

Borel Conjecture(s)

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countable versus uncountable

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Definition

A set $X \subset \mathbf{R}$ is **universally meager** if $f^{-1}(X)$ is meager in K for any continuous nowhere constant function $f : K \longrightarrow \mathbf{R}$, where K is a Baire space.

This is a variation on the notion of **universally Baire** in which we require that $f^{-1}(X)$ has the Baire property. All universally meager sets are universally Baire, and so they have the usual regularity properties.

Theorem (Todorćević)

Assume that there exists a compact cardinal. Then $X \subset \mathbf{R}$ is universally meager iff X is countable.

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Theorem (Todorćević)

Assume that there exists a compact cardinal. Then $X \subset \mathbf{R}$ is universally meager iff X is countable.

Suppose that \mathcal{I} is a translation invariant σ -ideal on \mathbf{R} . Define

$$\mathcal{I}^* = \left\{ X \subset \mathbf{R} : \forall A \in \mathcal{I} \ X + A = \bigcup_{x \in X} (A + x) \neq \mathbf{R} \right\}.$$

Clearly all countable sets of reals are in \mathcal{I}^* .

Theorem (Solecki)

There exists a translation invariant σ -ideal \mathcal{I} such that
 $\mathcal{I}^* = [\mathbf{R}]^{\leq \aleph_0}.$

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Definition (Blass)

Suppose that $\mathbf{A} = (A_-, A_+, A)$, where A is a binary relation between A_- and A_+ .

Let

$$\mathfrak{d}(\mathbf{A}) = \{Z \subseteq A_+ : \forall x \in A_- \exists z \in Z A(x, z)\}.$$

$$\mathfrak{b}(\mathbf{A}) = \{Z \subseteq A_- : \forall y \in A_+ \exists z \in Z \neg A(z, y)\}.$$

$$\|\mathbf{A}\| = \min\{|Z| : Z \in \mathfrak{d}(\mathbf{A})\}.$$

Define $\mathbf{A}^\perp = (A_+, A_-, A^\perp)$, where $A^\perp = \{(z, x) : \neg A(x, z)\}$. Note that $\mathfrak{b}(\mathbf{A}) = \mathfrak{d}(\mathbf{A}^\perp)$.

Note that $\|\mathbf{A}\|$ is the smallest size of the “dominating” family in A_+ and $\|\mathbf{A}^\perp\|$ is the smallest size of the “unbounded” family in A_- .

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For an ideal \mathcal{I} of subsets of \mathbf{R} we have:

- $\text{cof}(\mathcal{I}) = \|(\mathcal{I}, \mathcal{I}, \subseteq)\|$,
- $\text{add}(\mathcal{I}) = \|(\mathcal{I}, \mathcal{I}, \subseteq)^\perp\| = \|(\mathcal{I}, \mathcal{I}, \not\subseteq)\|$,
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Suppose that $\mathbf{A} = (A_-, A_+, A)$ is given.

$X \subset \mathbf{R}$ is **big** if there is $f : X \longrightarrow A_+$ such that $f[X] \in \mathfrak{d}(\mathbf{A}) = \{Z \subseteq A_+ : \forall x \in A_- \exists z \in Z A(x, z)\}$.

The following observation is obvious:

In $\mathbf{ZFC} + \|\mathbf{A}\| = \aleph_1$ we have

$$X \text{ is big} \iff X \text{ is uncountable.}$$

To make it interesting we will require that f is Borel/continuous or otherwise definable.

Definition

Suppose that $\mathbf{A} = (A_-, A_+, A)$ is given.

