C|}{2^{|I|}} \leq \varepsilon$ such that if $X \subseteq 2^I$, $|X| \geq N_{\varepsilon,\delta}$ then

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This allows us to construct a forcing notion which preserves non-null sets and adds a big sequence.

Next using $\Diamond$ for a given uncountable set of reals we can find a subforcing $\mathbb{P}_X$ such that

1. $\mathbb{P}_X$ is ccc,
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3. $\mathbf{V}^{\mathbb{P}_X} \models X \notin \mathcal{SM}$.

We will build the required forcing as a increasing chain of approximations $\{\mathbb{P}_\alpha : \alpha < \omega_1\}$ and put $\mathbb{P}_X = \bigcup_{\alpha<\omega_1} \mathbb{P}_\alpha$. In order to guarantee that $\mathbb{P}_X$ satisfies ccc we will use an oracle that will tell us that whenever $\mathcal{A}$ is a maximal antichain in $\mathbb{P}$ then $\mathcal{A}$ is frozen at some stage α.

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Thus we will have two disjoint stationary sets S_0 and S_1 and a sequence of countable models $\{M_\alpha : \alpha \in S_0 \cup S_1\}$ which witness $\Diamond$ on S_0 and S_1.

We will be making two types of commitment by requiring that for stationary many α:

1. If $\mathcal{A} \in M_\alpha$ is a maximal antichain in $\mathbb{P}_\alpha$ then $\mathcal{A}$ is a maximal antichain $\mathbb{P}\}$,
2. $\Vdash_{\mathbb{P}} x_\alpha$ is random over $M_\alpha[\dot{G}]\}$ for a fixed set $Y = \{x_\alpha : \alpha \in S\}$ such that x_α is random over M_α.

The forcing $\mathbb{P}$ will be constructed from $\omega_1 \times \omega_2$ countable pieces. The ω_2 axis will correspond to the ω_2-iteration while the ω_1 axis will correspond to the single task of making a given $\aleph_1$-set not strongly meager. In general, $\mathbb{P}_{\alpha+1}$ will be of the form $\mathbb{P}_\alpha \star \mathbb{P}_X$.

New type of iteration:
instead of preservation theorems we have commitments.
The task at the limit step will be to extend the construction rather than to prove a preservation theorem.