A **Borel Conjecture for \mathbf{A}** ($\mathbf{BC}(\mathbf{A})$) is the statement:

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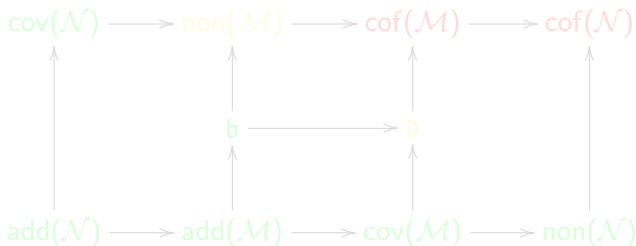
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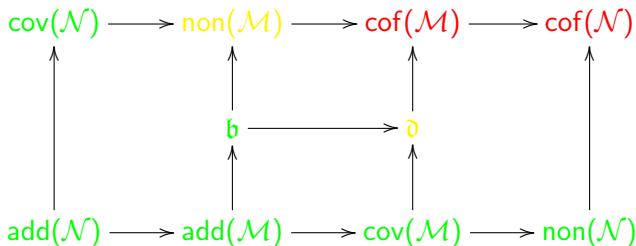
The following diagram show that status of Borel Conjecture for the cardinal characteristics from the Cichon's diagram.



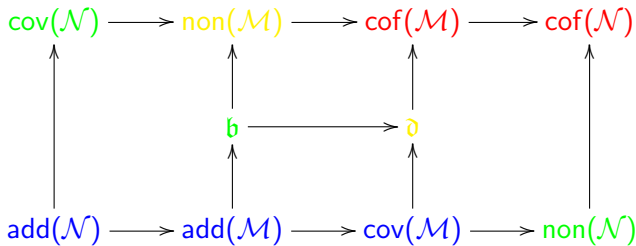
Green means that Borel Conjecture is consistent, red that it is not and yellow that the question is open.

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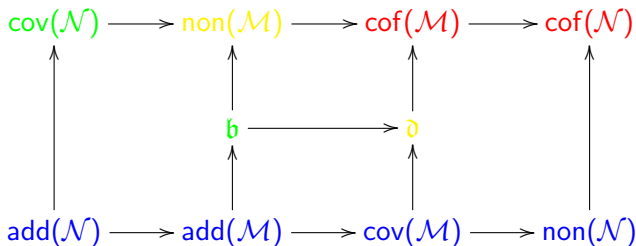
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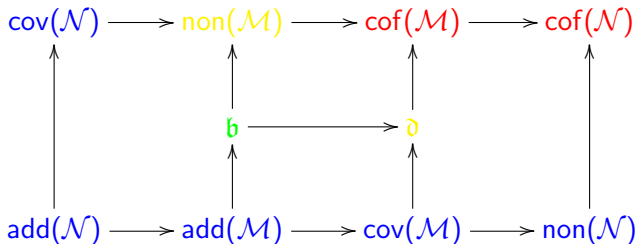


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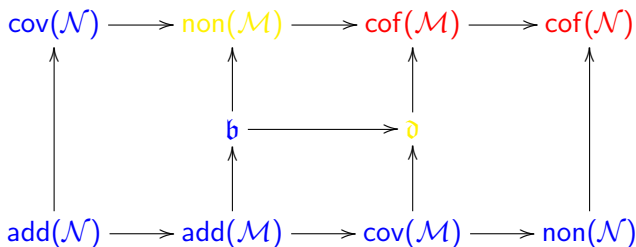
$\text{BC}(\text{non}(\mathcal{N}))$ (Bartoszyński-Shelah)



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$BC(\mathfrak{b})$ (Miller)

Theorem (Pawlikowski)

Borel Conjecture for $(\mathcal{M}, \mathcal{M}, \subset)$ is false.

If $\text{cof}(\mathcal{M}) > \aleph_1$ then no \aleph_1 set is in $\mathfrak{d}((\mathcal{M}, \mathcal{M}, \subset))$. If $\text{cof}(\mathcal{M}) = \aleph_1$ then there is a Lusin set. No Borel image of a Lusin set is a dominating family (in ω^ω) and so it is also not in $\mathfrak{d}((\mathcal{M}, \mathcal{M}, \subset))$.

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Theorem (Miller)

Borel Conjecture for \mathfrak{b} , that is $\text{BC}((\omega^\omega, \omega^\omega, \leq^))$, is consistent with **ZFC**. Specifically, it is consistent that whenever X is an uncountable set of reals then there exists a Borel mapping of X onto an unbounded family in ω^ω .*

This holds in a model where every uncountable set has a subset which is a G_δ but not F_σ .

Conjecture (Hurewicz)

Suppose that $X \subset \mathbf{R}$. The following conditions are equivalent:

- ① *For every continuous function $F : X \longrightarrow \omega^\omega$, $F[X]$ is \leq^* -bounded,*
- ② *X is σ -compact.*

Theorem (Just, Miller, Scheepers, Szeptycki)

Hurewicz Conjecture is false. In fact there is a set $X \subset \mathbf{R}$ of size \mathfrak{b} whose every continuous image into ω^ω is bounded.

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Definition (Borel)

A metric space X has **strong measure zero** if for every sequence of positive reals $\{\varepsilon_n : n \in \omega\}$ there exists a sequence $\{X_n : n \in \omega\}$ such that each set X_n has diameter $< \varepsilon_n$ and $X \subseteq \bigcup_{n \in \omega} X_n$.
Let \mathcal{SN} be the collection of all strong measure zero sets.

Theorem (Laver)

Borel Conjecture is consistent with ZFC. In particular BC implies $\text{BC}(\text{cov}(\mathcal{M}))$.

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The following are equivalent for $X \subset 2^\omega$:

- 1 $X \in \mathcal{SN}$,
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A metric space X has **strong measure zero** if for every sequence of positive reals $\{\varepsilon_n : n \in \omega\}$ there exists a sequence $\{X_n : n \in \omega\}$ such that each set X_n has diameter $< \varepsilon_n$ and $X \subseteq \bigcup_{n \in \omega} X_n$.
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These properties make sense in the general context of an abelian Polish group.

Theorem (Miller, Steprans)

Let $\kappa_G = \min\{|X| : X \subset \mathbf{G} \text{ \& } \exists F \in \mathcal{M} \ X + F = \mathbf{G}\}$. It is consistent that $\kappa_{2^\omega} < \kappa_{Z^\omega}$.

Theorem (Elekes)

Suppose that \mathbf{G} is locally compact Polish group and \mathcal{E} is the ideal of compact null subsets of \mathbf{G} . Then

$\lambda_G = \min\{|X| : X \subset \mathbf{G} \text{ \& } \exists E \in \mathcal{E} \ X + E = \mathbf{G}\}$ does not depend on \mathbf{G} .

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Let \mathbf{m} be Laver real over \mathbf{V} . Let $\{s_n : n \in \omega\} \in \mathbf{V}[\mathbf{m}]$ be such that for all $n \in \omega$, $s_n \in 2^{[\mathbf{m}(n), \mathbf{m}(n+1))}$.

Then in $\mathbf{V}[\mathbf{m}]$, $|\{x \in \mathbf{V} \cap 2^\omega : \exists^\infty n \ s_n \subset x\}| \leq \aleph_0$.

Thus, if $X \subset 2^\omega$ is uncountable then in $\mathbf{V}[\mathbf{m}] \models X \notin \mathcal{SN}$.

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Question

*Is it consistent with **ZFC** that every uncountable set of reals can be Borel mapped onto a non-meager set?*

Theorem (Bartoszynski, Shelah)

*It is consistent with **ZFC** that every uncountable set of reals can be mapped onto a non-null set by a uniformly continuous function.*

Lemma

There exists a proper forcing notion \mathbb{P} which adds a uniformly continuous function $F : 2^\omega \rightarrow 2^\omega$ such that if $X \subseteq \mathbf{V} \cap 2^\omega$, $X \in \mathbf{V}$ and $X \notin \mathcal{SN}$ then in $\mathbf{V}^\mathbb{P}$, $F[X] + \mathbb{Q} = 2^\omega$.

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Definition

We say that a set of reals X is **strongly meager** ($X \in \mathcal{SM}$) if $X \in \mathcal{N}^*$, that is for every $G \in \mathcal{N}$, $X + G \neq 2^\omega$.

Dual Borel Conjecture DBC says that $\mathcal{N}^* = [\mathbf{R}]^{\leq \aleph_0}$.

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We say that a sequence of clopen subsets of 2^ω , $\{C_n : n \in \omega\}$ is big over N , if

- 1 C_n 's have pairwise disjoint supports,
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The following are used in all constructions of the models for DBC — one needs a forcing notion \mathbb{P} which satisfies a strong form of ccc and adds a big sequence.

The following is the key observation.

Theorem (Lorenz)

For every $\varepsilon > 0$ and a sufficiently large finite set $I \subset \omega$ there exists $N_\varepsilon \in \omega$ (not depending on I) such that if $X \subseteq 2^I$, $|X| \geq N_\varepsilon$ then there exists a set $C \subseteq 2^I$, $\frac{|C|}{2^{|I|}} \leq \varepsilon$ and $C + X = 2^I$.

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Towards Borel Conjecture+ Dual Borel Conjecture consider a smaller goal: to construct a model for DBC without adding Cohen reals.

The key fact is the following strengthening of the Lorenz Theorem.

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For every $\varepsilon, \delta > 0$ and a sufficiently large finite set $I \subseteq \omega$ there exists $N_{\varepsilon, \delta} \in \omega$ (not depending on I) and a family \mathcal{A}_I consisting of sets $C \subseteq 2^I$, $\frac{|C|}{2^{|I|}} \leq \varepsilon$ such that if $X \subseteq 2^I$, $|X| \geq N_{\varepsilon, \delta}$ then

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This allows us to construct a forcing notion which preserves non-null sets and adds a big sequence.

Next using \diamond for a given uncountable set of reals we can find a subforcing \mathbb{P}_X such that

- ① \mathbb{P}_X is ccc,
- ② $\mathbf{V}^{\mathbb{P}_X} \models \mathbf{V} \cap 2^\omega \notin \mathcal{N} \cup \mathcal{M}$
- ③ $\mathbf{V}^{\mathbb{P}_X} \models X \notin \mathcal{SM}$.

We will build the required forcing as a increasing chain of approximations $\{\mathbb{P}_\alpha : \alpha < \omega_1\}$ and put $\mathbb{P}_X = \bigcup_{\alpha < \omega_1} \mathbb{P}_\alpha$. In order to guarantee that \mathbb{P}_X satisfies ccc we will use an oracle that will tell us that whenever \mathcal{A} is a maximal antichain in \mathbb{P} then \mathcal{A} is frozen at some stage α .

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Thus we will have two disjoint stationary sets S_0 and S_1 and a sequence of countable models $\{M_\alpha : \alpha \in S_0 \cup S_1\}$ which witness \diamond on S_0 and S_1 .

We will be making two types of commitment by requiring that for stationary many α :

- ① If $\mathcal{A} \in M_\alpha$ is a maximal antichain in \mathbb{P}_α then \mathcal{A} is a maximal antichain \mathbb{P} ,
- ② $\Vdash_{\mathbb{P}} x_\alpha$ is random over $M_\alpha[\dot{G}]$ for a fixed set $Y = \{x_\alpha : \alpha \in S\}$ such that x_α is random over M_α .

The forcing \mathbb{P} will be constructed from $\omega_1 \times \omega_2$ countable pieces. The ω_2 axis will correspond to the ω_2 -iteration while the ω_1 axis will correspond to the single task of making a given \aleph_1 -set not strongly meager. In general, $\mathbb{P}_{\alpha+1}$ will be of the form $\mathbb{P}_\alpha \star \mathbb{P}_X$.

New type of iteration:

instead of preservation theorems we have commitments.

The task at the limit step will be to extend the construction rather than to prove a preservation theorem.